On topological properties of Legendre projections in contact geometry of wave fronts

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There exists an interesting relation between the global topological problem of coexistence of singularities on wavefronts and the Sturm type theory of oscillations of linear combinations of eigenfunctions of differential operators.

One can use this relation in both directions.In this paper the Sturm-Tabachnikov theorem on the oscillations of linear combinations of eigenfunctions of differential operators on a circle is used to minorate the number of cusps of a generic wave front close to a circle collapsing to a point.

The classical four vertices theorem for curves in the euclidean plane is shown to be a particular case of a general fact of contact topology. This observation suggests the higher-dimensional generalizations of the four vertices theorem and also suggests some nontraditional generalizations of the Sturm theory for the functions on spheres.

The simplest of these generalizations corresponds to the theorem on the four umbilical points on a closed convex surface in three-dimensional euclidean space.The general formulation in terms of the symplectic and contact topologies open the way for the higher-dimensional noneuclidean generalizations of these classical results of differential geometry of the euclidean space (dealing with more general situations,than that of the Riemannian geometry).

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1 The necessity of four cusps on a collapsing front.

Consider,as an example,the family of circles of radia $|t|$ centred at a fixed point of an euclidean plane.Here t is a parameter running from $-\infty$to $+\infty$.The front shrinks while t is growing from $-\infty$ to 0 ,collapses to a point at $t = 0$ and then starts to grow (fig. 1).

(Fig.1 here)

The interior normals of the fronts that are shrinking reappear as the exterior normals of the expanding fronts.In this sence the initial front is reversed inside out during the time evolution.This family of fronts is called *the standard Legendre collapse.*

The collapsing to a point is not a typical singularity of moving fronts.In a generic family the reversing transformation of the shrinking front into an expanding one consists of several steps.The front consisting of one point is not occurrring at any step.The intermediate fronts have several singularities.

A typical singularity of a front on a plane is a (semicubical) cusp point.The main result of the present paper implies that *there exist fronts having at least four cusps in any generic deformation of the standard Legendre collapse.*

Example. Start from an initial elliptical front,for instance from an almost circular one.Consider the system of fronts consisting of the equidistant curves of this ellipse (fig.1).Some of these equidistant curves have four cusps.Our theorem implies that this is not by chance-it would be impossible to perturbe the standard collapse creating less cusps.

The reasons of the necessity of at least four cusps are essentially topological and belong to the contact topology domain.From the calculus point of view they are topological counerparts of the Sturm type theorems on the zeroes of the eigenfunctions and of their linear combinations.

The contact geometry is the odd-dimensional twin of the symplectic geometry.Unlike the symplectic geometry which is one of the most basic tool of mathematical physics,the contact geometry is less known. The relation between the symplectic and contact geometries is similar to that between the linear algebra and projective geometry.Any calculation in contact geometry is reducible to a calculation in terms of the usual hamiltonian formalism of Poisson brakets of homogeneous functions.But in the topological problems the homogeneous functions approach makes things complicated and the use of the language of contact geometry is practically unavoidable.

In the next two sections the main notions of contact geometry are presented in the form in which they will be used and with the examples which will be used in the proofs of the theorem on the four cusps of a front.

2 Contact manifolds

Definition. *A contact element* on a manifold is a hyperplane (a linear subspace of codimension one)in a tangent space to the manifold.

We shall mostly work with the *cooriented*contact elements (equiped with a transversal direction,defined,say,by a unit normal vector in some metric).All the cooriented contact elements on an n-dimensional manifold B^n form a manifold ST^*B^n of dimension $2n-1$- the space of the spherizied cotangent bundle. The fiber over a point of B^n is the sphere S^{n-1},formed by all the cooriented contact elements at this point.Thus ST^*B^n is the manifold of the directions of all the momenta vectors of the phase space over the configuration manifold B^n.

Example. The manifold of cooriented contact elements of the plane $\mathbf{R}^2$ is a three-dimensional manifold,diffeomorphic to the interior part of a filled torus,$S^1 \times \mathbf{R}^2$. We shall use as coordinates in this space (or in its universal covering three-space)the functions (x, y, φ),where x and y are the usual carthesian coordinates of the point where the contact element is tangent to the plane while φ is the angle between the positive direction of the x axis and the coorienting normal,counted in the direction from the positive direction of the x axis to the positive direction of the y axis.

(Fig.2 here)

Definition. *The standard contact structure* of the manifold of contact elements of B is the field of tangent hyperplanes of the manifold of contact elements o fB,defined by the following scating condition: the velocity vector of a motion of a contact element belongs to the hyperplane of the contact structure if the velocity of the point of contact to B belongs to the contact element.

Example. The standard contact structure of the manifold of cooriented contact elements of the plane is given in the coordinates introduced above by the equation $\alpha = 0$,where α is the differential one-form

$$\alpha = cdx + sdy$$

(here and below $c = \cos\varphi, s = \sin\varphi$.

Remark 1. The form α itself does not enter in the definition of the contact structure:it might be muliplied by any everywhere nonzero function,this will not change the structure $\alpha = 0$.

Remark 2.A *contact structure* on an odd-dimensional manifold is a field of hyperplanes in the tangent spaces,for which the restriction of the form $d\alpha$ to the hyperplane $\alpha = 0$ is a nondegenerate bilinear form (a linear symplectic structure). This nondegeneracy condition is intrinsic:it does not depend on the

particular choice of the form defining locally the contact structure. Globally a contact structure might admit no defining one-form, since the hyperplanes field might be noncoorientable. In our example the nondegeneracy condition holds.

Any contact structure in a neighbourhood of any point of a manifold of a fixed dimension $2m+1$ is equivalent to any other (the Darboux theorem).In a suitable system of the *Darboux coordinates* any contact structure is defined by the equation $\alpha = 0$, where

$$\alpha = dz - pdq(p = (p_1, \ldots, p_m), q = (q_1, \ldots, q_m)).$$

Example. The standard contact structure of the manifold of the cooriented contact elements of the plane is *globally* standard on the universal covering three-space (can be reduced to the Darboux normal form by a suitable choice of *global* coordinates).Indeed, the form α can be written as

$$cdx + sdy = d(cx + sy) + (sx - cy)d\varphi,$$

hence

$$z = cx + sy, p = -sx + cy, \varphi$$

form a system of Darboux coordinates.

3 Legendre submanifolds and their fronts

Definition.*A Legendre submanifold* of a contact manifold is an integral submanifold of the contact structure of maximal possible dimension (equal to m in a contact ,manifold of dimension $2m+1$).

Example. All the contact elements tangent to a smooth plane curve form a Legendre submanifold in the manifold of contact elements of the plane.All the contact elements at a given point of the plane also form a Legendre submanifold in the same manifold.

Definition. The *front* of a Legendre submanifold of the manifold of the contact elements of B is the projection of the Legendre submanifold to the base manifold B.

Example. The front of a generic Legendre submanifold of the manifold of contact elements of the plane is a plane curve whose only singularities are semicubical cusps and transversal selfintersections points.These singularities are stable.They persist under the deformations of the Legendre curve provided that the deformed curve remains Legendrian.

Example. The equidistant curves of an ellipse,shown in Fig.1, are the fronts of certain Legendrian curves.These Legendrian curves are obtained from the initial one (which is formed by the tangent elements of the ellipse) by the transformations of the geodesic flow of the plane (which naturally acts on the manifold of the cooriented contact elements).

Remark. Since our contact elements will always be cooriented, it is convinient always to equip the fronts with an *armement* or *framing* consisting of the cooriented unit normal vectors. A framed front defines its Legendrian submanifold in ST^*B.

Example.A point of B armed with the vectors of all directions, defines a Legendrian submanifold,namely the fiber of the spherical cotangent bundle of B over the point. Any submanifold V^k of any dimension k in B^n generates a Legendrian submanifold in ST^*B^n,namely the manifold L^{n-1} of the contact elements tangent to V^k.The projection of this Legendrian submanifold to the base space B^n defines the mapping

$$L^{n-1} \to V^k$$

which is a *Legendre collapse over* V^k, provided that k is smaller then $n-1$ (which means that V is not a hypersurface in B).

Probably the study of the Legendre collapse over a point below can be extended to the case of Legendre collapses over other submanifolds (the family of Legendre submanifolfs is always formed by the images of L^{n-1} under the geodesic flow action).

Definition.*A Legendre fibration* or *foliation* is a fibration (foliation) whose fibers (leaves)are Legendrian.

Example 1. The spherizied cotangent bundle is Legendrian.

Example 2. The projection

$$\mathbf{R}^{2m+1} \to \mathbf{R}^{m+1}$$

defined in the Darboux coordinates as the forgetting of p,

$$(p, q, z) \to (q, z)$$

is a Legendre fibration.

All the Legendre fibrations (foliations)of spaces of the same dimension are locally contactomorphic in a neighbourhood of a point of the fibration (foliation) space: they can be transformed one into another by a diffeomorphism transforming one contact structure to the other.

The projections of the Legendrian submanifolds of a Legendre fibration space to its base manifold are called the *fronts*. These fronts have locally the same properties as those obtained from the projection on B of the space of contact elements over B.

Example. Consider the mapping

$$ST^*\mathbf{R}^n \to S^{n-1} \times \mathbf{R}$$

sending each cooriented contact element of an euclidean space to the pair consisting of the unite vector of the coorienting normal and of the scalar product of the radius-vector of the contact point with this unit normal vector.This mapping is a Legendrian fibration.

Consider a cooriented hypersurface in the euclidean space.Its cooriented tangent elements form a Legendre submanifold in $ST^*\mathbf{R}^n$. The front of this submanifold is the graph of a function $K : S^{n-1} \to \mathbf{R}$,provided that the initial hypersurface bounds a convex body.The function K is then called *the supporting function* of the body.Hence the graph of the supporting function of a convex body is the front of a Legendre projection of a Legendre submanifold.

If we start from a nonconvex generic hypersurface in $\mathbf{R}^n$,then the front in $S^{n-1} \times \mathbf{R}$ of the Legendre submanifold in $ST^*\mathbf{R}^n$,formed by the tangent elements of the hypersurface, will not be a graph of a function.This front will have the ordinary singularities of fronts-it will be a hypersurface in $S^{n-1} \times \mathbf{R}$ with singularities (for instance,in the case $n = 2$ the front will be a curve on the surface of a cilinder having semicubical cusps and transversal selfintersections as singularities).Thus the generalised supporting functions of generic nonconvex hypersurfaces are multivalued and have the same singularities as the generic fronts of Legendre submanifolds and the Legendre transformations of the generic nonconvex functions (whence the name).

4 Standard Legendre collapse

Consider all the cooriented contact elements of the euclidean space $\mathbf{R}^n$,which are orthogonal to the vector connecting the contact point to the origin.

Proposition. These elements form a submanifold diffeomorphic to a cilinder $S^{n-1} \times \mathbf{R}$,embedded into the manifold of cooriented contact elements of the euclidean space.

Proof. The mapping

$$S^{n-1} \times \mathbf{R} \to ST^*\mathbf{R}^n$$

, sending the pair (ω, t) to the contact element with the cooorienting normal vector ω at the point ωt is an embedding.

Definition. The image of the embedding described above is called *the standard cilinder* over $\mathbf{R}^n$.

The standard cilinder is fibered into the spheres $t = \text{const}$. These spheres are Legendre submanifolds of the manifold of cooriented elements of $\mathbf{R}^n$.

Definition.*The standard Legendre collapse* over $\mathbf{R}^n$ is the projection of the standard Legendre cilinder from $ST^*\mathbf{R}^n$ to $\mathbf{R}^n$.

We shall consider this mapping as the family of mappings of the Legendre spheres $t = \text{const}$ onto their fronts.The sphere corresponding to $t \neq 0$ is mapped onto its front (which is the sphere of radius $|t|$ in $\mathbf{R}^n$) diffeomorphically. All the Legendre sphere $t = 0$ collapses to one point 0 of $\mathbf{R}^n$.

Example. The standard Legendre collapse over the plane is the reversal of the circle described in the section 1.

Now we shall consider the deformations of the standard Legendre collapse over $\mathbf{R}^2$.By definition such a deformation is a one- parameter family of Legendrian embeddings of a circle into $ST^*\mathbf{R}^2$,which should be not too far from the standard cilinder.

The family of equidistant curves of a deformed circle is the family of fronts representing a deformed standard Legendre collapse. We shall see that some of the deformed fronts have at least four cusps provided that the deformation is not too violent.

Remark. How small the deformations should be for the validity of this result is not clear.We shall see that the perturbed Legendrian curves have to be embedded.One may even suppose that the whole perturbed cilinder is embedded,but whether the last condition is really needed is not clear.

The situation is similar to that in the theory of fixed points of symplectomorphisms and of intersections of Lagrange submanifolds before the works of Conley and Zehnder [3],of Floer [4] e.t.c.

In this theory the infinitesimal perturbations of the standard objects can be studied by the Morse theory (applied to the Hamilton function describing the infinitesimal perturbation).

The minorations of the numbers of fixed points and of intersection points obtained from the Morse theory have been extended from the infinitesimal to the finite perturbations in [1],using the technique of generating families,which reduces to the Morse theory the study of finite perturbations (which should not be too large but do not need to be infinitesimally small).

The more modern methods of the study of the perturbations of the quadratic forms with infinite positive and negative indices have later succeded to delete all the restrictions of smallness of the perturbations (for tori in [3].for Kaelerian manifolds with zero second homotopy group in [4],for torical manifolds in [9], but still not for all symplectic manifolds of dimension higher then 2).

In the theory developed below the infinitesimal study leads instead of the Morse theory to the Sturm theory.The results of the present work (as well as those of its symplectic twin [2],see also [6]and [7]),reduce to the teorems of Sturm type the study of finite perturbations which are not too large.

However I was able to delete completly the restrictions on the size of the perturbations only in one model problem (whose relation to the fronts collapse perturbations will be explained below).The result (depending on the so-called *projective topology* ,see [8]),says that a closed simple curve,deviding the sphere into two parts of equal areas,has at least four inflection points.

5 The main theorem

Consider a one-parameter family of Legendre closed curves in $ST^*\mathbf{R}^2$.If this family is not too far from the standard cilindre,defining the standard Legandrian collapse ,then it verifies the conditions of the smallness of the perturbations- the *admissibility conditions*,formulated below.

To formulate the admissibility conditions let us introduce the following notations.Consider the *stanard projection*

$$ST^*\mathbf{R}^2 \to S^1 \times \mathbf{R},$$

sending the point $(x, y; \varphi \mathrm{mod} 2\pi)$ to the point $(\varphi \mathrm{mod} 2\pi, z)$,where $z = cx+sy$.

This mapping is a Legendre fibration.Indeed,$p = -sx + cy$ implies

$$\alpha = cdx + sdy = dz - pd\varphi.$$

The Legendre curves of the pertubed family project to the cilinder $S^1 \times \mathbf{R}$ as fronts.

Example. In the case of the standard nonperturbed Legendre collapse $x^2 + y^2 = t^2$ the fronts on the cilinder are the stanard circles $z = \mathrm{const} = t$.

Definition. A perturbation of a stanard Legendre collapse over the plane is called *admissible*, if the projections of the corresponding Legendre curves to the cilinder are the graphs of smooth functions $z = K(\varphi)$, the function K being 2π-periodic in φ.

This function K depends also on the variable t labeling the Legendre curves,The admissibility condition for this dependence is the uniform convergence of K to $+\infty$ and to $-\infty$ for $t \to +\infty$ and for $t \to -\infty$.

Remark. K is the supporting function of the momentary front at time t. Hence our admissibility conditions imply the local convexity of the fronts in the plane (x, y), whose coorienting normal vector rotates monotonnicaly (the front may have cusps and selfintersections). The second condition of admissibility means that the front on the plane (x, y) is reversed (the inside direction being transformed to the outside one) while t runs from $-\infty$ to $+\infty$.

Theorem 1. *Among the momentary wave fronts on the plane which are the projections of the Legendre curves of a generic admissible perturbation of the standard Legendre collapse there exist fronts having at least four cusps.*

Proof. The cusps of the momentary fronts are defined by the conditions $dx = dy = 0$. Along the Legendrian curve $dz = pd\varphi$. Hence if $z = K(\varphi), p = P(\varphi)$, then $P = K'$. ¿From the equations

$$cx + sy = K, \qquad -sx + cy = K'$$

we found

$$x = cK - sK' \qquad y = sK + cK'.$$

Hence along the Legendrian curve

$$dx = (-sK + cK' - cK' - sK'')d\varphi = -s(K + K'')d\varphi$$

and

$$dy = (cK + sK' - sK' + cK'')d\varphi = c(K + K'')d\varphi.$$

Hence the cusps of the front on the plane (x, y) are the points where $K + K'' = 0$.

Now we shall use the condition $K \to +\infty, -\infty$ for $t \to +\infty, -\infty$. This condition implies that the integral of the function $K + K''$ along the circle vanishes for some value t_0 of the parameter t.

We shall prove that the front on the plane,corresponding to this moment t_0,has at least four cusps.

Consider the Fourier series of the function $K + K''$. This function is orthogonal to $\sin\varphi$ and to $\cos\varphi$ for any value of t.For the choosen value t_0 it is also orthogonal to the constants.Hence its Fourier series starts from the second harmonics:for $t = t_0$we have

$$K + K'' = \sum a_n \cos n\varphi + b_n \sin n\varphi, \quad n \geq 2.$$

Accordingly to the Tabachnikov Sturm-type theorem (see [5] or [2]), any Fourier series starting from the N-th harmonics has at least 2N zeroes.Hence our function has at least 4 zeroes on the circle.

For the generic perturbations of the standard collapse all these zeroes are simple and correspond to the usual cusps of the fronts on the plane (x, y).The theorem is thus proved.

Remark. The existence of a front having at least 4 cusps implies the existence of at least four events of birth and death of pairs of cusps in the family of moving fronts (provided that $K + K'' \neq 0$ for large $|t|$,that is provided that the front evolution srtarts and ends at nonsingular fronts). It is sufficient for this that the standard cilinder be perturbed only in a compact domain,or that $|K''|$ be bounded.

This way one can deduce from the above theorem the classical theorem on the four vertices,accordingly to which any simple closed curve on the euclidean plane has at least four extremal points of curvature,and also the symplectical variant of this theorem (see [2],[6],[7]).

The theorem,proved above,contains (at least formally)more information,than the symplectic version,since the restrictions on the evolution of the fronts of the family in this theorem are less restrictive.Our fronts are not supposed to be equidistant in any sence and the corresponding Legendre manifolds are not supposed a priori to be the images of one of them under a flow of a Hamiltonian system,as it has been the case in [2].In the classical four vertices theorem this Hamiltonian system describes a free particle.

6 Minoration of number of inflection points on a spherical curve

The main result of section 5 can be reformulated in dual terms as a theorem on the inflection points of curves on a sphere.

Consider (φ, z) as the cilindrical coordinates on the surface of the cilinder $X^2 + Y^2 = 1$ in the three-space and project the cilinder to the sphere $X^2 + Y^2 + Z^2 = 1$ by the family of rays issued from the origin.Each curve on the cilinder projects to a curve on the sphere.Consider the projection of a curve defined by the equation $z = K(\varphi)$.

Proposition. *The points where $K + K'' = 0$ project to the inflection points of the curve on the sphere. Inversely,suppose that a curve on the sphere intersects a meridien $\varphi = \mathrm{const}$ transversally at its inflection point (different from a pole of the sphere).Then $K + K'' = 0$ at the corresponding point of the projection of this curve to the surface of the cilinder.*

Proof. The graphs of the solutions of the differential equation $K + K'' = 0$ are the sections of the surface of the cilinder by the nonvertical planes.Hence their projections to the sphere are its nonvertical greate circles.

Consider a point where $K + K'' = 0$, for any function K.There exists a solution of the differential equation $K + K'' = 0$ whose graph is tangent to the graph of our function at that point.The order of tangency of the two graphs at that point is at least equal to *two*,since both functions verify the same equation $K + K'' = 0$ at that point.Hence the spherical curve is at least second order tangent to the greate circle tangent to it at the point of the sphere corresponding to our point on the cilinder.The proposition is proved.

Corollary. *Consider a function on a two-sphere, whose derivative along any meridien is positive, except at the two poles,(which are the maximum and the minimum points),and otherwise generic.Then at least one level line of this function has at least four inflection points.*

Proof. Projecting the level lines to the cilinder,we arrive to the situation of the proof of the main theorem.On the surface of the cilinder our curves are defined by the equations of the form $z = K(\varphi)$,and $K \to +\infty \text{or} -\infty$ when the spherical curve shrinks to a pole.

The inflection points of the curves on the sphere are the projections of the points where $K + K'' = 0$ on the surface of the cilinder.Hence some of the curves on the sphere have at least four inflection points.

We shall see now,that the curve dividing the sphere into two parts of equal areas alwais has this property.Moreover,the requirement of the transversality to the meridiens may be deleted ,since the following result holds:

Theorem 2. *Any closed smooth curve on the sphere, dividing it into two parts of equal areas,has at least four inflection points .*

Proof. This follows from a general result in the projective topology,which might be of independent interest:

Theorem 3. *Any closed connected smooth curve embedded into the two-sphere and having at most three inflection points lies in an open hemisphere.*

Remark. Any closed connected locally convex hypersurface embedded into a sphere of any dimension and any embedded segment having no interior inflection points in the two-sphere lie in an open hemisphere accordingly to [8].

Theorem 3,proven in the next section,implies theorem 2.Indeed, if the number of inflection points were smaller then four, the curve would lie in an open hemisphere and the area of the domain bounded by the curve in this hemisphere would be smaller than the area of the complementary domain.

Inflection points in the above theorems are the zeroes of the geodesic curvature,that is the points of at least second order tangency of the curve with a greate circle.If there are exactly three such points on a closed curve,then at least one of them is degenerate (being the local maximum or local minimum of the curvature).We may eliminate such a point reducing theorem 3 to the case of at most two inflection points.In this form we shall prove theorem 3 below.

7 Spherical curves having two inflection points.

The proof of theorem 3 is based on the results of projective topology from [8].To avoid the duplication of the arguments of [8] I shall omitt some details.For instance I shall suppose that the given spherical curve has exactly two points of zero geodesical curvature,both nondegenerate. This genericity condition may be eliminated by elementary continuity arguments.

The nondegenerate inflection points A and B (fig.3)subdivide the curve into two locally convex segments connecting A to B.

(Fig. 3 here)

These curved segments are convex to the same side.This means that the orientations of the sphere defined by the direction from A to B and by the direction to the center of the curvature on the sphere are the same for both segments.

Accordingly to [8],any of the two curved segments AB lie in an open hemisphere.The projection $S^2 \to \mathbf{R}P^2$ sends the curved segment to an *affine* locally convex curve AB (that is to a curve ,which does not intersect some projective line,called line at infinity).

The main difficulty of the proof of theorem 3 is to see that this line can be choosen to be the same for the two curved segments AB on the sphere.Such a choice is indeed possible,and here are the reasons why it is so.

Consider a convex affine curve AB (fig.4).The boundary of its convex hull consists of a convex arc CD and of a segment DC of a straight line connecting the endpoints of the arc (this result,implicitly proved in [8],will not be proved here).

(Fig. 4 here)

Point C may coincide or not with A and D with B (in fig.4 $C = A$ but $D \neq B$).The domain,bounded by the curve CDC on the affine plane,define on the initial sphere a domain,containing the interval AB of the greate circle,shorter than π.I shall call this domain *the spherical convex hull* of the convex segment AB of the initial spherical curve.

For the second convex segment AB of the initial spherical curve one defines similarly its spherical convex hull.

Main lemma. *One of the two spherical convex hulls constructed above contains the other.*

Theorem 3 follows,since even the largest of the two spherical convex hulls lie in a hemisphere.

Proof of the maim lemma. Consider the greate circle through AB, call it the *equator*.Its lenght is 2π and each of the two convex hulls intersects it along an interval containing the interval AB and shorter than π.

The equator divides the sphere into the North and the Southern hemispheres.The segment CD of the geodesic on the boundary of the spherical convex shell cannot intersect the equator (except at A or B in the cases $A = C$ or $B = D$ and except the case $AB = CD$,when all the segment belongs to the equator). Hence one of the hemisheres intersects the boundary of the spherical convex hull along a strictly convex curve (curve KL in fig.5).I shall call this hemisphere *the Southern hemisphere* and this curve KL - *the Southern curve.*

(Fig.5 here)

Lemma. *The Southern hemisphere is the same one for both convex segments into which the inflection points A and B belonging to the equator subdivide the initial spherical curve.*

Proof. This hemisphere lies at the side of the segment AB of the eqator for which the orientation of the sphere defined by the direction from A to B and the direction toward the hemisphere is opposit to the orientation defined by the direction of the convex segment from A to B and the direction to its center of curvature on the sphere.

But the last orientation is independent on the choice of one of the two convex segments AB(this is evident in a neighbourhood of A and hence is true everywhere).Hence the Southern hemispheres ,defined by both convex segments, coincide,and the lemma is proved.

The Southern arcs KL of one and $K'L'$ of the second convex hulls both lie in the Southern hemisphere. They do not intersect each other since the initial curve was embedded.Hence the end points of the two arcs on the equator do not alternate and thus one of the short arcs of the equator containing AB,say KL,contains the other,say $K'L'$.It follows that the whole convex segment AB,containing the points K' and L' lies inside the spherical convex hull $KLDC$ (in fig.5 this segment is shown as a broken line).

The proof of the statement,that the broken convex segment AB is confined to the convex hull $CKLD$ (which is sufficiently evident from fig.5) can be obtained from the arguments in [8] that I shall not reproduce here. It follows from this confinement that the convex hull of the broken curve lies entierly inside the convex hull of the affine curve $ACKLDB$.Thus the main lemma is proved,and hence also is theorem 3 which implies theorem 2.

8 Fronts with two inflection points

A Legendre curve,which is far from the circles forming the standard collapse,can also be projected to the cilinder with coordinates (φ, z) and then to the sphere.However the projections are in general neither graphs of functions $z = K(\varphi)$ nor smooth curves on the sphere.

These projections (of generic Legendre curves) have the usual singularities of wavefronts-the selfintersection points and the cusps (of semicubical type).Moreover the tangents to the branches of these fronts are nonvertical both at the usual points and at the points of selfintersection and at the cusp points: along these tangents one $d\varphi \neq 0$.

The question arise what is the minimal number of inflection points of the front having the maximal number of inflection points among the fronts of a family connecting a small paralel cetred at the South pole to a small parallel centred at the North pole of the sphere (provided that the family is generic and verifies the above conditions,implying that all these fronts are cooriented by the direction of the meridiens from the South to the North).

The fronts of the family depend on one parameter and experience under its variation the standard perestroikas (namely the quadratic selftangencies,the triple selfintersections with distinct tangents,the transversal crossings of a cusp point by a regular branche of the front and the birthes and deathes of pairs of cusps).

The example described below shows that one can satisfy all these conditions in a family of fronts,containing no fronts having more than *two* inflection points.

Construct first the front which divides the sphere into two parts of equal areas.This front has no selfintersections and provides by itself a counterexample to an attempt of extention of theorem 2 to singular curves.

Consider two antipodal points A and A' on the equator of the sphere (fig.6).

(Fig.6 here)

The point A is an inflection point of the front.The parts of the front are convex till the cusp points B (where the tangent is horizontal,that is touches the parallel $Z = -1/2$) and C (where the tangent touches the parallel $Z = 1/2$.)The coordinate φ is monotonously growing along the interval BAC of the front.

The segment $B'A'C'$ is central symmetrical to BAC.The parts of the front CB' and $C'B$ are closed to the parallels $Z = 1/2$ and $Z = -1/2$.Along the segment AC the direction to the center of curvature on the sphere is at the Southern side of the front,along BA at the Northern side.Thus A and A' are the only inflection points.

Include the front thus constructed into a family,interpolating a parallel closed to the South pole to a parallel closed to the North pole.I shall describe the deformation contracting the symmetrical front to the South pole.The deformation contracting it to the North pole is symmetrical.

First rotate the arcs BC and $B'C'$ toward each other along the sphere as rigid bodies (the rotation axis connecting the South and North poles of the sphere).The almost parallel BC' shrinks and the almost parallel BC' is dilated.At some moment of this process the points B and C' on the longer almost parallel have to go through each other both tangents being horizontal.(This is a codimention two nongeneric event in one-parameter families,which can be avoided by a small deformation of the family, making it generic).

After this moment two selfintersection points occur on the front in the neighbourhood of B and C'. The upper selfintersection point X starts to mount along BA and $C'A'$.At some moment it goes through the inflection points A and A' (coinceiding at that moment) and then we stop the rotation of the arcs (fig.7).

(Fig. 7 here)

The next step is the killing of the triangle $XB'C$ (accompagnying the standard death of two cusps B' and C).The resulting curve has only two cusps B and C' and still have two inflection points A and A'.These two inflection points can be killed by a smooth deformation of the arc BC' (fig.8.).The remaining two cusps die together with the lowest selfintersection point of the front.

(Fig.8 here)

The resulting front is close to the parallel $Z = -1/2$ which it is easy to shrink to the South pole.

Remark. The fronts of the generic family that we have constructed have momentary selftangencies (at some moments close to the coincidence of two cusps in the nongeneric family that we were perturbing)c.The corresponding Legendre curves have selfintersections.Unlike the embedded Legendre curves of the standard collapse they are only immersed to $ST^*\mathbf{R}^2$.

Hence the above example does not exlude the necessity of at least four cusps on a front of a family of reversing inside out fronts on the plane connecting a nonsingular front shrinking along the coorienting directions to an expanding one,provided that the fronts of the family are required to have no selftangencies.This nontangency property holds, for instavce,if the perturbed cilinder is *embedded* into $ST^*\mathbf{R}^2$, as the standard one is.

It is interesting to study the particular case, when this perturbed cilinder is connected to the standard one by a regular hopotopy,under which it remains all the time embedded and fibered into closed Legendre curves.This holds,for instance, if the perturbed cilinder is the image of the standard one under a contactomorphism,which is connected to the identity by a smooth path in the space of contactomorphisms of the manifold $ST^*\mathbf{R}^2$ to itself,different from the identity only in a compact domain.

However even in this particular case the necessity of at least four cusps on one of the fronts during the reversal of the coorientation process is not proved.

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Figures captions.

Fig.1.Standard Legendre collapse and its generic deformation.

Fig.2.Coordinates of a cooriented contact element of the plane.

Fig.3.The structure of a curve with two inflection points.

Fig.4.The convex hull of a convex segment AB.

Fig.5. The South curves KL and $K\ ^{prime}L'$.

Fig.6.A front with two inflection points dividing the sphere into two parts of equal areas.

Fig.7.The contraction of the arc $B'C$.

Fig.8. The contraction of the arc BC'.