FORSCHUNGSINSTITUT FÜR MATHEMATIK ETH ZÜRICH

Plane Curves, Their Invariants, Perestroikas and Classifications

by

Vladimir Arnold

Steklov Mathematical Institute, Moscow

May 1993

Plane curves, their invariants, perestroikas and classifications

Vladimir Arnold

Steklov Mathematical Institute, Moscow*

A *curve* below means a smooth mapping of a circle to a plane whose derivative nowhere vanishes (an immersion).

A generic immersion has only ordinary double points of transversal selfintersections. All nongeneric immersions form in the space of immersions a hypersurface which I shall call the *discriminant hypersurface* or for short the *discriminant.*

The goal of this article is the study of the geometry and of the topology of this hypersurface. This study leads immediately to nontrivial information on the generic immersions. For instance, the results of this paper imply the following facts.

Consider a generic closed path in the space of immersions.

Theorem. *The number of values of the parameter for which the immersed curve has a triple point is even.*

Definition. A selftangency point of an immersed curve is called a point of *direct tangency* if the velocity vectors are pointing to the same direction; otherwise it is called a point of *inverse tangency.*

A moment of selftangency of a curve on a generic path in the space of curves is *positive (negative)* if the number of double points increases (decreases) while the path goes through this moment.

Theorem. *The difference between the numbers of positive and negative direct selftangency moments does not depend on a generic path connecting two generic immersions, but only on the two immersions connected by the path.*

---

* This paper has been written while visiting Isaac Newton Institute for Mathematical Sciences, Cambridge (UK), Université Paris-Dauphine and Mathematics Research Institute, ETH Zürich. The author thanks these institutions for their hospitality and Rahel Boller for fasttyping a difficult manuscript.

A similar result holds for the inverse selftangency moments.

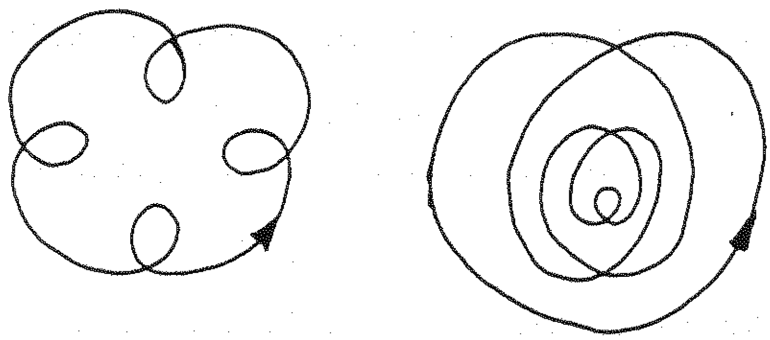

Figure 1:
Two immersions connectable by a path in the space of immersions

The two immersions shown in Fig. 1 can be connected by a path in the space of immersions.

Theorem. *Any generic one-parameter family of immersions, connecting the two immersions shown in Fig. 1, has at least 6 paramter values corresponding to immersions with triple points, at least 6 paramter values corresponding to immersions with direct selftangencies, and at least 6 parameter values corresponding to immersions with inverse selftangencies.*

Moreover, the difference between the numbers of moments of positive and negative direct tangencies on a path from the right curve to the left one is equal to 6. For the inverse tangencies the difference is equal to -6.

For the triple points there also exists a (nontrivial) way to define the positive and negative moments. The number of the moments of triple points, counted with these signs (described below) on any generic path from the left curve to the right one, is equal to 6.

The invariants of immersions, responsible for these and many other results of this paper, are dual to different strata of the discriminant hypersurface.

The description of all the invariants (that is, of all functions locally constant on the complement of the discriminant hypersurface) is equivalent to the classification of the immersions up to diffeomorphisms of the plane and of the circle. Such a description seems to be very complicated (more complicated than knot theory).

We shall see, however, that some simple and rather natural axioms are verified only by a small number of simplest invariants which seem to be the most important ones. It is strange that these invariants — in particular the most fundamental three of them which I denote St, J^+, J^- — have not been introduced earlier.

The initial goal of my study of plane curves was an attempt to prove some symplectic topology generalisations of the 4-vertex theorem. This generalisation leads to the following conjecture.

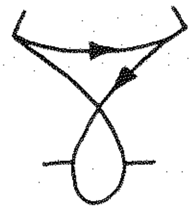

Figure 2:
A cooriented and oriented front with two cusps

Consider the plane cooriented curve with two semicubical cusp points shown in Fig. 2. There exists a path in the space of cooriented curves with two cusps, connecting this curve to the same curve with the opposite coorientation (Fig. 3)

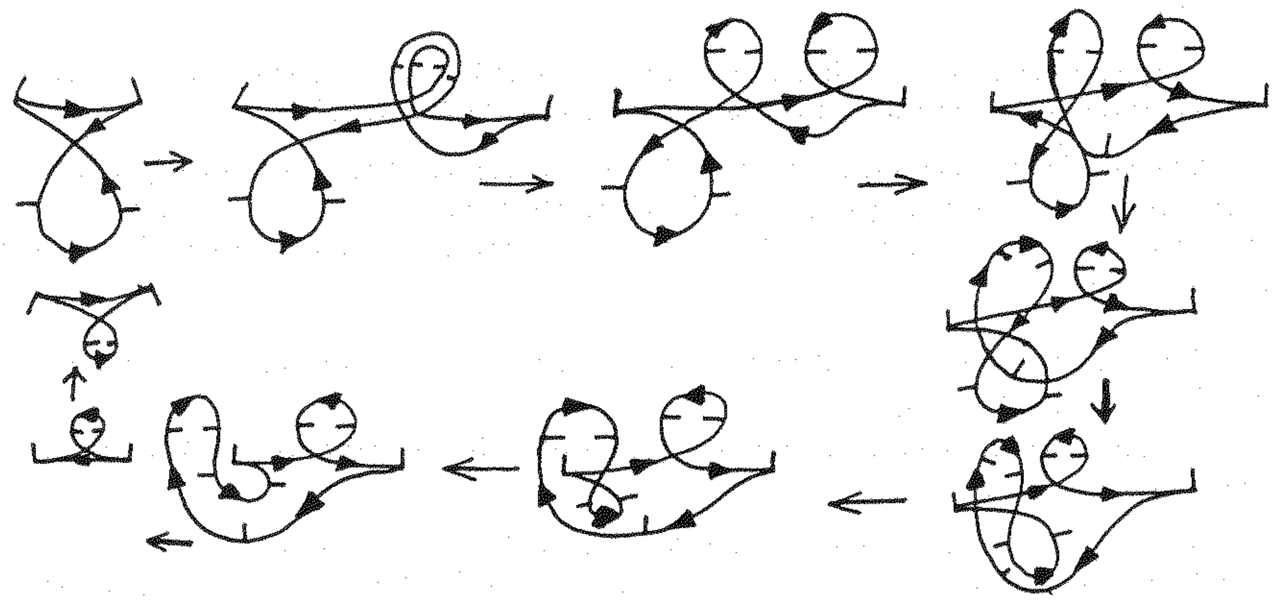

Figure 3:
A sequence of perestroikas of a front, reversing its coorientation

Conjecture. *Any such generic path contains at least one moment of selftangency with coinciding coorientations.*

The number of such moments is even (this can be proved by the methods of the present article). However, I have not (yet?) constructed an invariant of cooriented curves with two cusps and no equally cooriented selftangencies, taking different values on the curve of Fig. 2 and on the curve with opposite coorientation. It seems that such an invariant must violate the "natural" axioms verified by our invariants.

§1 The three basic invariants

These invariants are dual to the three parts of the discriminant hypersurface, formed by the immersions having triple points, having direct selftangencies, and having inverse selftangencies respectively.

Lemma. *Each of these three parts of the discriminant hypersurface is coorientable.*

Comments. A *coorientation* of a smooth hypersurface in a functional space is the choice of one of the two parts, separated by this hypersurface in a neighbourhood of any of its points. This part is called *positive.*

The coorientation of the smooth part of a singular hypersurface is called *consistent* if the following consistency condition holds in a neighbourhood of any singular point of any stratum of codimension 1 on the hypersurface (of codimension 2 in the ambient functional space):

The intersection index of any generic small oriented closed curve with the hypersurface (defined as the difference between the numbers of positive and negative intersections) *should vanish.*

A hypersurface (a subvariety of codimension one) is called *cooriented* if a consistent coorientation of its smooth part is chosen, and *coorientable* if it exists.

The proof of the Lemma is based on the explicit construction of a coorientation. This coorientation is in fact *unique* (up to the complete change of sign) in the class of *local* coorientations, defined above, and does not depend neither on the orientation of the plane nor of the curve.

Definition. A transversal crossing of a self tangency is *positive* if the number of double points grows (by 2).

A transversal crossing of a triple point is *positive* if the new-born vanishing triangle is positive.

A *vanishing triangle* is the triangle, formed by the three branches of a curve, corresponding to a sub-critical or to a super-critical value of the parameter near a triple point of a critical curve.

The *sign of a vanishing triangle* is defined by the following construction. The orientation of the immersed circle defines a cyclical ordering of the sides of the vanishing triangle (it is the order of the visits of the triple point by the three branches). Hence the sides of the triangle acquire orientations induced by the ordering. But each side has also its own direction which might coincide, or not, with the orientation defined by the ordering.

For each vanishing triangle we define a quantity (which takes four values 0, 1, 2, 3)

q = number of sides equally oriented by the ordering and by their directions (Fig. 4).

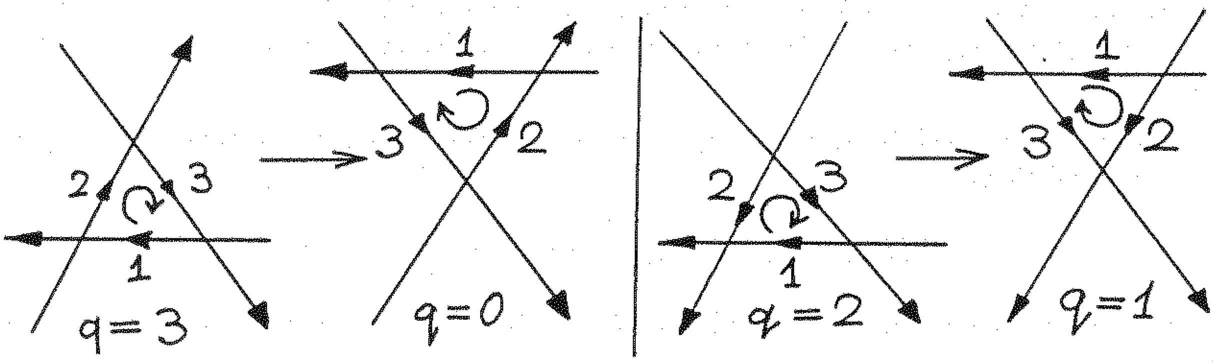

Figure 4:
Positive and negative vanishing triangles

Definition. The *sign* of a vanishing triangle is $(-1)^q$.

Example. A triangle is *positive* if the directions of the sides are all *opposite* to the orientation defined by the ordering.

Remark. The reversing of the orientation of the curve does *not* change the quantity q, since it reverses both the cyclic order and the directions of the sides. Hence, the coorientation of the triple points component of the discriminant, chosen above, does not depend on the orientation of the circle.

The orientation of the plane has not been used in the construction at all.

Theorem. *The above coorientation of the smooth part of the discriminant is consistent.*

The proof is a rather long routine check of what happens at the codimension one strata of the discriminant. In the case of the selftangencies this proof is very easy, since the number of double points is a well-defined invariant. In the case of triple points some real computations are needed; we shall present them in §5 (where the unicity of the coorientation is also proved).

Definition. The *index* of an immersion (of an oriented circle into an oriented plane) is the number of turns of the tangent vector (the degree of the mapping, sending a point of the circle to the direction of the derivative of the immersion at this point).

The index is the only invariant of immersions remaining constant along all the paths in the space of immersions. In other words, the connected components of the space of immersions are counted by the indices of the immersed curves (Whitney's theorem, [1]).

Consider one of these components, that is, the space of immersions of a fixed index. In §2 we shall prove the following three theorems.

Theorem 1. *There exists a unique (up to an additive constant) invariant of generic immersions of fixed index whose value remains unchanged while the immersed curve experiences a selftangency perestroika, but increases by 1 under the positive crossing of a triple point.*

This invariant will be denoted by St (from Strangeness), when normalized by the following conditions:

$$St(K_0) = 0\,, \quad St(K_{i+1}) = i \qquad (i = 0, 1, \ldots)\,,$$

where K_0 is the figure eight curve (∞) and K_{i+1} is the simplest curve with i double points (Fig. 5).

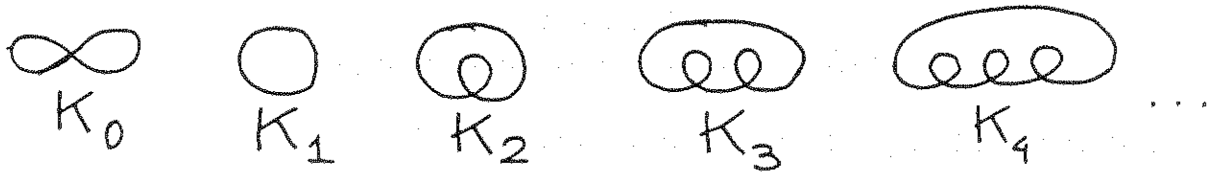

Figure 5:
The standard curves of indices $0, \pm 1, \pm 2, \ldots$

The reason for this normalization is discussed above in §5. The curves K_0 and $K_1, K_2, \ldots$ with different orientations have indices $0, \pm 1, \pm 2, \ldots$.

Theorem 2. *There exists a unique (up to an additive constant) invariant of generic immersions of fixed index whose value remains unchanged while the immersed curve experiences an inverse selftangency perestroika or a triple point crossing, but increases by a constant number a_+ under a positive (increasing the number of double points) crossing of a direct tancency perestroika.*

This invariant will be denoted by J^+ when normalized by the choice $a_+ = 2$ and by the following choices of the values on the curves K_i of Fig. 5 (oriented arbitrarily):

$$J^+(K_0) = 0\,, \quad J^+(K_{i+1}) = -2i \qquad (i = 0, 1, 2, \ldots)\,.$$

The reasons for these choices are explained below.

Theorem 3. *There exists a unique (up to an additive constant) invariant of generic immersions of fixed index whose value remains unchanged while the immersed curve experiences a direct tangency perestroika or a triple point crossing, but increases by a constant number a_- under a positive (increasing the number of double points) crossing of the inverse tangency perestroika.*

This invariant will be denoted by J^- when normalized by the choices $a_- = -2$ and $J^-(K_0) = -1$, $J^-(K_{i+1}) = -3i$ $(i = 0, 1, \ldots)$ for the curves K_i of Fig. 5, oriented arbitrarily. The reasons for these choices are explained below.

St	1	0	0	0	0	1	2	3	...
J^+	0	2	0	0	0	-2	-4	-6	...
J^-	0	0	-2	-1	0	-3	-6	-9	...

Figure 6:
The definitions of the basic invariants St, J^+, J^-

The normalizing constants are summarized in Fig. 6, where $a(\cdot)$ means the jump of an invariant under the positive crossing of the discriminant at the stratum $(\cdot)$.

To calculate the value of an invariant on a generic immersion it suffices to join it by a generic path with the standard immersion of the same index and to count the jumps at the crossings of the discriminant.

Example: There exist exactly five classes of immersions with two double points, as shown in Fig. 7 (two immersions are in the same class if one can be transformed into the other by diffeomorphisms of the circle and of the plane; both diffeomorphisms might reverse the orientations).

The values of the basic invariants for these five immersions are shown in Fig. 7.

The values of the basic invariants on the curves L (to the left) and R (to the right) in Fig. 1 are equal to

$$St(L) = 4 \;, \quad J^+(L) = -8 \;, \quad J^-(L) = -12$$
$$St(R) = 10 \;, \quad J^+(R) = -20 \;, \quad J^-(R) = -24 \,.$$

These values imply the theorems formulated in the introduction.

The values of the basic invariants on the curves with three and four double points are presented at the end of this article. These tables imply thousands of theorems similar to that of the introduction.

					
$\lvert ind \rvert$	1	1	1	3	3
St	0	0	1	2	3
J^+	0	0	−2	−4	−6
J^-	−2	−2	−4	−6	−8

Figure 7:
The basic invariants of immersions with two double points

§2 Proof of the existence of the basic invariants

This proof is based on a method of S. Smale [2].

The group of the euclidean motions of the plane, $\mathsf{R}^2 \times S^1$, acts naturally on the space of circle immersions into the plane. This action has no fixed points in the space of immersions.

The space of immersions is fibered into the orbits of this action. This action has a section: the set of immersions sending a fixed point $(*)$ on the circle to a fixed point (0) of the plane and the orienting vector of the circle to a vector of a fixed direction, tangent to the plane at 0. We shall call such immersions *normed* immersions.

The above action preserves all our discriminants, classes of immersions, and invariants. Hence it suffices to study all these objects only for the normed immersions. From now on all the immersions are supposed to be normed. The Smale theory implies the

Theorem. *The space of normed immersions of a fixed index has trivial homotopy groups of dimension $i > 0$.*

Proof. This follows from the exact sequence of the Serre fibration studied by Smale,

$$N \longrightarrow (\mathsf{R}^2 \backslash 0) \times \mathsf{R}^2 ,$$

sending a normed immersion of a segment $[0,1]$ (with a fixed direction at the starting point 0) to the velocity vector at the end point 1.

The homotopy groups of the space N are trivial. Indeed, an immersion f of the segment $0 \leq t \leq 1$ may be contracted to its small part by the homotopy $f_s(t) = f(st)$ with a parameter $s_0 \leq s \leq 1$, s_0 very small. The small resulting normed immersion may be then interpolated with a standard small normed immersion linearly.

We shall use the fact that the space N_i of normed immersions of a circle of index i is simply connected. This fact implies the following

Theorem. *The intersection index of a closed curve in the space of normed immersions with any of the three branches of the discriminant, described in §1, is equal to zero.*

Proof. Since the normed immersions space is simply connected, one can find a disc bounded by the curve. This disc (and the curve itself) may be made transversal to the stratification of the discriminant. As a chain, the disc is the sum of small simplices, also transversal to the discriminant.

If a transversal simplex is sufficiently small, it lies in a two-dimensional versal deformation plane of a singular point of codimension zero or 1 on the discriminant. In this case, the intersection index of its boundary with the discriminant vanishes, since the coorientation of the smooth part of the discriminant is consistent (see §1). The theorem is thus proved. It implies the existence of the invariants of theorems 1–3 of §1.

At the same time we have proved the uniqueness of the invariants (up to the choice of the additive constants, depending only on the index), since the space of immersions of fixed index is connected.

§3 Properties of the invariant St

Definition. The *connected sum* of two immersions of a circle (the first into the left, the second into the right halfplane) is the new immersion shown in Fig. 8.

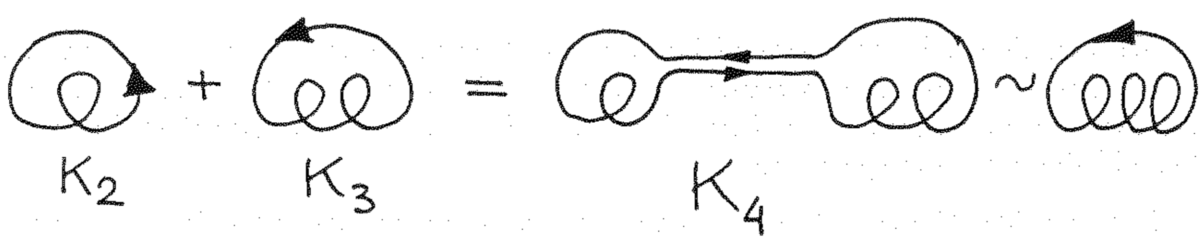

Figure 8:
The connected sum of two immersions

Remark. The connected summation is not an operation on the immersion classes. Some pairs of immersions can not be added (example: two standard circles with opposite orientations). When the summation is possible, the class of the sum depends in general on the representatives of the classes of the summands.

Theorem. *The invariant St is additive:*

$$St(a+b) = St(a) + St(b) .$$

Proof. Choose a standard representative of each class of immersions of index i, orienting the curves of Fig. 5 (the figure eight curve with both orientations belongs to the same class). It follows from Fig. 8 that St is additive for the standard immersions of positive indices $i > 0$, $j > 0$:

$$St(K_i) = i-1\ , \quad St(K_j) = j-1\ , \quad K_i + K_j = K_{i+j-1}\ .$$

One proves by a direct computation that St is additive for the standard summation with the figure eight curve (this follows from the fact that the chain of perestroikas shown in Fig. 9 decreases St by 1).

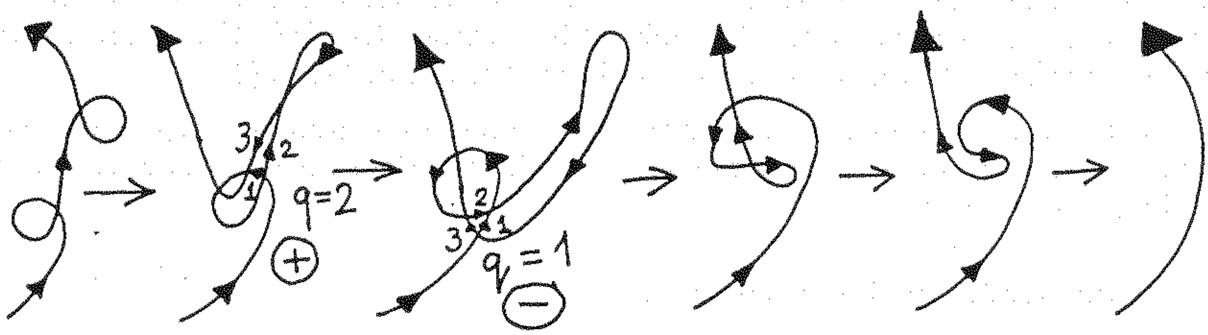

Figure 9:
Annihilation of counteroriented loops

We can choose in any class of immersions of given index i a special immersion L_i whose exterior contour has intervals oriented in both directions. Two immersions of this kind always admit summations. Using Fig. 8 and Fig. 9, we see directly that the invariant St is additive for these special cases.

Now consider the sum $X+Y$ of any two generic immersions of a circle. We connect the left curve X with a standard curve L of the same index by a generic path in the space of immersions. We can suppose that the immersions forming this path are all identical in the neighbourhood of the point sent to the connection place on the left curve (the other branches of the left curve may cross this point during the deformation of the immersion). We also can suppose that the whole path consists of immersions into the left halfplane.

Similarly we connect the right curve Y with a standard curve R in the right halfplane. Combining both paths, we obtain a generic path connecting $X+Y$ with $L+R$.

The increasing of St along this path equals the sum of its increasings along the left and the right paths. Indeed, neither the orientations of the vanishing triangles nor the cyclic orderings of their sides do change under the summation. Hence, the signs of the triple point crossings of the left and the right curves do not change. Besides these points, the new path crosses the triple points while the double points of the moving left or right curve cross the joining band. These crossings occur in pairs of opposite signs and do not contribute to the increment of St, since the corresponding vanishing triangles differ by the direction of one of the sides.

Thus, we find

$$St(X+Y) - St(L+R) = St(X) - St(L) + St(Y) - St(R)\,.$$

Hence, the additivity for the summation of any two curves follows from the additivity for the summation of the special curves

$$St(L+R) = St(L) + St(R)$$

which is already proved by direct calculations.

Definition. The *strange summation* of two immersions of a circle (one into the left, the other into the right halfplane) is defined by an immersion of a segment joining them such that the immersions coorient the immersed segment differentily at the end points. To obtain the new circle immersion, one doubles the segment and smoothens the angles as shown in fig 10.

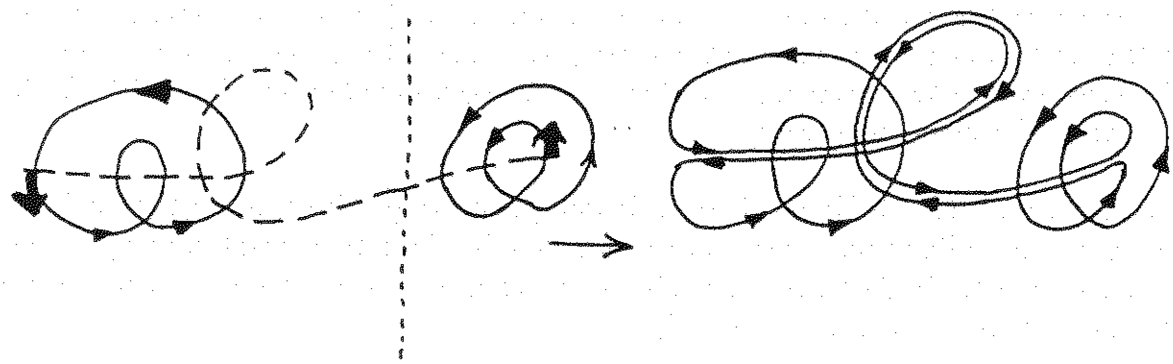

Figure 10:
The strange summation of immersions

Theorem. *The invariant St is additive under the strange summation of immersions.*

Proof. The strange summation can be reduced to the ordinary one if we first push appendices from each curve toward the other (Fig. 11).

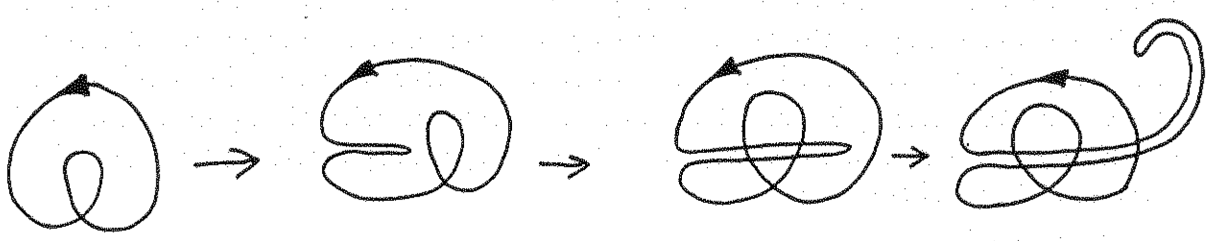

Figure 11:
Pushing of an appendice from the left curve

This pushing does not introduce triple points, hence does not change the value of St. Thus the strange summation additivity follows from the ordinary additivity.

§4 Properties of the invariants $J^{\pm}$

The invariants $J^{\pm}$ are not additive under the strange summation, but they are additive under the usual one. This follows, for instance, from the explicit formula for the combinations $J^{\pm}+3St$ in terms of the rotations of the radius-vectors connecting the double points of the curve to its moving point.

Choose such a representative of a generic immersion (or such a metric on the plane) that the intersecting branches in all double points are orthogonal. Choose the ordering of the directions of the outgoing branches (1,2) at each double point for which the frame (1,2) orients the plane positively.

Definition. The *halfindex* i_1 of a double point (respectively i_2) is the angle of the rotation of the radius-vector connecting this double point to a point moving along the branch 1 (respectively 2) from the double point to itself divided by $\pi/2$. The *index* of a double point is the difference $i = i_1 - i_2$.

The index does not depend on the orientations of the curve and of the plane (Fig. 12).

Definition. The *invariants* $I^{\pm}$ are defined by the summation of the indices of all the n double points, namely,

$$I^{\pm} = \frac{\sum i \pm 2n}{4}.$$

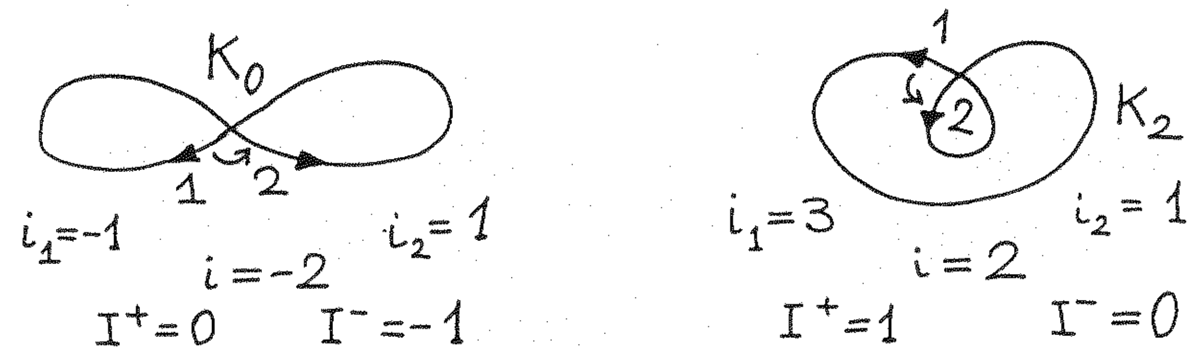

Figure 12:
The halfindices and indices of double points

Theorem. *The invariants $J^{\pm}$ are related to $I^{\pm}$ by the relation*

$$J^{\pm} = I^{\pm} - 3St\,.$$

Proof. For the special curves K_i of Fig. 5, one can check it directly. For instance, $St(\infty) = 0$, $St(K_2) = 1$. Hence Fig. 12 corresponds to the values

$$J^+(K_0) = 0\,, \quad J^-(K_0) = -1\,, \quad J^+(K_2) = -2\,, \quad J^-(K_2) = -3\,.$$

To prove the theorem for the curves of an arbitrary index it suffices hence to calculate the increments of $I^{\pm}$ under the elementary perestroikas.

Case 1. Direct selftangency. Consider the crossing of a direct selftangency generating two double points A and B. We wish to prove that the value of I^+ increases by 2 while that of I^- does not change.

Lemma. $i(A) + i(B) = 4$.

Proof. First calculate the sum for one example (Fig. 13).

The lemma for the general case follows. Indeed, any other example is different from this one only outside the neighbourhood of the newborn point. The replacement of any of the branches of the curve outside this neighbourhood can be considered as an adding to this branch of a closed curve which does not intersect the neighbourhood. Such an addition changes the halfindices by the (quadrupled) number of the turns the added curve makes around the double point A or B. Or these two numbers are equal, since the added curve does not intersect the neighbourhood. However, the ordering of the branches at A and at B are opposite, and the increment of $i_1(A)$ takes place together with an equal increment of $i_2(B)$ (similarly for $i_2(A)$ and $i_1(B)$). Thus the increment of the sum

$$i(A) + i(B) = i_1(A) - i_2(B) + i_1(B) - i_2(A)$$

under the addition of the closed curve vanishes.

Hence I^+ and I^- behave correctly under the direct selftangency perestroika.

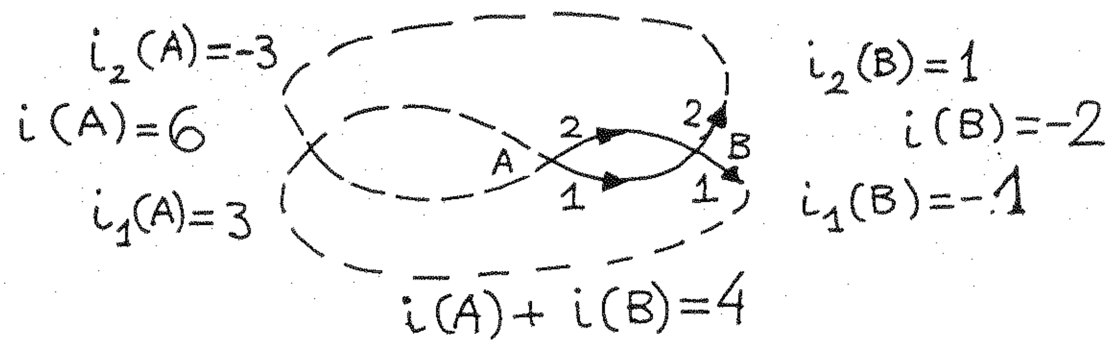

Figure 13:
The increment of the sum of the indices of
double points under the direct selftangency perestroika

Case 2. Inverse selftangency.

Lemma. $i(A) + i(B) = -4$.

Proof. A particular example is shown in Fig. 14.

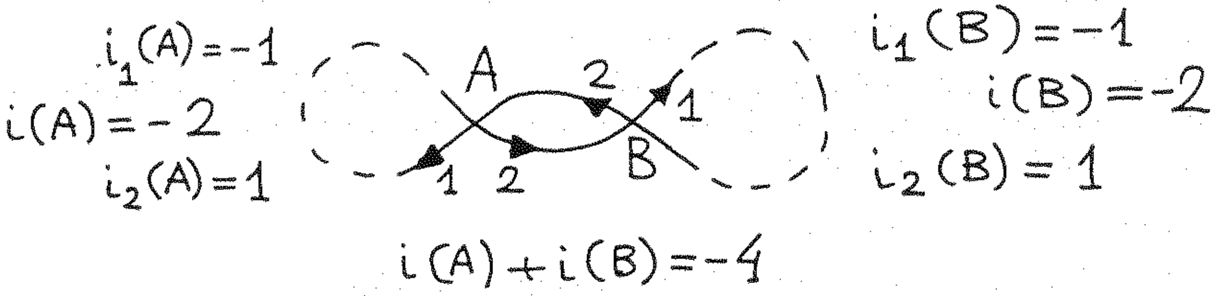

Figure 14:
The increment of the sum of the indices of double
points under the inverse selftangency perestroika

The general case is reducible to that of Fig. 14 by the same arguments as in the direct selftangency case.

Hence I^+ does not change its value under the crossing of an inverse selftangency giving birth to two double points while the value of I^- decreases by 2.

Case 3. A triple point. The number n of double points does not change under a crossing of a triple point.

Lemma. *The index of each of the three vertices of the vanishing triangle increases by 4 under a positive crossing of a triple point.*

Proof. We can fix two of the three branches of the curve and move the third one. Let A_1, A_2 be the fixed branches leaving the fixed intersection point A along the vectors (1,2) orienting the plane positively. The third visit of the triple point takes place on one of these two branches. The rotation of the radius-vector will change under the crossing of the triple point for one of the branches, namely for the one containing the third visit moment. The increment of the angle of the rotation will be $\pm 2\pi$. It will be positive if, after the crossing, the third branch will define a frame (radius-vector from A to a point of the third branch, velocity vector of the third branch) positively orienting the plane.

This increment of the argument divided by $\pi/2$ is equal to the increment of the index of the point A, with the *plus* sign if the third visit point belongs to the branch A_1 (leaving A in the direction 1).

There exist four possibilities for the occuring vanishing triangles (for which the increments of the argument along the branch and of the index of the point A are both positive). They are shown in Fig. 15.

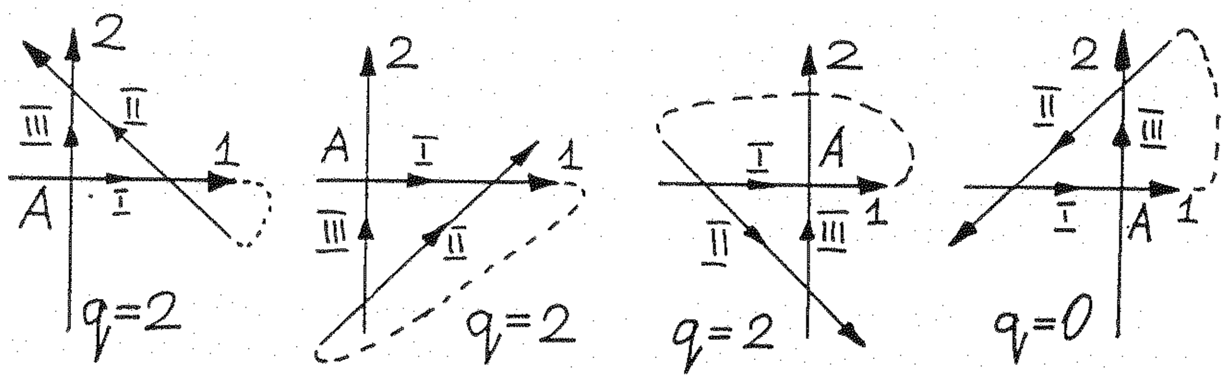

Figure 15:
Vanishing triangles whose births imply increments
of arguments and of the index of point A

We observe that all these four vanishing triangles are *positive*.

The case of the third visit on the branch 2 can be considered similarly. But one also can avoid this, since J^+ and i do not depend on the orientations.

Thus, the quantities $I^{\pm} - 3St$ and $J^{\pm}$ behave the same way under all the three elementary perestroikas. Since their values on the standard curves K_i of Fig. 5 also coincide, the theorem is proved.

Corollary. *The invariants $J^{\pm}$ are additive with respect to the usual connected sum of immersions.*

Proof. The invariants St and n (the number of double points) are additive. Hence we only have to check the additivity of the sum of the indices of the double points.

The double points of the sum are just the double points of the left and of the right curve. The addition preserves any halfindex of any double point of the left (right) curve, since the added part of the curve may be contracted in the right (left) halfplane to the end of the joining segment.

Remark. Of course,

$$J^+ - J^- = I^+ - I^- = n \ , \text{ the number of double points,}$$

is also an additive invariant. However, the preceeding theorems show that it is better to consider J^+ and J^- as the basic invariants. The invariant J^+ measures some kind of self-linking of the Legendre curve formed by the directions of the plane curve in the 3-dimensional space of plane contact elements.

§5 The axiomatic description of the basic invariants

Definition. An invariant (of an immersion of a circle into a plane with no triple points) is *local* if its jump at the generic crossing of the hypersurface of immersions with triple points depends only on the behaviour of the family in the neighbourhoods of the three points sent to the triple point by the critical immersion.

It means that the jump will not change if we replace our family by any other family outside the above mentioned neighbourhoods.

Theorem. *The jump of any local invariant of immersions with no triple points is proportional to the jump of the invariant St (with a coefficient independent of the immersion).*

Proof. The only strata of codimension one on the triple points discriminant are the strata corresponding to the following singularities of immersions:

(a) a triple point with a tangency of two branches;

(b) a quadruple point;

(c) two triple points.

The jump of an invariant which is local does not change along any of the two intersecting branches of the discriminant when one crosses the other branch moving along the first in case (c).

The jumps near the strata (a) and (b) must be consistent. We shall see that the consistency condition at the (a) stratum leaves for the jump only one possibility — the one we have described above in the definition of St.

The versal deformation of the singularity of type (a) is a two-parameter family of curves on a plane (x,y) with parameters u and v:

$$x = 0\,, \quad y = 0\,, \quad y = x^2 + ux + v\,.$$

The discriminant is shown in Fig. 16.

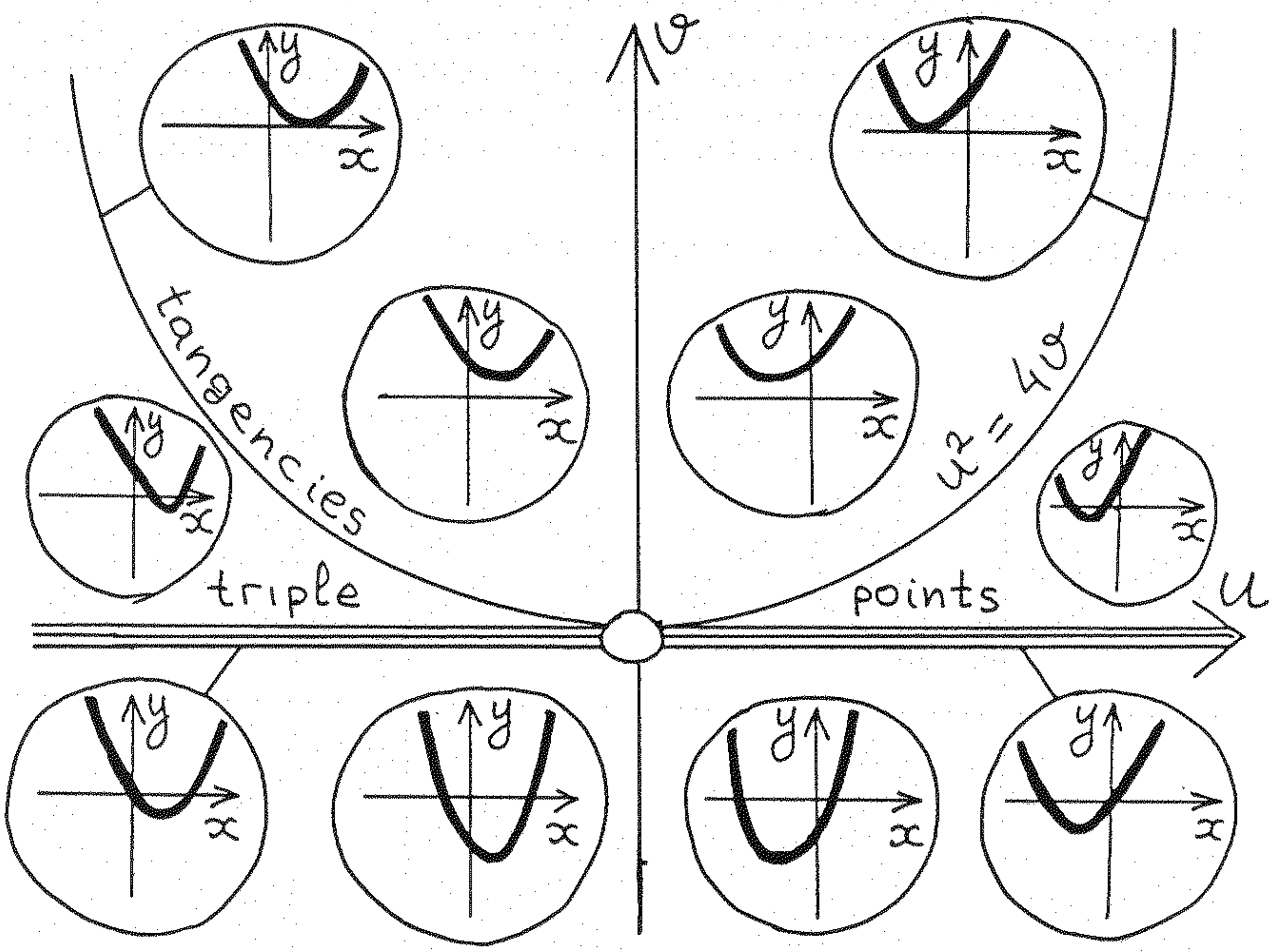

Figure 16:
The versal deformation of a degenerate triple point

The stratum of triple points is hence smooth in the neighbourhood of the singularity of type (a) — the event happening there is the tangency to the stratum of selftangency.

The consistency condition says that the jumps of the invariant due to the transversal crossing of the line $v = 0$, say from $v > 0$ to $v < 0$, should be the same on the left ($u < 0$) and on the right ($u > 0$) part of Fig. 16.

The local situation in the neighbourhood of a triple point crossing is described by the vanishing triangle formed by three oriented lines ordered cyclically.

There exist eight types of such triangles in the plane (up to orientation preserving diffeomorphisms of the plane). Indeed, the angles between the three lines may be reduced to 60^0. Then the angles between their directions are either all equal to 120^0 (case A) or are $(60^0, 60^0, 120^0)$ (case B).

The cyclical ordering of the three rays leaving the triple point defines an orientation of the plane which may be positive or negative. We thus define four types of triple points, A_-, A_+, B_-, B_+. In each case we have two types of vanishing triangles — one from each side of the discriminant hypersurface. We shall denote these by the types of the triple point with a superior index q (equal to the number of sides directed conformally to the cyclical order).

All the eight resulting triangles are shown in Fig. 17.

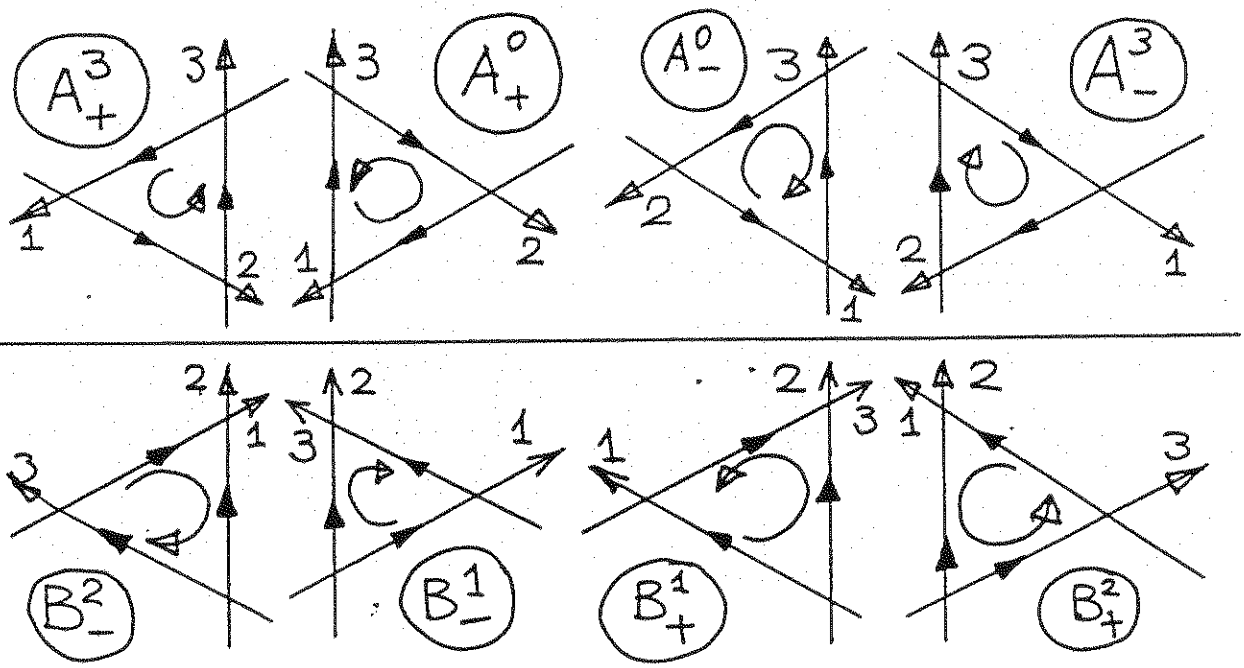

Figure 17:
Classification of the vanishing triangles

Remark. The pairity of q always changes under the crossing of the discriminant of the triple points. Indeed, the orientation of the plane, defined by the cyclical ordering of the sides of the triangle, changes while the directions do not change, hence $q(-)+q(+)=3$ for any two neighbouring points $(-)$ and $(+)$ separated by the discriminant.

Lemma 1. *The pairs of the types of the vanishing triangles in the domain $v<0$ of Fig. 16 may be only*

$$A_+^3 \text{ and } B_+^1 \,,\; A_-^3 \text{ and } B_-^1 \,,\; A_-^0 \text{ and } B_-^2 \,,\; A_+^0 \text{ and } B_+^2 \,,$$

$$B_-^2 \text{ and } B_+^2 \,,\; B_-^1 \text{ and } B_+^1 \,,$$

depending on the orientation and ordering of the three curves of Fig. 16.

To prove this it suffices to calculate the types of both curvilinear triangles formed by the parabola and the coordinate axis for different orientations and orderings. Two crucial examples are shown in Fig. 18.

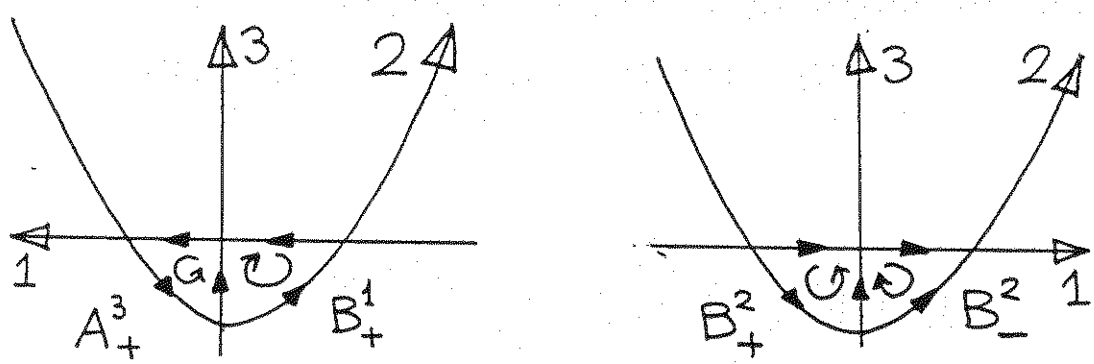

Figure 18:
Coexisting vanishing triangles

Thus, if there exists a local invariant, then, in the corresponding coorientation of the stratum of triple points, the points corresponding to the four coexistent triangles $A^0_- \sim B^2_- \sim B^2_+ \sim A^0_+$ should all be on one side of the discriminant, and those corresponding to $A^3_- \sim B^1_- \sim B^1_+ \sim A^3_+$ on the other. Thus q is even for all the triangles on one side and odd for those on the other side, conformally to the coorientation rule defining the invariant St.

Remark. The independence of this coorientation of the choices of the orientations of S^1 and $\mathbb{R}^2$ follows thus from the localness of the jump.

Lemma 2. *The coorientation of the triple points discriminant is consistent at the points of the stratum of quadruple points.*

Proof. The (topological) versal deformation is the two-parameter family of quadruples of lines in the plane, given by the table

I	II	III	IV
$x = 0$	$y = 0$	$y = x + u$	$y = -x + v$,

where u and v are parameters. The stratum of triple points intersects the plane (u, v) along four lines corresponding to the 4 types of triple points (Fig. 19)

$(I\ II\ III)$	$(I\ II\ IV)$	$(I\ III\ IV)$	$(II\ III\ IV)$
$u = 0$	$v = 0$	$u = v$	$u = -v$.

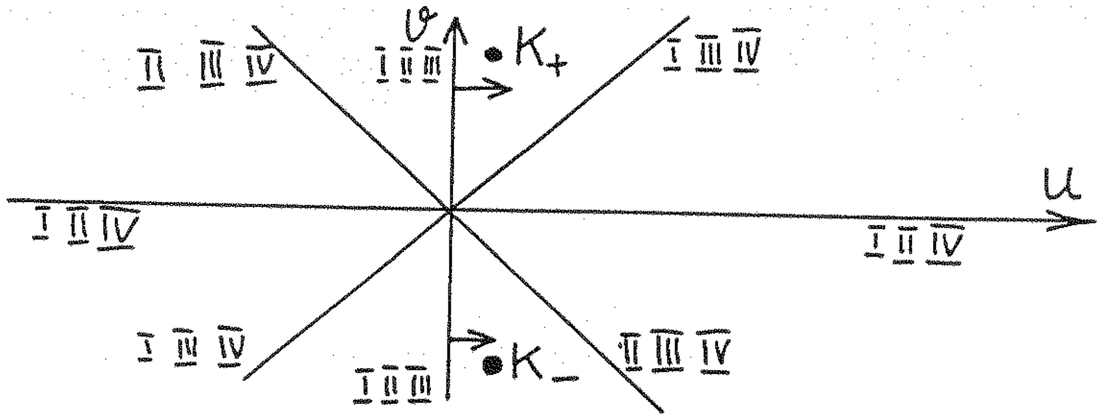

Figure 19:
The versal deformation of a quadruple point

Proposition. *The coorientations of the triple points discriminant defines on each of the four lines the same coorientation along both rays into which this line is divided by the origin.*

Proof. Consider for example the discriminant line (I II III), $u = 0$. The configurations of the branches at the $u > 0$ side of this line for $v = 0$ and for $v < 0$ are shown in Fig. 20.

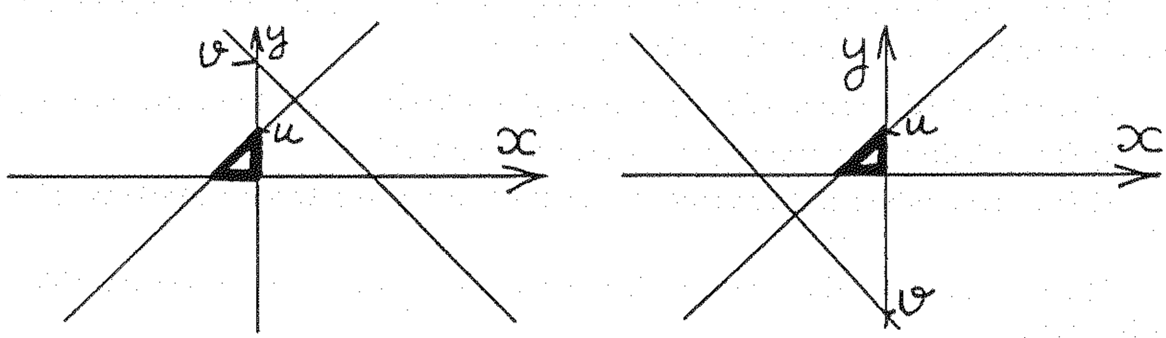

Figure 20:
The consistency of coorientations at a quadruple point

The vanishing triangles for fixed and very small $u > 0$ simply coincide, they are the same for $v > 0$ and $v < 0$. Hence $q(K_+) = q(K_-)$ and both points K_+ and K_- lie on the same side of the stratum of triple points.

Theorem. *There exists exactly one invariant, St, of immersions* $S^1 \to \mathbb{R}^2$ *without triple points with the following properties:*

1) *it is local* (the jump only depends on the behaviour near the selftangency point),

2) *it is additive* (under the ordinary connected summation),

3) *it is orientation independent* (at least for one of the orientations: that of S^1 or that of $\mathbb{R}^2$).

The conditions 1–3 define the invariant up to a multiplicative constant. To fix the constant it is sufficient to fix the value at one curve, where the value of St is not zero. We choose the condition

4) *The invariant takes the value 1 at the immersion of index 2 with one selfintersection point* (K_2 in Fig. 5).

Remark. We already know that St has all these properties (and moreover that it is additive under the more general *strange* summation and independent on *both* orientations). So we only have to prove its uniqueness.

Proof. The jump of an invariant which is local is proportional to the jump of St, according to the preceeding theorem. Hence the invariant is defined by its values on the standard immersions K_i of index i of Fig. 5 ($i = 0, \pm 1, \pm 2, \ldots$), and the difference of two such invariants is constant along the set of immersions of index i. Let this difference be $f(i)$. Then we have

$$f(i_1 + i_2 - 1) = f(i_1) + f(i_2),$$

hence $f(i) = c(i-1)$. The independence of orientation implies $f(1) = f(-1)$, hence $c = 0$, which proves the theorem.

Theorem. *There exists exactly one invariant, $J^+(J^-)$, of immersions with no direct (inverse) selftangencies with the following properties:*

1) *it is local* (the jump only depends on the behaviour near the selftangency point),

2) *it is additive* (for ordinary connected summation),

3) *it is orientation independent* (for at least one of the two orientations, of S^1 or of $\mathbb{R}^2$).

These conditions define the invariant up to the multiplicative constant which we fix by the condition

4) $J^+(K_2) = -2 \quad (J^-(\infty) = -1)$.

Remark. The multiplicative constant has been chosen in such a way that a crossing of the direct selftangency discriminant increases J^+ by the *same* number as it increases the number n of double points, while the crossing of the *inverse* selftangency discriminant increases J^- by the *opposite* of the increment of n. These choices imply $J^+ - J^- = n$. Note that the jumps of $J^\pm$ at the crossings of the corresponding discriminants are thus ± 2 and not ± 1.

§6 The "pushing away" formula

The calculation of the invariant St is greatly simplified by the following method: one can control the increment of St when a fragment of a curve is pushed away, as it is shown in Fig. 21.

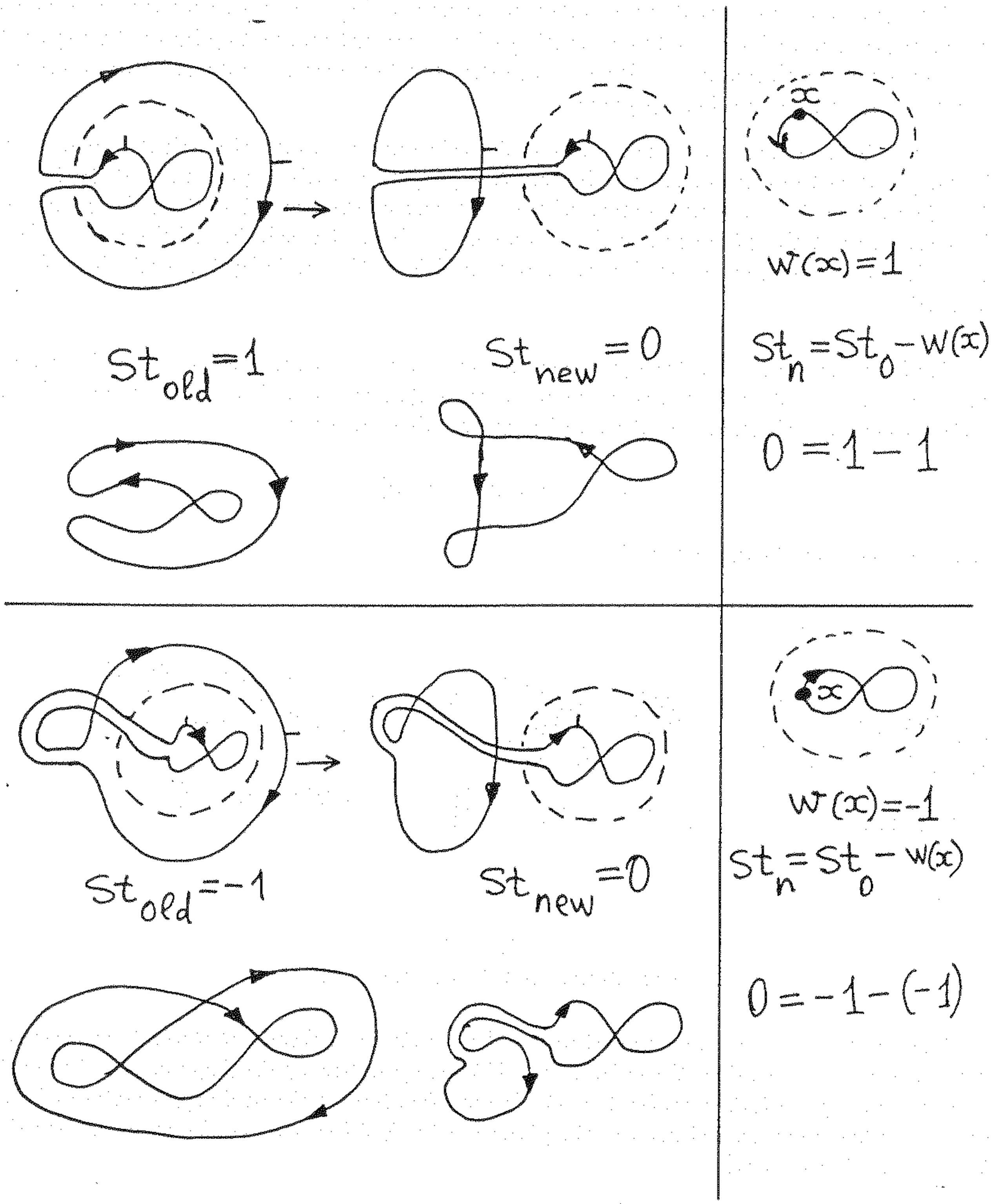

Figure 21:
The "pushing away" perestroika

The *fragment* encirceled by an interrupted line in Fig. 21 is an arbitrary immersed closed curve, equiped with a *tail* consisting of two parallel segments joining the segment to an other *main* closed curve. We suppose that the tail does not intersect the fragment at other points than the joining place.

Let x be a regular point of an oriented circle immersed into the oriented plane. A double point of the immersion is called *positive (negative)* with respect to x, if the frame (1,2) formed by the velocity vectors of the first and of the second visit of the point by the curve orients the plane positively (negatively). (Here the ordering of the visits refers to the immersion of the oriented interval obtained from the circle by the exclusion of the point x).

Definition. The *Whitney function* defined at the ordinary points x of the circle immersion is the difference between the numbers $w_+(x)$ and $w_-(x)$ of double points of the circle immersion, which are positive and negative with respect to x:

$$w(x) = w_+(x) - w_-(x) .$$

The examples are shown in Fig. 22.

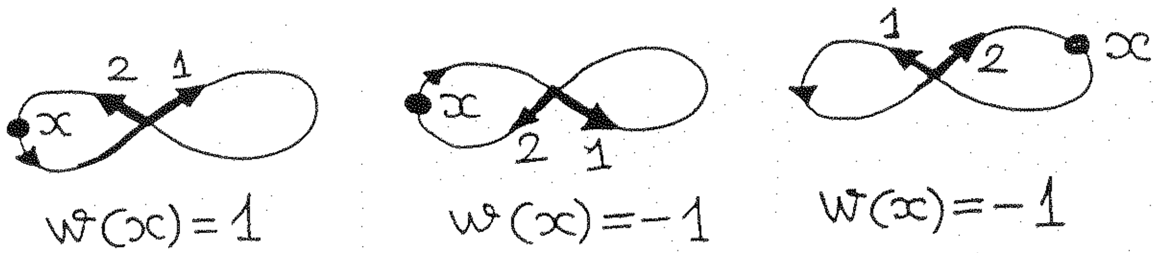

Figure 22:
Calculation of Whitney function

Theorem. *The increment of St under the pushing of a fragment through an interval of a main curve is equal to the value of the Whitney function of the fragment at the point of the joining of the tail, provided that the frame (direction of the pushing, direction of the interval of the main curve) orients the plane positively (if it orients the plane negatively the increment is $-w(x)$).*

The examples are shown in Fig. 21.

Remark. The standard orientation of the plane might be excluded from the formulation. We should count the positive (negative) points of the curve while defining the Whitney function using the orientation of the plane defined by the pushing direction and the main curve interval direction. This way we see that the increment of St is independent of the choice of the standard orientation of the plane.

Proof of the theorem. While we push the fragment, each of its double points will contribute ±1 to the increment of St. The sign depends on the sign of the vanishing triangle.

The cyclic ordering of the sides of the vanishing triangle is (1,2,3): the first visit, the second visit, the interval of the main curve visit.

The vanishing triangle, born at the crossing of the triple point is positive if and only if the orientation (1,2) coincides with the orientation (3,4), where 3 is the pushing direction and 4 the main curve interval direction.

This fact is checked directly by inspection of the eight possible cases (we have done it before, see Fig. 15). Summation over all the vanishing triangles provides now the theorem.

It is clear that the jump of the Whitney function at any double point of the curve equals ±2.

Theorem. *The value of the Whitney function at an ordinary point x of an immersed circle is*

$$w(x) = i(x) - \mathrm{ind}\,,$$

where $i(x)$ is the number of half-turns of the vector connecting x to a point y moving along the curve from x to x, and where ind *is the number of turns of the tangent vector.*

Example. These values for a curve with two selfintersection points are shown in Fig. 23.

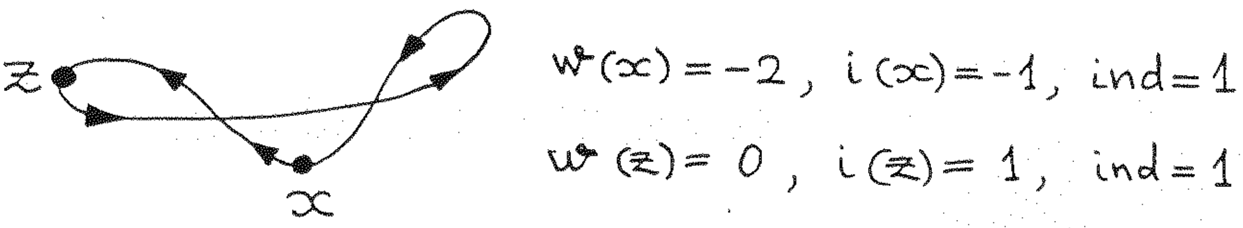

Figure 23:
Whitney function and indices

Proof. The jumps (±2) of $w(x)$ and $i(x)$ at each double point of the curve are equal. Hence the difference is independent of x and is an invariant of the immersed curve.

This invariant does not change its value under the three basic perestroikas. Indeed, one may choose the point x far away from the perestroika place. Both $w(x)$ and $i(x)$ will then be continuous and hence constant under the perestroika.

The constancy of $w(x)$ when two double points are born follows from the fact that the orientations (1,2) at these two points are different. Thus, the difference $w(x) - i(x)$ is constant along the space of immersions of a given index. For the standard curves of Fig. 5, this difference is equal to the index, whence the theorem follows.

Remark. Applying this theorem to the points of the exterior contour, we obtain the following Whitney formula for the index. Connect a point of the exterior contour to infinity by a transversal ray which does not intersect the curve at other places.

Theorem. *The index of an immersed circle is equal to* $a + \sum$, *where* $a = \pm 1$ *and* $\sum$ *is the sum of* ± 1 *over the double points of the curve.*

The signs are defined by the following rule: $a = i(x)$, where x is the exterior point, $\sum = -w(x)$. In other words, $a = 1$ iff the frame (ray's direction, curve's direction) is positive, and the contribution of a double point to $\sum$ is positive iff the frame (2,1) is positive (Fig. 24).

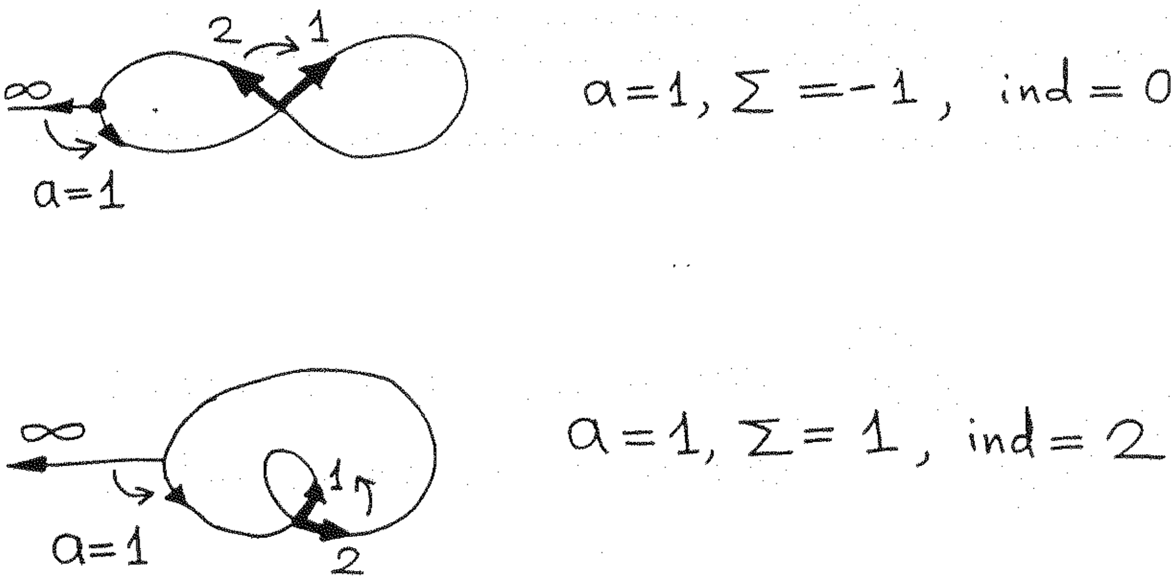

Figure 24:
Calculation of the index by the summation of the double points

Corollary. *The modulus of the index of a generic curve with* n *selfintersection points* *is at most* $n+1$ *and takes all the values between* $-1-n$ *and* $1+n$ *congruent to* $1+n$ mod 2:

$$|\text{ind}| \le n+1\,, \qquad \text{ind} - n \equiv 1 \bmod 2\,.$$

§7 The extremal curves

Definition. A generic curve with n double points is *extremal* if the absolute value of its index i takes the maximal possible value: $|i| = n+1$.

The extremal curves with four double points are shown in Fig. 25.

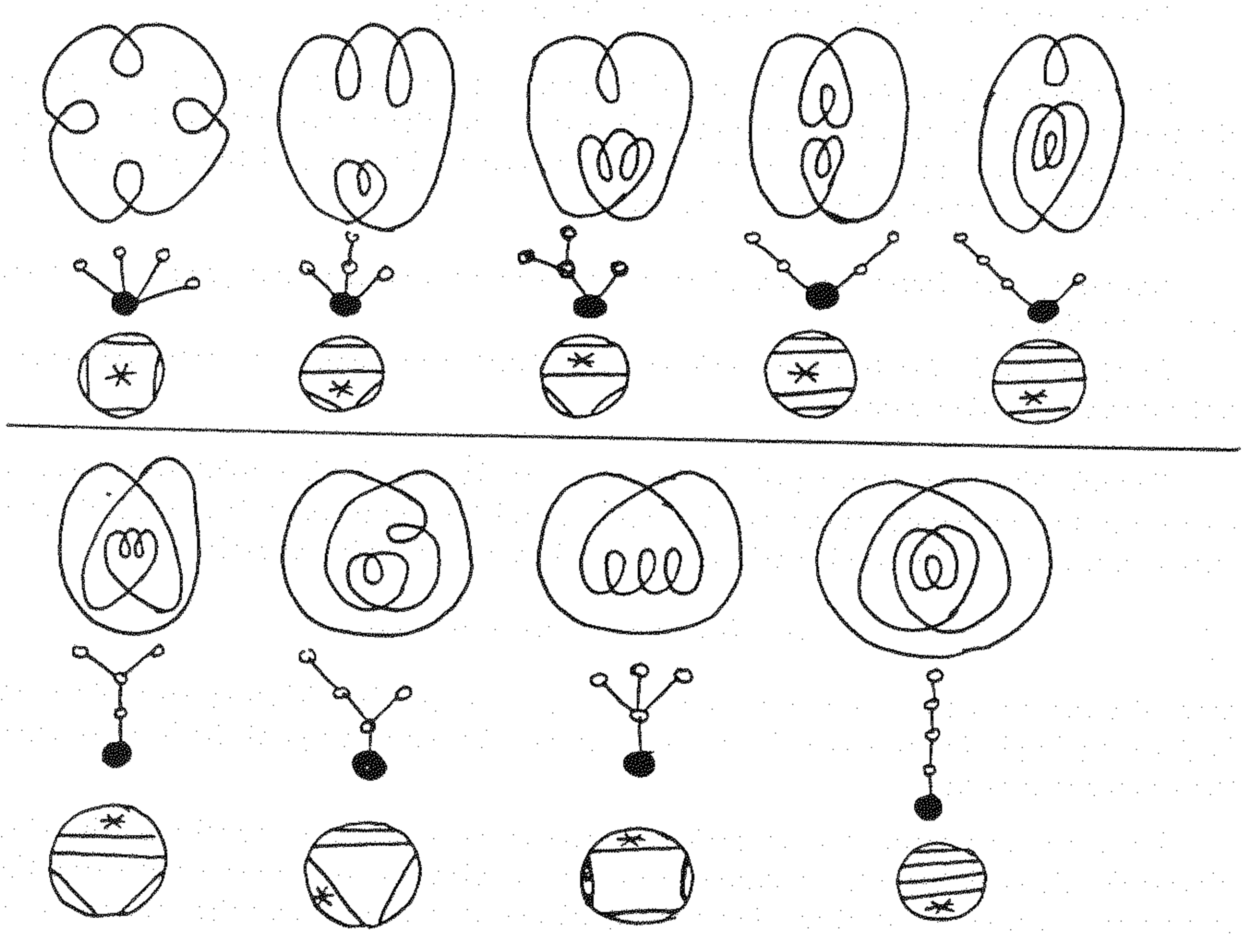

Figure 25:
Extremal curves with $n = 4$ double points

Theorem. *There exist natural bijections between*

1) *the set (of classes) of immersed circles of index $n+1$ with n double points*;

2) *the set (of classes) of plane rooted trees with n edges*;

3) *the set (of classes) of the decompositions of a disc into parts by n chords with no common points and one part distinguished.*

Remark. Two curves (trees, decompositions) are here *equivalent* if one can be transformed into the other by a homeomorphism of the plane (which may preserve, or not, the orientation).

A similar result holds also in the oriented case, when the equivalences preserve the orientation of the plane.

The construction of the bijections uses the following

Definition. The *Gauss diagram* of a generic immersion of a circle into a plane is the system of chords of the immersing circle, connecting the points sent by the immersion to the same double point of the immersed curve. (Gauss has studied the question, which chord diagrams correspond to immersions).

Lemma. *The Gauss diagram of an extremal immersion is planar (consists of nonintersecting chords).*

In other words, the two loops into which a double point breaks an extremal curve do not intersect at other points.

Proof. The smoothening of the immersed circle at a double point transforms the curve into two immersed branches. The indices and the numbers of double points of the initial curve and of the two branches are related by the following equations

$$i = i_1 + i_2 , \quad n = 1 + n_1 + n_2 + n_{12} .$$

Since the curve is extremal, $i = n + 1 = 2 + n_1 + n_2 + n_{12}$. The Whitney inequalities (§6) $i_k \leq 1 + n_k$ imply $i = i_1 + i_2 \leq 2 + n_1 + n_2$. Hence the number n_{12} of the intersections of the branches vanishes.

Thus we have associated to each extremal curve a disc decomposition (the Gauss diagram). To each bounded component of the complement to the extremal curve there corresponds a component of the disc decomposition (it is bounded by the arcs sent to the boundary of the component of the complement by the immersion).

Lemma. *Among the bounded components of the complement to an extremal curve in the plane exactly one component has common boundaries with the unbounded component.*

Proof. The Whitney formula implies that: (1) The exterior contour of an extremal curve is well oriented by the immersion. (2) The loops connected to the exterior contour at the double points lie inside the domain bounded by this exterior contour (which bounds a topological disc).

These loops, as we have seen, do not intersect. Hence, the complement to the domains bounded by these loops in the above disc is connected. This proves the lemma.

We distinguish the part of the decomposition given by the Gauss diagram, which corresponds to the above distinguished bounded component (the circle arcs bounding this part are sent to the exterior contour by the immersion). We have thus associated a decomposition with a distinguished part to an extremal curve.

Choose a point in each part of the decomposition. Connect these points by edges when two parts have a common chord. We obtain a plane rooted tree. We have thus constructed the required mappings from extremal curves to plane rooted trees and to disc partitions.

The fact that these mappings are bijective is proved inductively on the number n of double points. The branches of the tree encode the places where loops should be attached to the exterior contour, and these loops are themselves standard by the inductive conjecture, since they have less double points.

Remark. All the nine plane rooted trees with 4 edges (Fig. 25) are also different as abstract rooted trees.

For trees with ≥ 5 edges the situation is different. The number of nonequivalent plane rooted trees with 5 edges is 21, while that of abstract rooted trees is only 20.

Theorem. *The invariant St of an extremal curve is equal to the sum of the distances of the vertices of the corresponding tree from the root.*

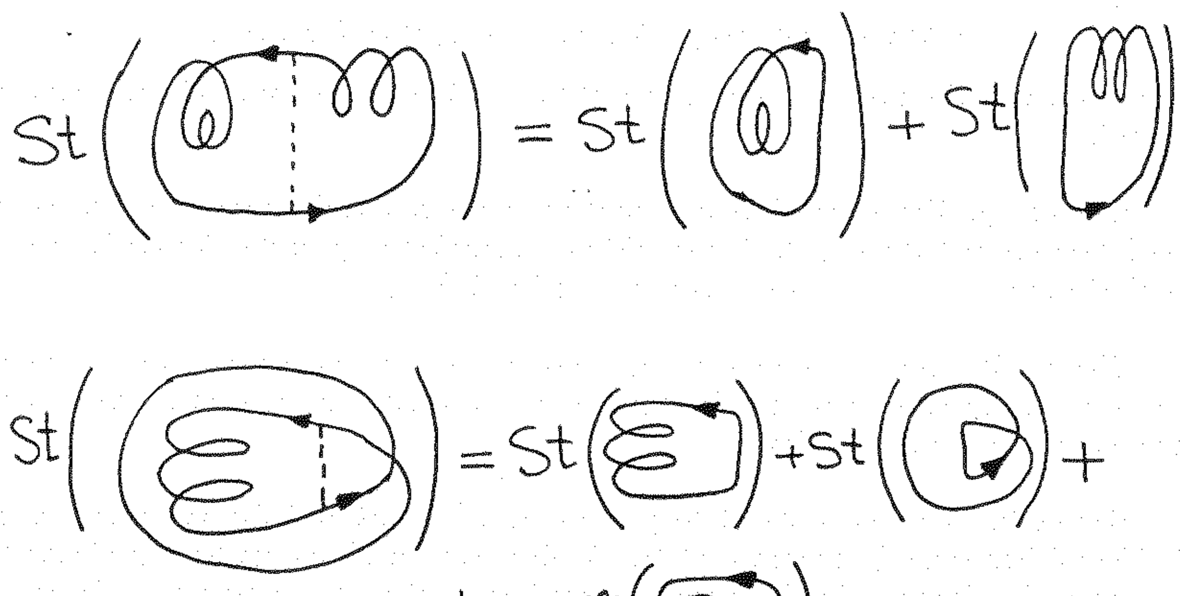

Figure 26:
Calculation of the invariant St for the extremal curves

Proof. Induction on the number n of double points. We see directly that the result is true for $n = 1$. If the exterior contour contains several double points, we use the additivity (Fig. 26).

If there is only one double point, we push away the loop attached at that point, as it is shown in Fig. 26 below. The resulting curve is the connected sum of two extremal curves (since the "(1,2)" orientation at all the double points is as it should be). Using the additivity of St and knowing its values for the summands, we can compute the St of the sum.

The value of St at the initial curve is greater than this sum (accordingly to the "pushing away" theorem of §6) by the value $w(x)$ of the Whitney function at the attachement point. This value, in our case, is equal to the number of double points of the fragment since all the "(1,2)" orientations are as they should be. Thus we have an expression of the value of St in terms of the value of St at the main curve and at the fragment and of the number of double points of the fragment.

On the level of trees our operation is the decomposition of the tree into two: we cut off a branch at distance one from the root and preserve the vertice at the cut place both on the initial tree and on the branch (where it becomes the root vertice).

The sum of the distances from the vertices to the roots decreases under this operation by the number of edges of the branch we cut off, i.e. by $w(x)$. Thus

$$St(\text{initial curve}) = St(\text{mutilated tree}) + St(\text{branch}) + w(a) =$$

$$= S(\text{mutilated tree}) + S(\text{branch}) + w(x) = S(\text{initial tree}) ,$$

where S means the sum of the distances of the vertices to the root. The theorem is thus proved.

Corollary. *The value of the invariant St on the extremal curves with n double points lies between n and $n(n+1)/2$. The equalities are reached only on the standard curves A_{n+1} (with the tree $\odot - \cdot - \cdots - \cdot$) and K_{n+1}* (Fig. 27).

The results and conjectures on the maximal and minimal values of the invariants over the set of immersions of a fixed index with a fixed number n of double points are summarized in the table below.

Remark. It is not difficult to prove that $J^+ + 2St = 0$ for any extremal curve. F. Aicardi has recently proved that this is also true for all curves having planar Gauss diagrams. Her formula for St of such curves implies that the minimum of St for curves with planar Gauss diagrams with n double points is attained on a curve having an index which is close to $n/3$.

invariant	ind	min	max	min curves	max curves
St	$n+1$	n	$n(n+1)/2$	K_{n+1}	A_{n+1}
St	$n-1$	0	$n(n-1)/2$	*Fig.* 28	*Fig.* 28
St	$n-3$	$2-n$	?	*Fig.* 29	?
J^+	$n+1$	$-n^2-n$	$-2n$	A_{n+1}	K_{n+1}
J^+	$n-1$	$-n^2+n$	0	*Fig.* 28	*Fig.* 28, 30

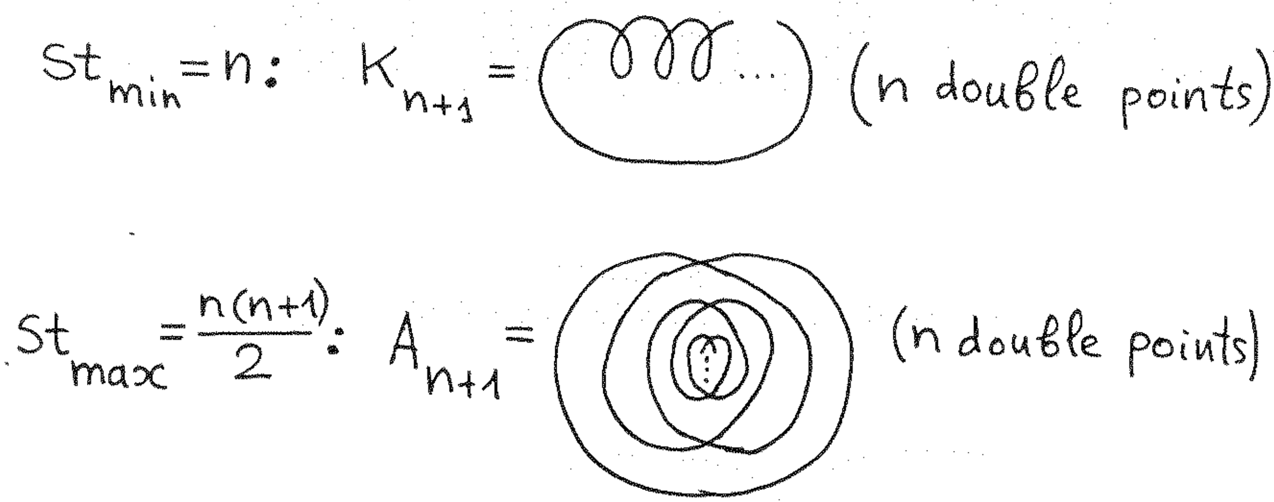

Figure 27:
The extremal curves of index $n+1$ having the maximal and minimal St values

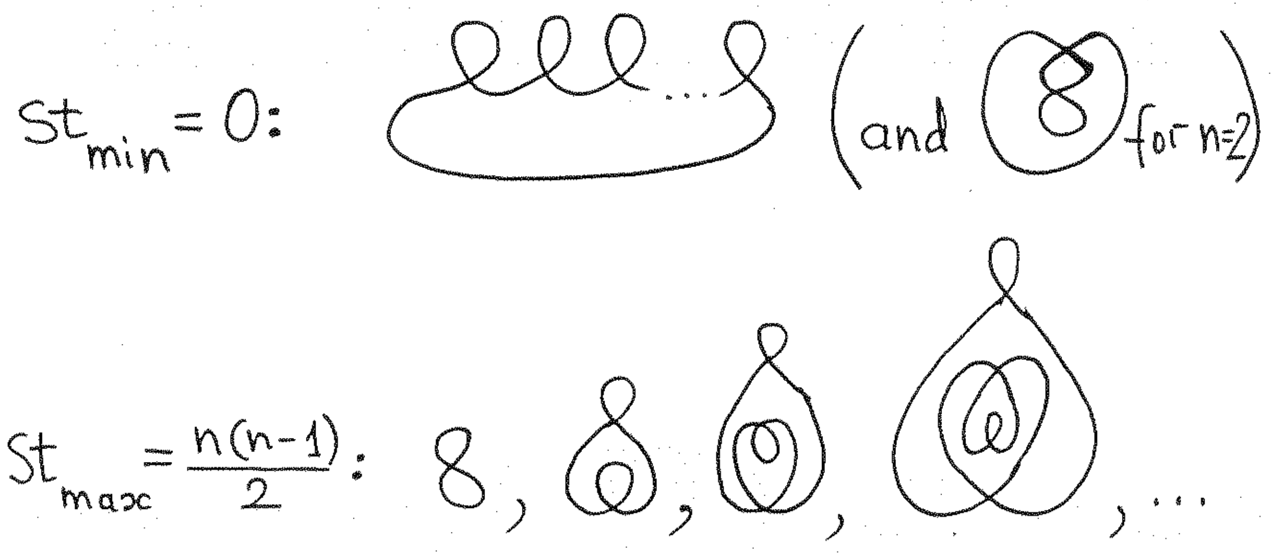

Figure 28:
Curves of index $n-1$ maximizing and minimizing St

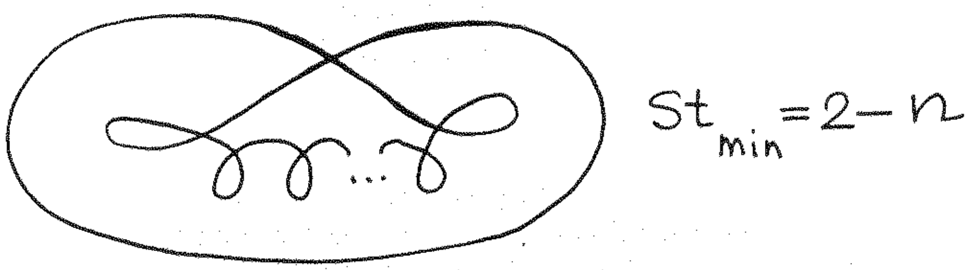

Figure 29:
Curve of index $n-3$ minimizing St

Recall that $J^- = J^+ - n$, and hence the theorems and conjectures for J^- follow. Consider a curve with n double points whose index $n+1-2k \geq 0$ is by $2k$ smaller than the maximal possible value. Call k the *defect* and consider the maximal and the minimal values of an invariant U over the set of all generic immersions having a fixed number n of double points and a fixed defect k. We shall denote them $U_{\max}(k)$ and $U_{\min}(k)$; these numbers depends on n.

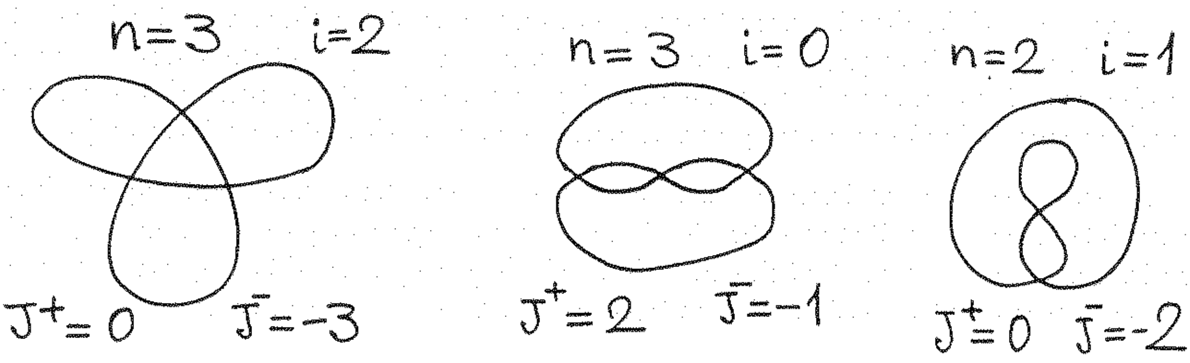

Figure 30:
Exceptional curves

Conjecture. *The numbers $St_{\max}(k)$ and $J^{\pm}_{\min}(k)$ are monotonic in k (decreasing for St, increasing for $J^{\pm}$).*

Conjecture. *The maximal value of St on all generic curves with n double points is attained only on the curve A_{n+1} of Fig. 27*

$$St \leq n(n+1)/2\,.$$

Conjecture. *The minimal values of $J^{\pm}$ on all generic curves with n double points is attained only on the curve A_{n+1} of Fig. 27:*

$$J^{+} \geq -n^2 - n\,, \qquad J^{-} \geq -n^2 - 2n\,.$$

Conjecture. *The minimal value of St and the maximal values of $J^{\pm}$ on all generic curves with n double points are attained on the same curve if n is sufficiently large.*

The minimum of St for $n = 2$ and the maximum of J^{+} for $n = 3$ are also attained at the exceptional curves of Fig. 30.

It would be interesting to study the curves with the extremal values of the invariants $I^{\pm}$ of §4.

§8 The cobordisms

The final goal of our work is the study of the components of the complement of different strata of the discriminant in the space of immersions. The classification of the additive invariants, constant on such components, provides the dual objects — the abelian groups which we may call the K-theories of the corresponding classification problems.

The classification problems I have in mind are, for instance, the classification of the components of the complement of the whole discriminant (it coincides with the classification of the immersions up to orientation preserving diffeomorphisms of the plane and of the circle), or the classification of the components of the complement to one of the three branches of the discriminant (corresponding to one of the three perestroikas — triple points (∗), direct selftangency (+), inverse selftangency (−)), or of the components of the complement of the union of two branches. There exist also other possibilities, since each branch consists of infinitely many components (which we shall discuss later in §9).

In any case none of these K-theories is calculated. As a very rough approximation I calculate here the oriented cobordism theories corresponding to different parts of the discriminants (0, 1, 2 or 3 of the branches (∗), (+), (−)). In this way we obtain eight theories, according to the lists of permitted and forbidden perestroikas.

Definition. An immersion of a circle (or of a finite set of disjoint oriented circles) into the plane is called *cobordant* in the sense of one of the eight theories to an other immersion (of different system of circles) if one can join these two immersions by a chain of isotopies of immersions and of the perestroikas of the types mentioned in the name of the theory together with the (oriented) Morse perestroikas (shown in Fig. 31).

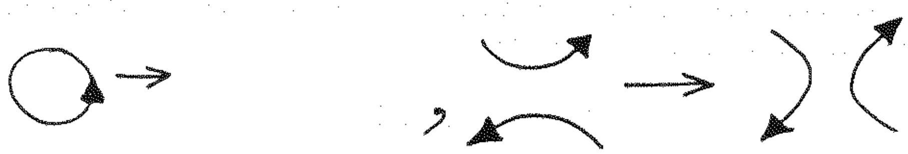

Figure 31:
Oriented Morse perestroikas of immersed curves

The addition of two cobordism classes is defined as the disjoint union of two immersions: one into the left half-plane, the other into the right one.

The eight commutative semigroups formed by the cobordism classes (most of them are groups) are connected by twelve natural homomorphisms and form a commutative cube (the growing of the list of permitted perestroikas reduces the (semi-) group of cobordism classes).

Denote by X_i the *basic immersion* of total index i shown in Fig. 32 ($i = 0, \pm 1, \pm 2, \ldots$).

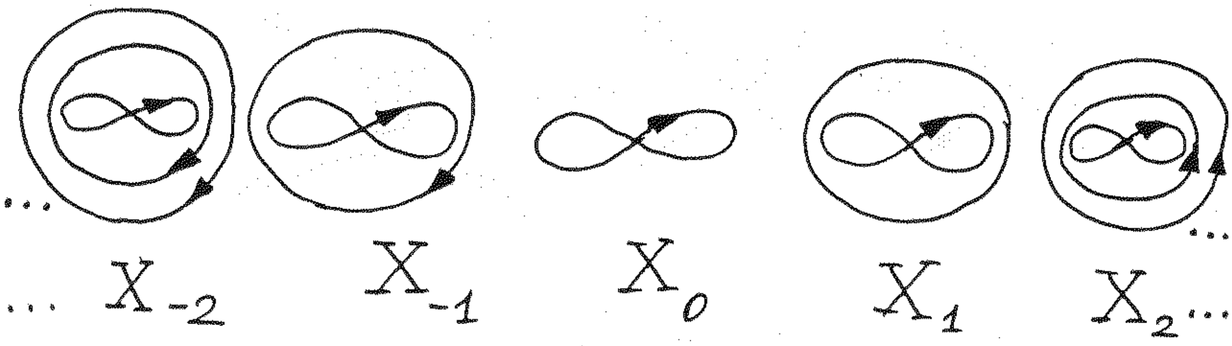

Figure 32:
The basic immersions

Theorem. *The eight (semi-) groups of cobordism classes of curve immersions are given by the following table:*

permitted		$-$	$+$	$*$	$*-$	$*+$	$+-$	$*+-$
forbidden	$*+-$	$*+$	$*-$	$+-$	$+$	$-$	$*$	
answer generators	$\mathbb{Z}_+^\infty$ X_i	$\mathbb{Z}_2^\infty$ X_i	$\mathbb{Z}_2^\infty$ X_i	H X_i	$\mathbb{Z}_2$ X_0	$\mathbb{Z}_2$ X_0	$\mathbb{Z}_2^\infty$ X_i	$\mathbb{Z}_2$ X_0

where

$\mathbb{Z}_+$ is the semigroup of nonegative integers

$\mathbb{Z}_2$ is the group consisting of two elements

H is the semigroup with generators X_i, $i = 0, \pm 1, \pm 2, \ldots$

and relations

$$3X_i = 3X_j\ , \quad 2X_i + X_{i+1} = X_i + 2X_{i+1}\ .$$

Lemma 1. *Each curve is Morse cobordant to a sum of the curves X_i (none of the perestroikas $(*)$, $(+)$, $(-)$ being permitted).*

Proof. 1°. Each double point might be cut off of the curve by a pair of Morse perestroikas (Fig. 33).

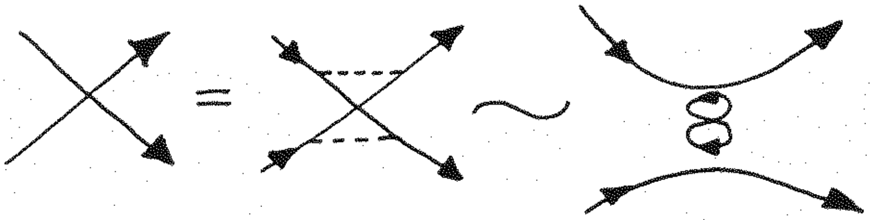

Figure 33:
The cobordism isolating the double points

The new born Figure eight curves are void, other components of the resulting curve do not intersect any other embedded circle.

2°. Each disc containing several inclusions can be transformed by a Morse perestroika into several discs, each containing only one inclusion (Fig. 34).

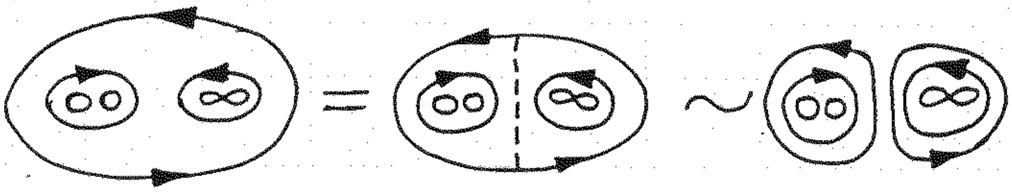

Figure 34:
The cobordism isolating the inclusions

3°. Iterating these two procedures we obtain a curve consisting of the nests of two types: ending by 8 or by 0 (Fig. 35).

Figure 35:
The two types of nests

4°. The nest II is cobordant to the void curve (a series of Morse perestroikas).

5°. The nest I is cobordant to one of the nests X_i, where all the embedded circles are oriented the same way (Fig. 36). Thus the semigroup of cobordism classes is generated by the X_i.

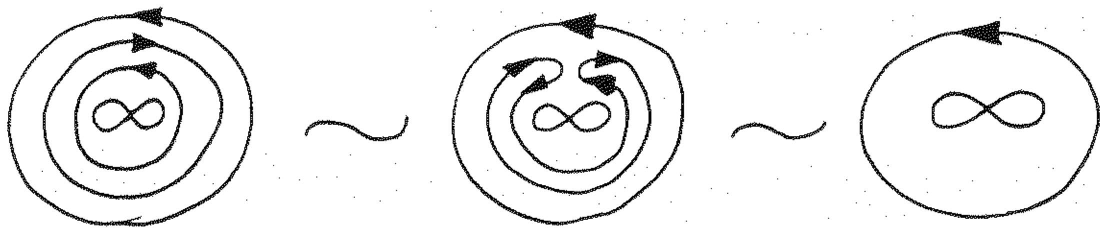

Figure 36:
The cobordism of a nest I to the standard one

Definition. The *index* of a double point (of a generic immersion of a union of oriented circles into the plane) is the number of turns around the origin of the radius-vector from this double point to the point moving along the oriented curve which is obtained from the given immersed curve by isolation of the double point with a subsequent elimination of the newborn figure eight curve (Fig. 37).

Example. The double point of the curve X_i has index i.

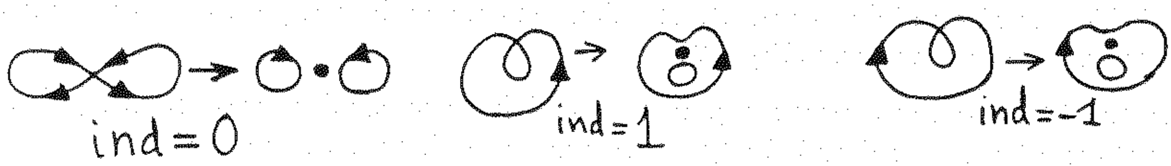

Figure 37:
The index of a double point

Theorem. *The classes of the X_i ($i \in \mathbb{Z}$) in the largest semigroup of cobordism classes ("everything forbidden") are independent. Hence this semigroup is the direct sum of the $\mathbb{Z}^+ X_i$, $i \in \mathbb{Z}$.*

Proof. The double points are not born or killed by Morse perestroikas and isotopies. The indices of the existent double points are the integrals of $(2\pi i)^{-1}\, d\,\ell n(z - z_{\text{sing}})$ along the curve obtained as the result of the isolation of z_{sing}. Hence the number of double points of index i is an invariant of our cobordisms, which proves the theorem.

Theorem. *If the selftangency perestroikas are permitted (but the triple points are not), the cobordism classes semigroup is the group* $\mathbb{Z}_2^\infty = \oplus \mathbb{Z}_2 X_i$.

Proof. 1°. The class of X_0 has order 2 (Fig. 38).

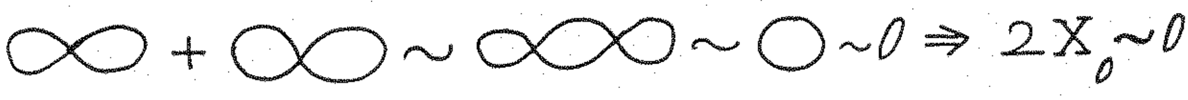

Figure 38:
Calculation of $2X_0$ in the "(+), (−) permitted" case

2°. The class of any X_i has order 2 (Fig. 39).

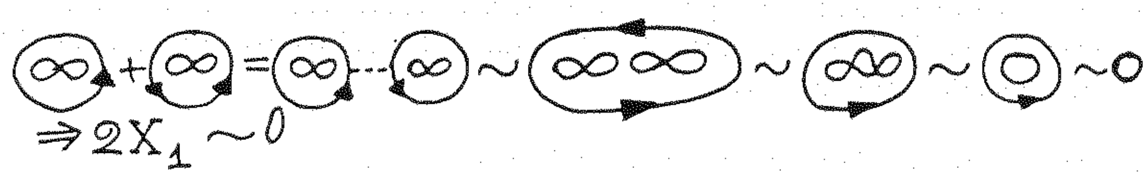

Figure 39:
Calculation of $2X_i$ in the "(+), (−) permitted" case

3°. The numbers of double points of index i change under the permitted selftangency perestroikas by even numbers (Fig. 40).

In the (+) case both newborn points, A and B, have equal indices. In the (−) case the newborn island is void and the two newborn points have equal indices.

The indices of other points do not change, since these points are far from the perestroika region and the integral along the island is small. Hence, the classes of the X_i are the generators, and $\{2X_i = 0\}$ generates all the relations of our group.

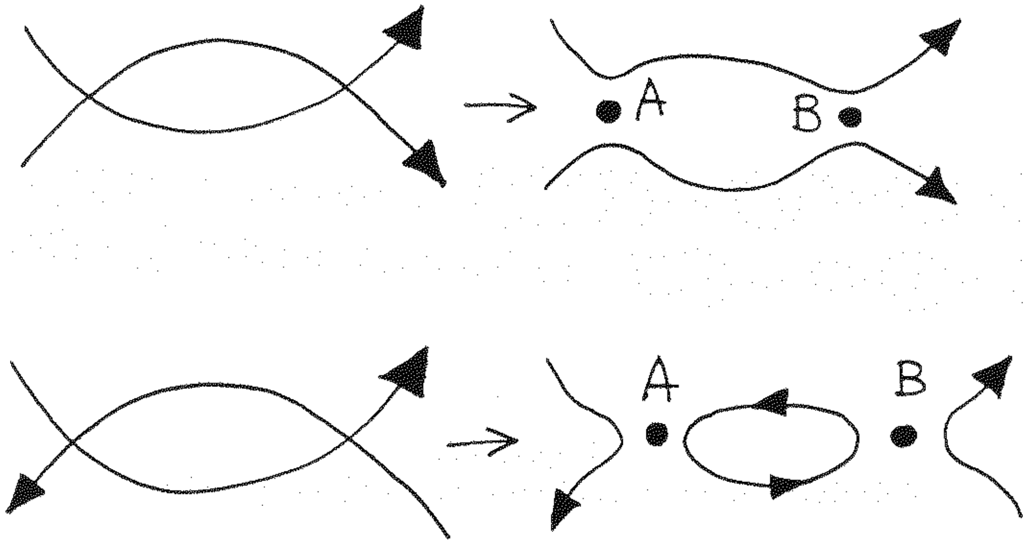

Figure 40:
The preservation of parity of the number of double points of index i

Theorem. *The cobordism classes "(−), (∗) permitted" form the group $\mathbb{Z}_2$ generated by X_0. The class of a curve is zero iff it has an even number of double points.*

Proof. 1°. $2X_0 = 0$ as above. 2°. $X_1 \sim X_0$ (see Fig. 41).

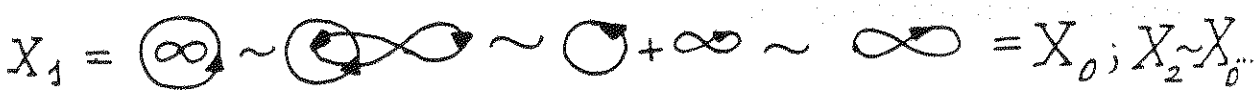

Figure 41:
Calculation of the cobordism group "(−), (∗) permitted"

3°. The parity of the number of double points is an invariant of all permitted perestroikas.

Theorem. *The cobordisms classes "(+), (∗) permitted" form a group $\mathbb{Z}_2$ (generator ∞, the class is zero iff the number of double points is even).*

Proof. See Fig. 42.

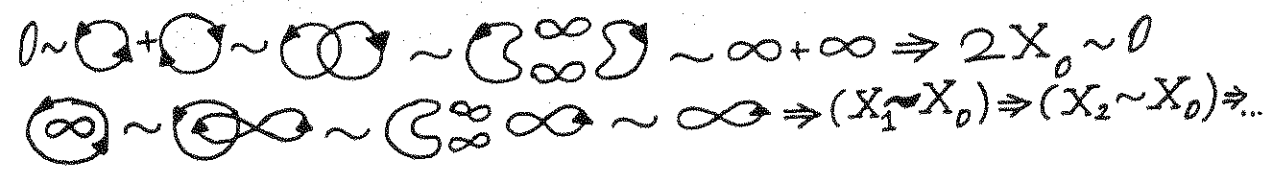

Figure 42:
Calculation of the cobordism group "(+), (*) permitted"

Theorem. *The increment D of the class of a curve in the group* Z_2^∞ *of the theory "(+), (−) permitted" under the crossing of a triple point has the following form*

$$D = X_j + X_{j+1} \, ,$$

where the pair $(j, j+1)$ *is*

$$(j, j+1) = (c_{\text{old}} + c_{\text{new}} \pm 1)/2 \, ,$$

and where c_{old} *and* c_{new} *are the indices of the curve with respect to the center of the vanishing triangle before and after the crossing of the triple point.*

Examples are shown in Fig. 43.

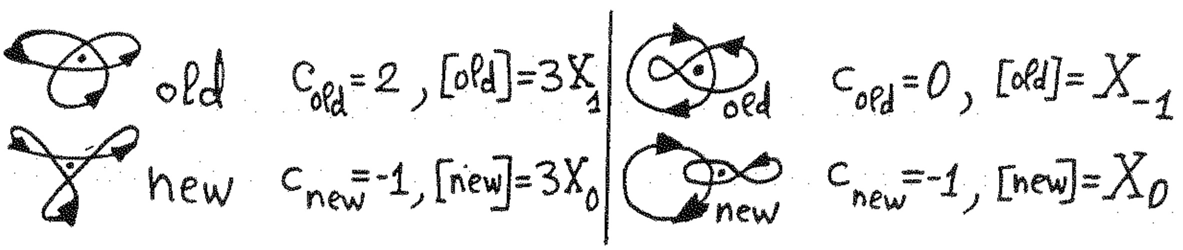

Figure 43:
The triple point crossing

Remark. In the semigroup of the theory "everything forbidden" the formula for the increment D has one of the following forms:

A) *case of symmetry of order 3* (all angles 120°):

$$D = 3X_{i_{\text{new}}} - 3X_{i_{\text{old}}} ,$$

where

$$i_{\text{new}} = \frac{2c_{\text{new}} + c_{\text{old}}}{3} , \qquad i_{\text{old}} = \frac{c_{\text{new}} + 2c_{\text{old}}}{3}$$

(*in this case* $i_{\text{new}} - i_{\text{old}} = (c_{\text{new}} - c_{\text{old}})/3 = \pm 1$.

B) *case of symmetry of order 2* (angles 60°, 60°, 120°):

$$D = X_{c_{\text{new}}} - X_{c_{\text{old}}}$$

(*in this case* $i_{\text{new}} - i_{\text{old}} = \pm 1$, $c_{\text{new}} - c_{\text{old}} = \pm 1$).

Proof (of the Remark and hence of the Theorem).

Case A. 1°. Consider the parts of the old and the new curves near the triple crossing. The results of the isolation of the three double points are shown in Fig. 44.

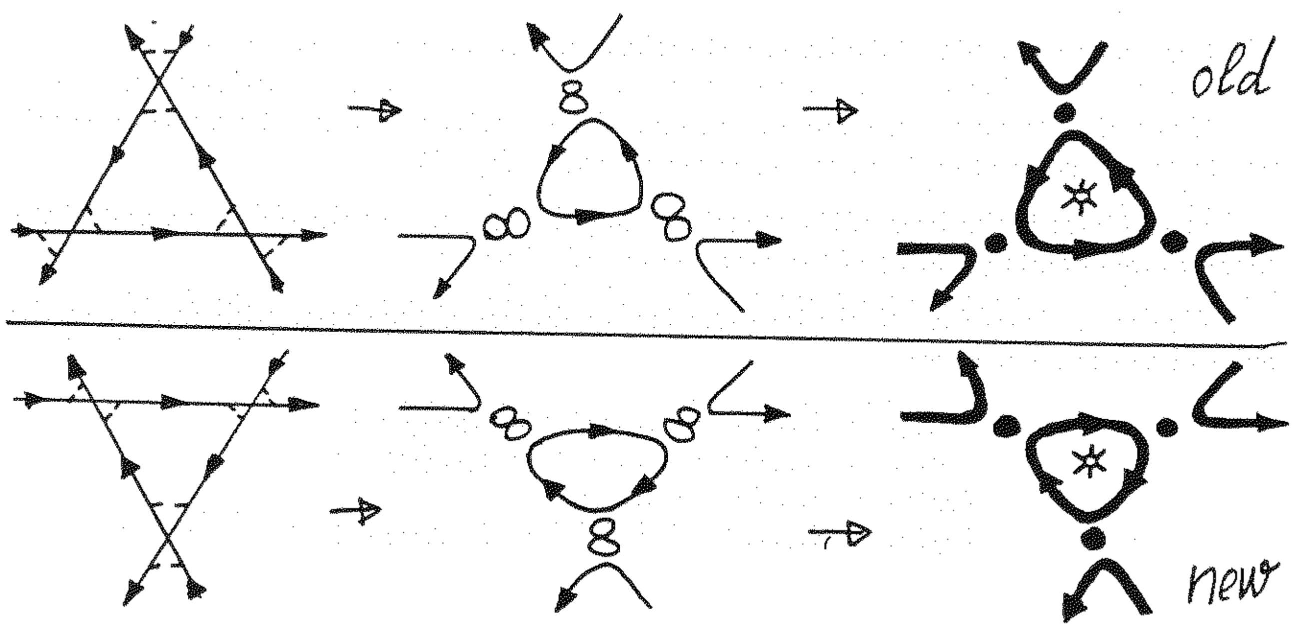

Figure 44:
The calculation of the increment of the class of the curve under a crossing of a triple point with 3-symmetry

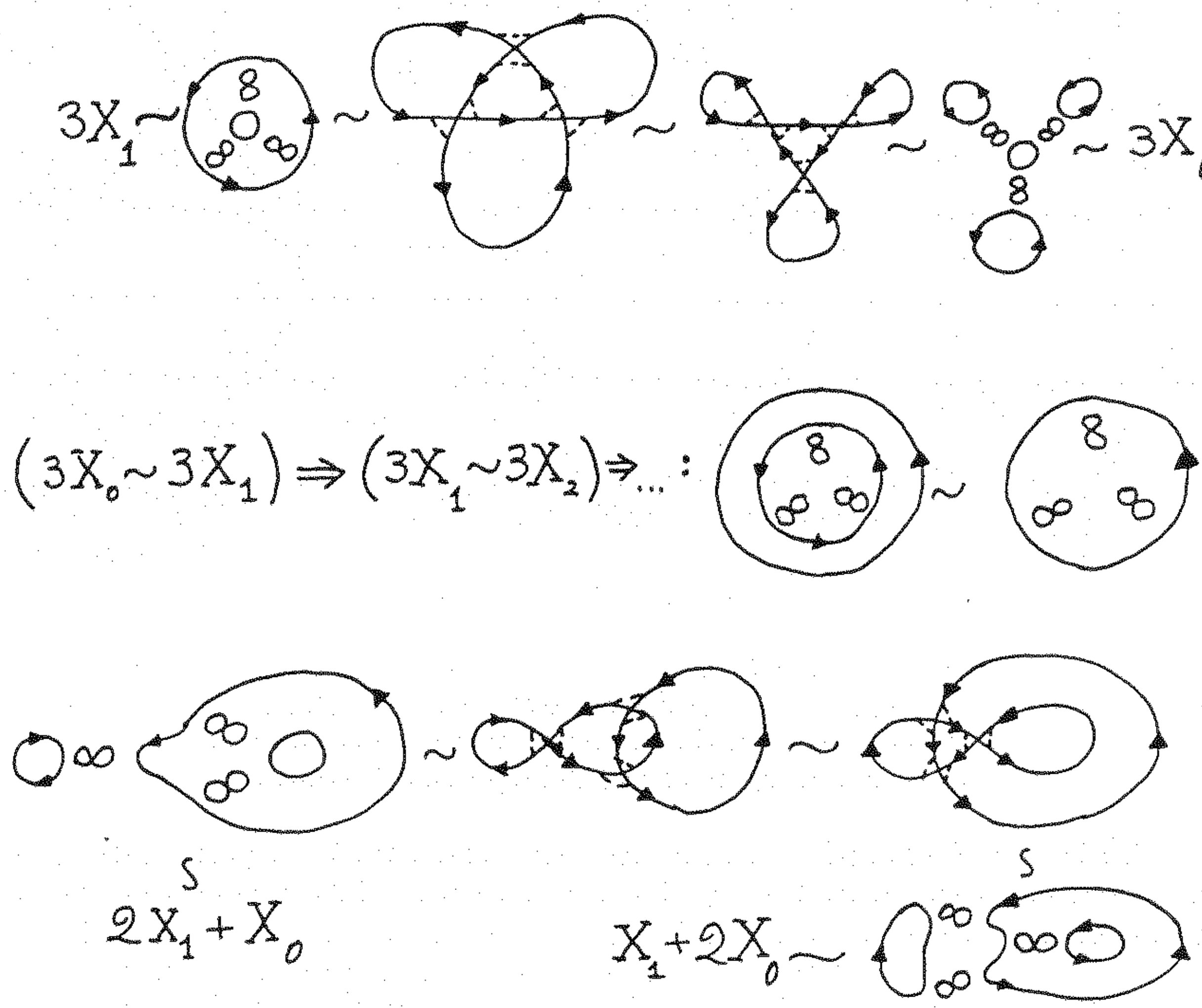

Figure 46:
The relations of the semigroup H

To prove that the list of relations is complete we consider any triple point crossing and compare two curves K_{old} and K_{new} obtained from the old curve C_{old} and the new curve C_{new} by isolation and elimination of the three double points. The reasoning of the preceeding proof gives the following cobordisms (Fig. 47).

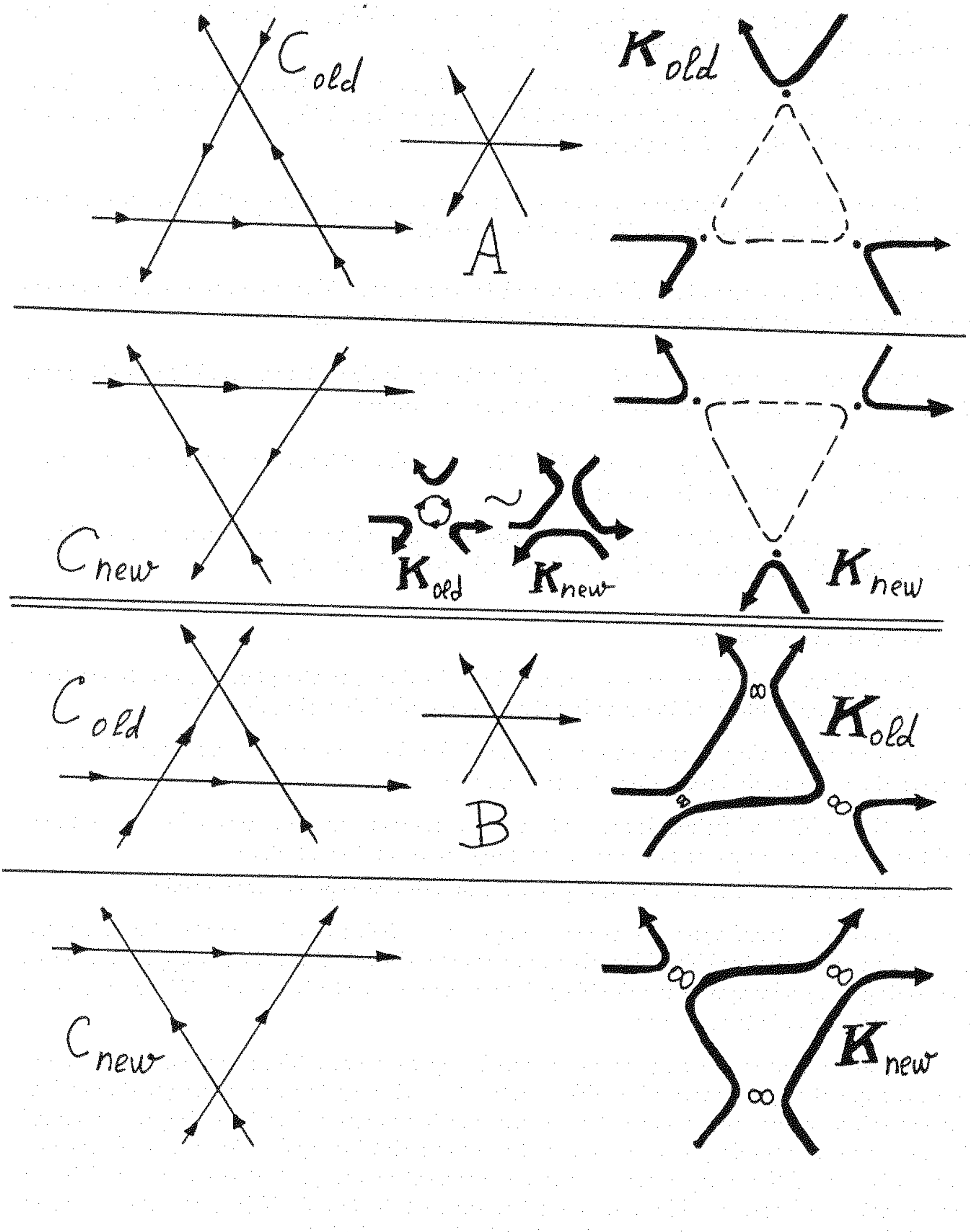

Figure 47:
The completeness of the list of relations

Case A. $C_{\text{old}} \sim K_{\text{old}} + 3X_{i_{\text{old}}}$, $C_{\text{new}} \sim K_{\text{new}} + 3X_{i_{\text{new}}}$. Since $K_{\text{old}} \sim K_{\text{new}}$, the cobordism $C_{\text{old}} \sim C_{\text{new}}$ follows from $3X_{i_{\text{old}}} \sim 3X_{i_{\text{new}}}$.

Case B. $C_{\text{old}} \sim K_{\text{old}} + 2X_i + X_{i\pm1}$, $C_{\text{new}} \sim K_{\text{new}} + X_i + 2X_{i\pm1}$. Since $K_{\text{old}} \sim K_{\text{new}}$, the cobordism $C_{\text{old}} \sim C_{\text{new}}$ follows from $2X_i + X_{i\pm1} \sim X_i + 2X_{i\pm1}$. Hence all the relations are generated by those of the theorem which is thus proved.

Theorem. *The cobordism classes of the theory "only (−) permitted" form the same group* $\mathbb{Z}_2^\infty$ *as that of the theory "(+) and (−) permitted".*

Proof. The perestroikas of Fig. 38 and 39 were either Morse or *inverse* selftangency perestroikas, direct selftangency perestroikas have not been used.

Theorem. *The cobordism classes of the theory "only (+) permitted" form the group* $\mathbb{Z}_2$ *(only the parity of the number of double points counts).*

Proof. We deduce $2X_0 = 0$ from Fig. 42, then $2X_i = 0$ from Fig. 39; the nontriviality follows from Fig. 40 (the first case).

We have thus calculated the seven (semi-) groups of curve cobordisms. The eighth group ("everything is permitted") has been calculated earlier [3].

§9 The long curves

Definition. A *long curve* is an immersion of a line into the plane which differs from the standard embedding of the x-axis only in a bounded domain (Fig. 48).

All the preceeding theory can be repeated for these curves. These ones are better objects than the circle immersions for two reasons:

(1) the "addition" (which is associative but in general noncommutative) is well defined, and the index (the number of turns of the tangent vector) is additive.

(2) the space of long curves of a given combinatorial type is contractible.

The combinatorial type of a generic long curve is defined by its oriented Vassiliev diagram. The (nonoriented) Vassiliev diagram consists of a set of arcs in the upper halfplane, connecting the preimages of the double points belonging to the boundary. The orientations are defined in the following way.

A generic long curve may be constructed step by step, adding the segments joining two consequential preimages of double points. At each step the new segment should join a given initial point with a given final point which belongs either to a simply-connected domain or to its boundary (to which also the initial point belongs). This simply-connected domain is one of the components of the complement of the part of the curve which has been constructed earlier. The last point lies inside this domain if it is the double point visited the first time or on its boundary if it is the second visit.

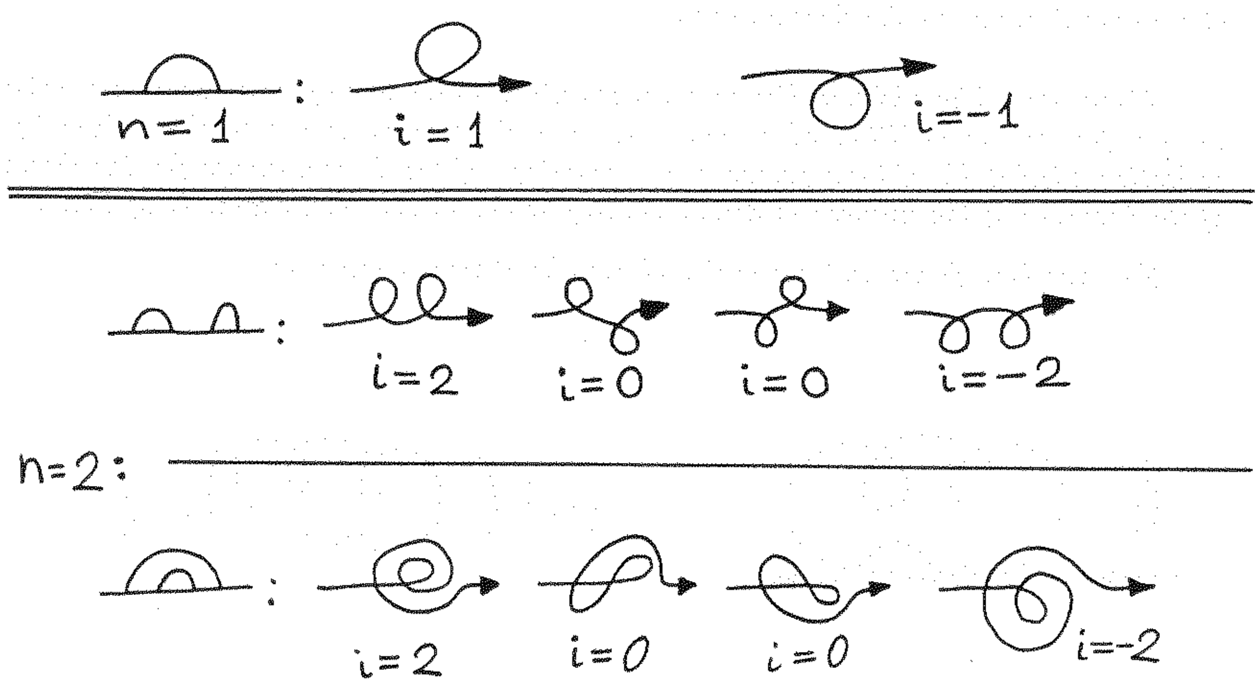

Figure 48:
Long curves with one and with two selfintersection points

The possible choices of the last point and of the path form a contractible set, unless the endpoint belongs to a segment of the boundary to which the above domain is adjacent from both sides (which may happen). In this case, the space of possible choices consists of two contractible components and they are distinguished by the "orientation" supplementary structure. A similar structure exists for the strata of any codimension in the discriminant.

Thus the combinatorics of the long curves, described by the oriented diagrams and by their adjacencies, provide us with a variant of a cell decomposition of the complement of the discriminant and of the stratification of the discriminant. In principle, our questions about the topology of the complement of different strata of the discriminant might be answered in terms of the algebra of the oriented diagrams. However, the corresponding invariants are yet to be calculated and we present below only the first steps in this direction.

We call below "the discriminant" the hypersurface formed by the immersions with no direct selftangencies in the space of long curves.

Definition. A *Vassiliev invariant* of order one is a function of a long curve which is constant on the components of the complement of the discriminant and whose jump under the positive crossing of the discriminant does not change under the crossing of the strata of codimension one in the discriminant.

Theorem. *The additive Vassiliev invariants of order one of long curves are uniquely defined (up to the addition of a zero order invariant proportional to the index) by the collection $\{f_i, i \in \mathbb{Z}\}$ of their jumps at the crossings of the discriminants (in the direction of the increase of the number of double points) at the special long curves ℓ_i of Fig. 49.*

The numbers f_i are independent and may be chosen arbitrarily.

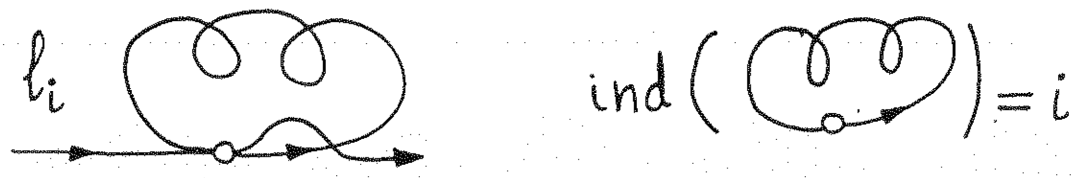

Figure 49:
Special long curve ℓ_i having a direct selftangency of index i

Remark. This theorem shows that the theory of plane curves is much more complicated than the theory of knots, where the first order Vassiliev invariants are trivial.

Proof. 1°. The first order condition implies that the jump might only depend on the following data:

(a) the indices j and i of the long and of the short branches of the curve with a direct selftangency (the index j vanishes for the standard curve ℓ_i);

(b) the orientation of the plane, defined by the pair (short branch, long branch) leaving the selftangency point.

2°. The additivity implies that the jump cannot depend on j.

3°. The independence of the jump of the orientation (b) follows from the existence of the perestroika of codimension one shown in Fig. 50.

This perestroika does not change the index i of the short component but changes the orientation (b).

4°. The existence of an invariant having jump 1 at the perestroikas of index i and jump 0 at the perestroikas of all other indices follows from the fact that the long curves with a direct selftangency of index i form a codimension 1 cycle in the simply connected space of long curves of a given index.

5^0. To make this invariant additive it is sufficient to add a constant depending only on the index.

Suppose that we wish to make additive the invariant F whose jump under the positive crossing of the discriminant at a direct selftangency of index 0 is equal to a (above, a is either 1 or 0).

We choose the values of the invariant at the long curves K_i of index i, shown in Fig. 51, to be

$$F(K_i) = -|i|\,a/2\,.$$

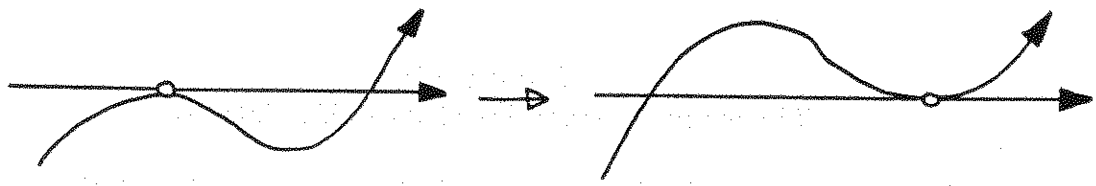

Figure 50:
Crossing of a stratum of codimension one in the discriminant

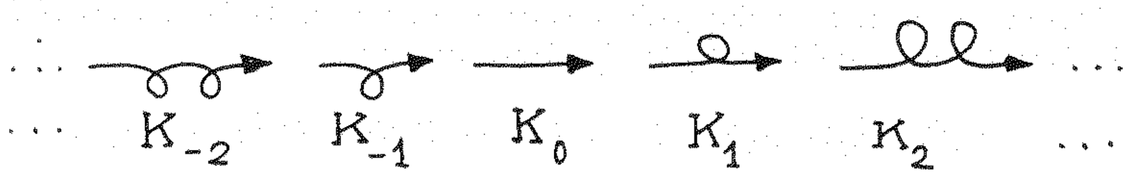

Figure 51:
The long standard curves K_i

Our choice implies the relation

$$F(K_i + K_j) = F(K_i) + F(K_j)\,.$$

This relation is evident for i and j of the same sign.

The curve $K_1 + K_{-1}$ can be transformed into K_0 by one positive direct selftangency perestroika of index 0 (Fig. 52).

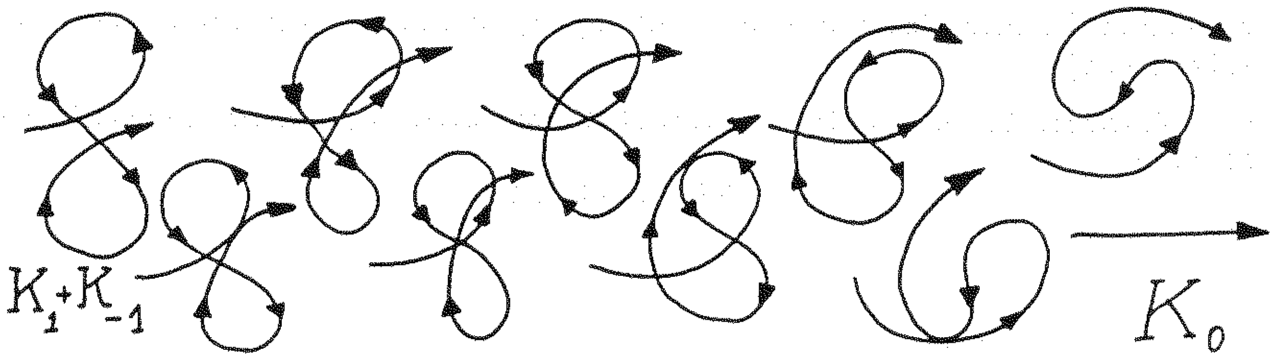

Figure 52:
The index 0 perestroika of $K_1 + K_{-1}$ into K_0

Hence $F(K_1 + K_{-1}) = F(K_0) - a = -a$, which is also the value of $F(K_1) + F(K_{-1})$. Iterating this argument, we prove that $F(K_i + K_j) = F(K_i) + F(K_j)$ for any i and j.

Any long curve $\tilde{K}_i$ of index i can be connected by a generic path with K_i in the space of long curves. Connecting also $\tilde{K}_j$ to K_j, we construct a path connecting a long curve $\tilde{K}_i + \tilde{K}_j$ to $K_i + K_j$.

The increment of F on this path is equal to the sum of the increments of F on the paths from $\tilde{K}_i$ to K_i and from $\tilde{K}_j$ to K_j. Hence we have

$$F(\tilde{K}_i + \tilde{K}_j) - F(K_i + K_j) = F(\tilde{K}_i) - F(K_i) + F(\tilde{K}_j) - F(K_j) .$$

Thus the additivity of F on the special curves K_i implies its additivity on any two curves: $F(\tilde{K}_i + \tilde{K}_j) = F(\tilde{K}_i) + F(\tilde{K}_j)$.

6^o. The uniqueness of an additive invariant with given jumps: the difference of two such invariants has no jumps and hence is a function of the index. The additivity implies that this function is linear.

7^o. The basic invariants we have constructed have the nice property that their values at a fixed curve are almost all zero (only a finite number of nonzero values is possible). Indeed, any curve can be obtained from the basic curve by a finite chain of perestroikas changing a finite number of invariants, while for the basic curves all the invariants but a finite number of them vanish.

It follows that *any* infinite formal linear combination of the basic invariants has a well defined value at each curve and hence represents a genuine invariant. The theorem is thus proved.

§10 Tables of curves

Here I discuss the classification of generic immersions of a circle into a plane or a sphere up to the diffeomorphisms of the plane or of the sphere and of the circle. These diffeomorphisms may preserve, or not, the orientation of the plane (sphere) and that of the circle. Hence there exist four different classification problems. The calculations (which are rather long and for which I am very gratefull to F. Aicardi, SISSA Trieste) lead to the following numbers of classes.

Plane closed curves

The number of types (taking or not taking into account the orientations of the plane and of the circle) of generic immersions with n double points are given by the following table:

oriented	$n=0$	1	2	3	4	5
$\mathbb{R}^2, S^1$	2	3	10	39	204	1262
$\mathbb{R}^2$	1	2	5	21	102	639
S^1	1	2	5	21	102	640
—	1	2	5	20	82	435

Remark 1. The standard orientation reversing involutions σ of the circle and Σ of the plane act on the space of immersions. Some immersion classes are, and some are not, invariant under the action of these involutions.

Denote by τ and by T the standard antipodal involutions of the circle and of the plane. An immersion $f : S^1 \to \mathbb{R}^2$ is *symmetric* with respect to the orientation reversals (of the circle and of the plane) if

$$f\sigma = \Sigma f \ \ \text{(or, equivalently,}\ \ \Sigma f\sigma = f) .$$

If the index of f vanishes, there are two other possibilities for the symmetry:

$$f\sigma = Tf \ \ \text{(or, equivalently,}\ \ Tf\sigma = f) ,$$

$$\Sigma f = f\tau \ \ \text{(or, equivalently,}\ \ \Sigma f\tau = f) .$$

An immersion is *asymmetric, symmetric*, or *supersymmetric* if the number of different classes among the classes of the immersions

$$f,\ \Sigma f\sigma,\ Tf\sigma,\ \Sigma f\tau$$

is 4, 2, or 1.

Example. The standard "8" immersion is supersymmetric. The standard "0" immersion is symmetric ($\Sigma f\sigma = f$, $Tf\sigma = \Sigma f\tau$). If an immersion has two of the three symmetries, it has the third and is supersymmetric.

An asymmetric curve contributes to the table of types a column (4, 2, 2, 1). A supersymmetric curve contributes (1, 1, 1, 1). A symmetric curve of type $\Sigma f \sigma = f$ contributes (2, 1, 1, 1). The symmetric curve of the types $T f \sigma = f$ and $\Sigma f \tau = f$ contribute (2, 2, 1, 1) and (2, 1, 2, 1), respectively.

Remark 2. If a class of a curve is transformed into itself by one of the three involutions $((\Sigma, \sigma), (T, \sigma), (\Sigma, t))$, then there exists an immersion of this class which is exactly invariant under this involution.

This is also true for the equivalence classes, defined by any of the three strata of the discriminant, studied above. These three strata are invariant under the involutions. Hence these involutions act on the set of classes which are the components of the complement. Whenever a class is invariant, it contains an invariant point.

It seems that this observation (not too difficult to prove for generic curves with only double points) is true in a very general situation of finite (or even compact) group actions on the stratified spaces of subvarieties (or on the stratified spaces of mappings).

The distribution of the oriented curves on the oriented plane by their indices

					1		1					
				1		1		1				
			2		3		3		2			
		4		10		11		10		4		
	10		35		57		57		35		10	
26		133		290		364		290		133		26

For instance, the number of types of curves having 5 double points and index 6 is equal to 26, of those of index 4, the number is equal to 133, and so on.

Curves on S^2. The number of types

oriented	$n = 0$	1	2	3	4	5
S^2, S^1	1	1	3	9	37	182
S^2	1	1	2	6	21	99
S^1	1	1	2	6	21	97
—	1	1	2	6	19	76

Definition. A circle immersion is *reducible* if any double point cuts the image into two non-intersecting loops.

The plane extremal curves are in this sense *completely reducible.*

Irreducible spherical curves with $n \leq 7$ selfintersection points

The following table gives the number N of types (no orientations taken into account).

n	0	1	2	3	4	5	6	7
N	1	0	0	1	1	2	5	9

The curves themselves are shown in Fig. 53.

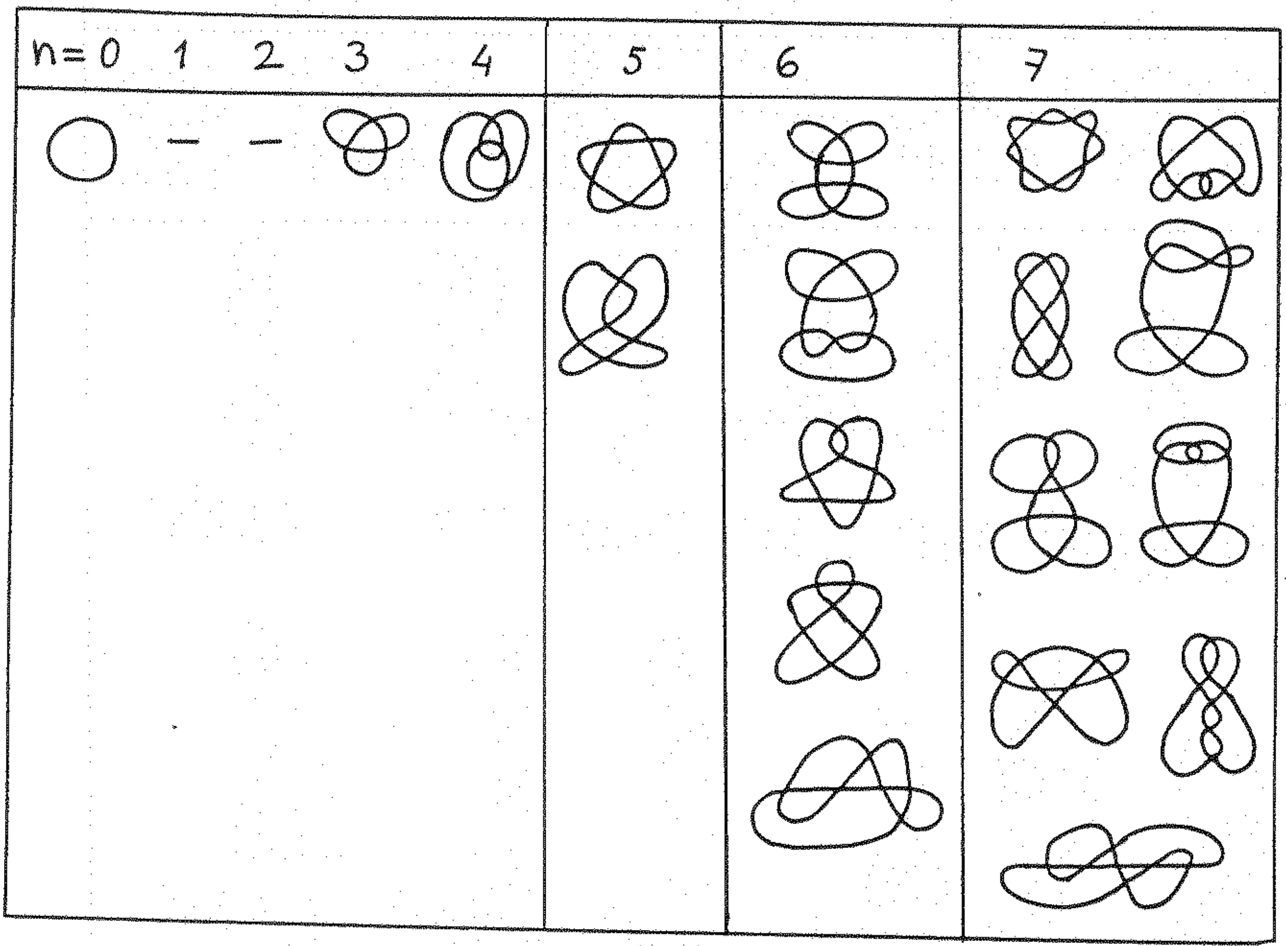

Figure 53:
The irreducible spherical curves

The classification of spherical curves is simpler, since the number of spherical curves with n double points is smaller than that of the plane curve (some of which are conformally equivalent). To obtain all the plane curves from the list of spherical curves it suffices to place the point ∞ inside all the components of the complement to the spherical curve.

The classifications of the spherical curves having $n \leq 4$ and $n = 5$ double points is shown in Fig. 55 and Fig. 56. The classification of plane curves with $n \leq 3$ and with $n = 4$ is shown in Fig. 57 and Fig. 58, where each curve is accompanied by the values of the invariants St, J^+, J^-.

As it is explained in the introduction, the knowledge of these values provides thousands of theorems on the generic immersions, similar to the three examples described in the introduction.

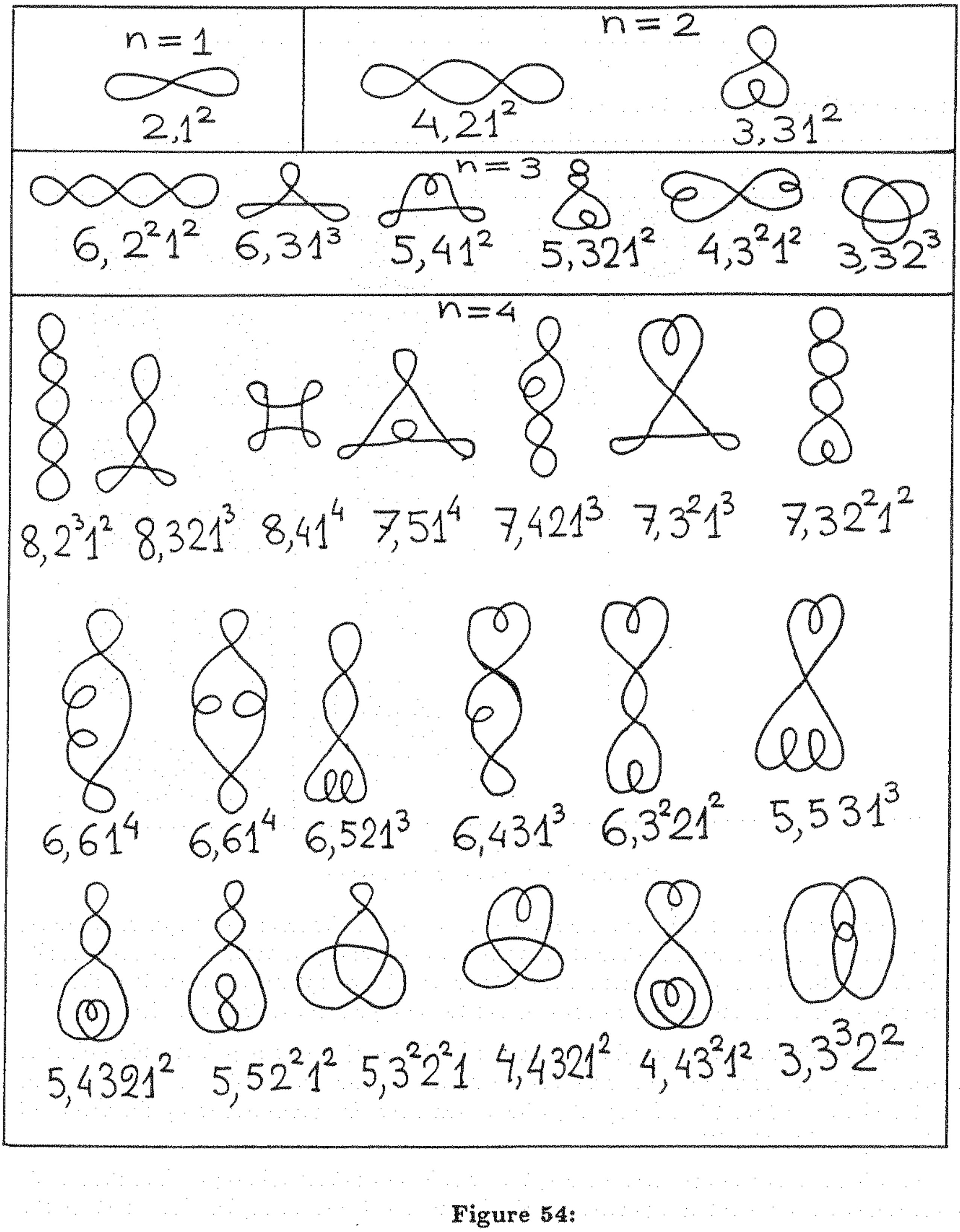

Figure 54:
Spherical curves with $n \leq 4$ double points

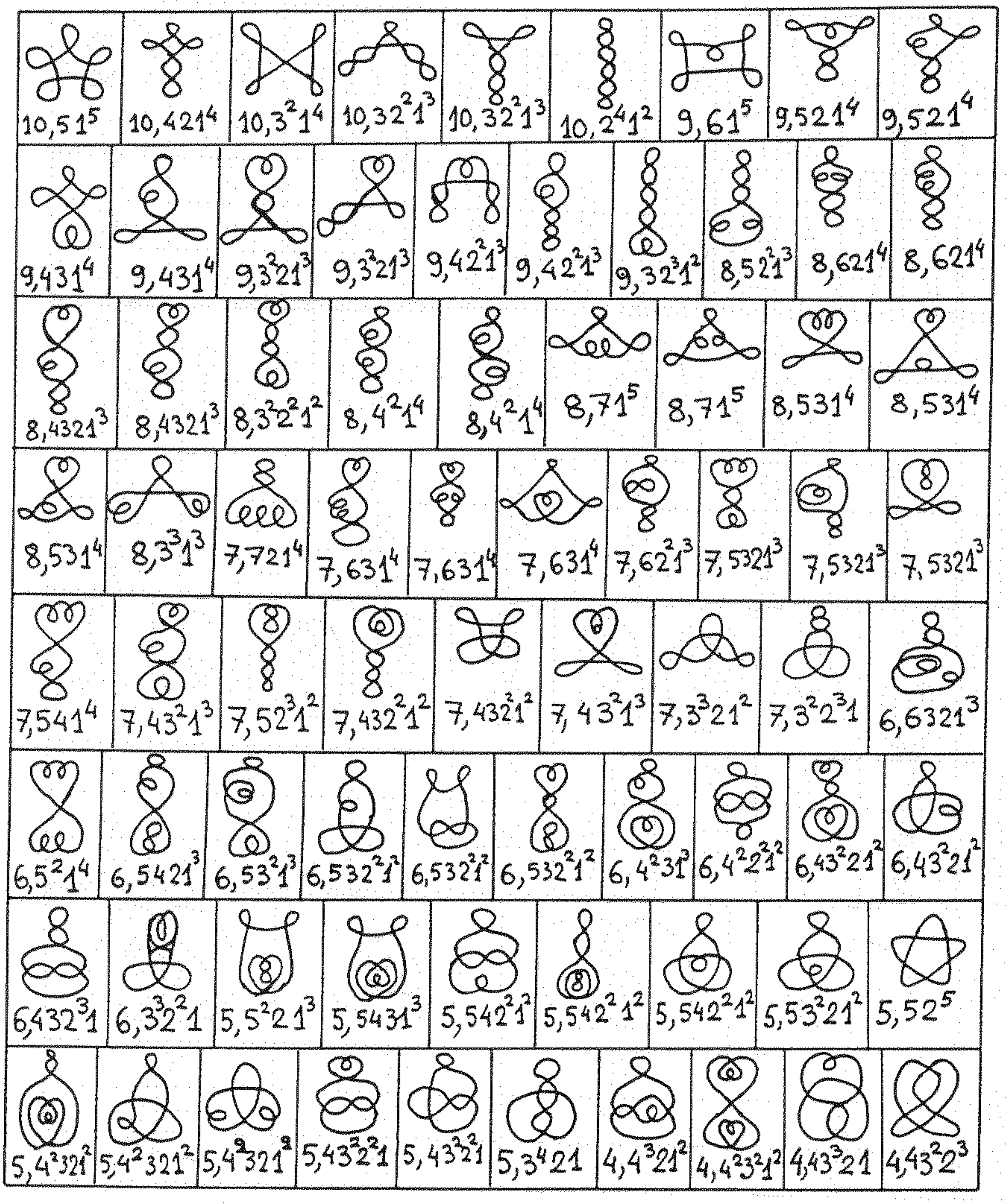

Figure 55:
Spherical curves with 5 double points

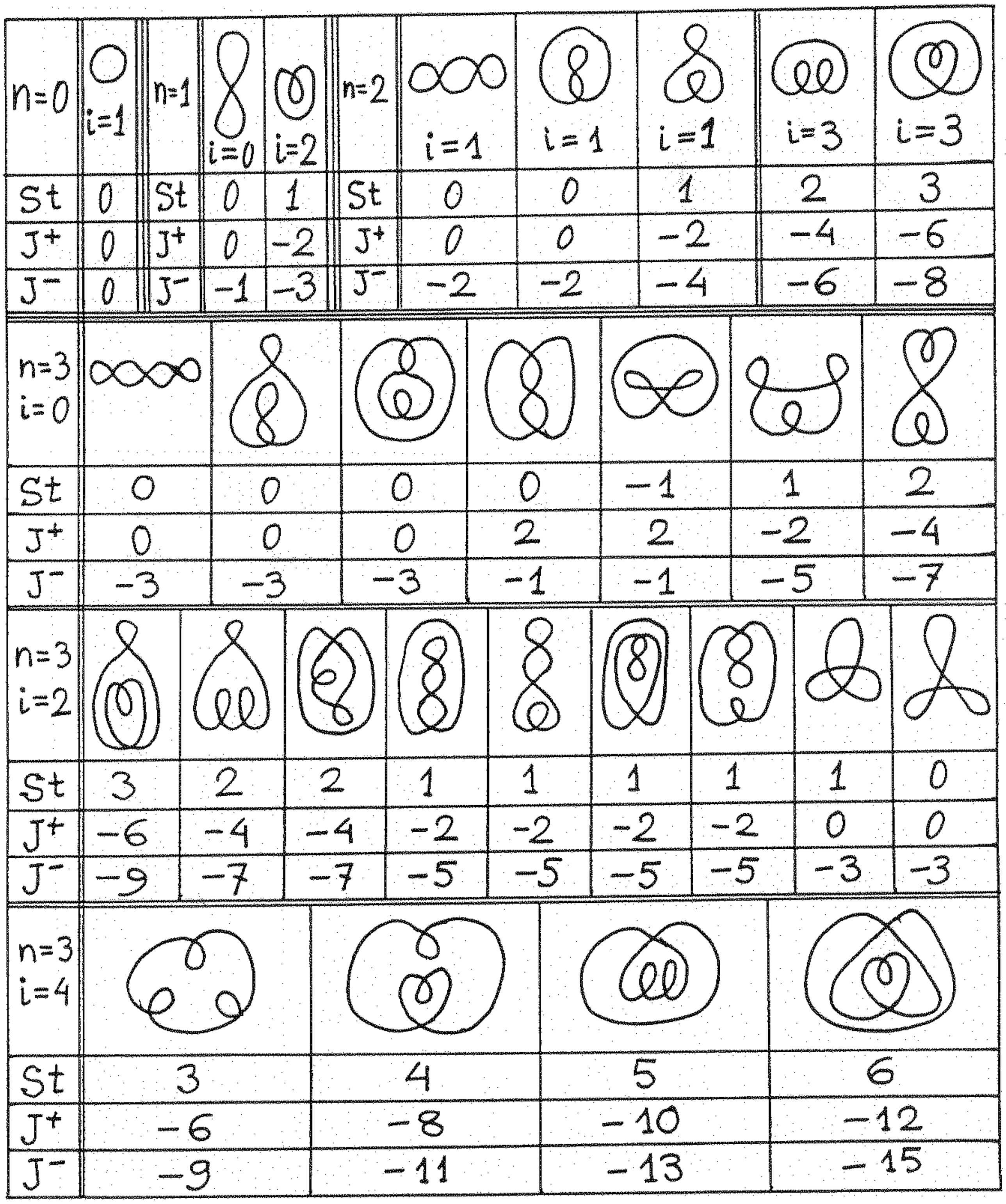

Figure 56:
Plane curves with $n \leq 3$ double points

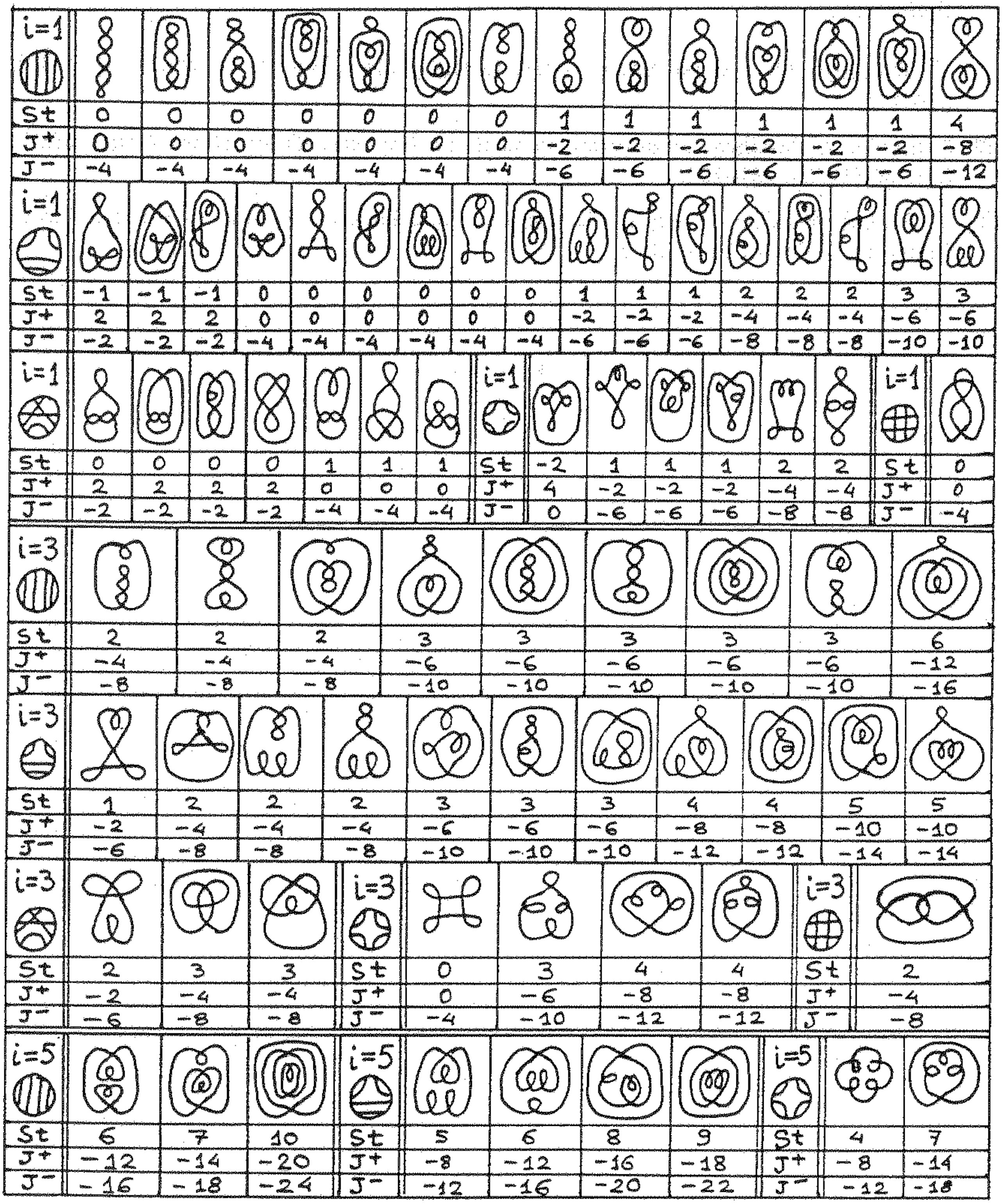

Figure 57:
Plane curves with $n = 4$ double points

References

[1] H. Whitney. On regular closed curves in the plane. Compositio Math. 4 (1937), 276–284.

[2] S. Smale. A classification of immersions of the 2-sphere. Transactions Amer. Math. Soc. 90 (1958), 281–290.

[3] V. Arnold. Vassiliev theory of discriminants and knots. Proceedings of the I Europ. Congress of Math., Paris 1992.

* * *

*