Simple singularities of curves

V.I.Arnold *
Steklov Mathematics Institute, Moscow
& CEREMADE, Université Paris-Dauphine

Dedicated to L.D.Faddeev

1 Introduction

A singularity of a curve below means a germ of a holomorphic mapping of the complex line into the complex space at a singular point (where the derivative of the mapping vanishes) considered up to the biholomorphic mappings of the image space.

The singularity is called *simple* if all neighbouring singularities belong to a finite set of equivalence classes (have no moduli). The simple singularities of curves in the plane have been classified by J.W.Bruce and T.Gaffney [1], that of curves in three-space – by C.Gibson and C.Hobbs [7].

The singularity is called *stably simple* if it is simple and remains simple when the ambient space is embedded into a larger space.

Two curves, obtained one from the other by such an embedding, are called *stably equivalent.*

We classify below the simple curve singularities in spaces of any dimension up to stable equivalence.

We find, that almost all those singularities, whose Tailor series starts from a term of degree 2 or 3 or has the form

$$x = t^4, \quad y = t^6, \quad z_i = 0 \pmod{t^7},$$

are simple, and that besides these curves there exist 32 *sporadic simple curves*, listed below.

"Almost all" curves means here and below "all curves except the infinitely - degenerate curves, forming a codimension infinity set in the space of all curves".

The *codimension* here and below means "the codimension in the space of Taylor series with zero constant terms."

The codimension of the set of all nonsimple singularities of germs of curves belonging to the N-space equals $6N-6$. This means, that the non-simple singularities never occur on curves of typical $6N-8$ - parametric families of curves, but can't be eliminated by a small deformation of the family, if the number of parameters reaches $6N-7$.

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All simple singularities, whose Taylor series starts from terms of degree n, are realised in the n-space. From the 32 sporadic singularities one requires the 6-space, four – the 5-space, 9 – the 4-space, 13 – the 3-space, five – can be realised in the plane.

¿From the point of view of algebra we study the "coartinian" subalgebras (of finite codimension in the algebra of power series in one variable). The main tools are *the singularity semigroup,* formed by the orders of the principal terms of the series forming the subalgebra (see [2, 3, 4]) and *the spectral sequence* of a quasihomogeneous filtration, similar to that used in the theory of critical points of functions (see [5]).

Two natural filtrations in the subalgebra and the set of the commuting nilpotent matricies representing our algebra suggest that the classification of the simple (having no continious invariants or moduli) coartinian subalgebras might be included into some theory similar to that of quivers. But up to now I was unable to find any relation of our classification problem to the known problems of this type (including the classifications of the simple Lie algebras, of the simple categories of vector spaces, of the simple caustics and wave fronts, of the regular polyhedra, of the Coxeter groups, of the simple meromorphic functions and the others " A, D, E – classifications").

The special role of the number 6 as of the boundary between the simple and non simple singularities suggest that the usual E_6, E_7, E_8 have something to do with our problem. This "six" is responsible for the tangent 7/6 of the upper boundary of the domain of the "good dimensions" of John Mather [6].

I was led to the classification of the simple singularities of curves by an attempt to construct the local algebraic (or better analytic) symplectic geometry in terms of the combination of the Poisson bracket and of the usual multiplication of functions in the algebra of functions on the phase space. I hope to describe elsewhere the symplectic corollaries of the results of the present paper.

2 The list of the stably simple and of the fencing singularities of curves

A monomial curve in $\mathbf{C}^N$, defined by the parametrical equations

$$x_1 = t^{a_1}, \quad \dots, \quad x_N = t^{a_N} \quad (a_1 < \dots < a_N),$$

will be denoted by the symbol $(a_1, \dots, a_N)$. If the monomial t^{a_i} is replaced by the binomial $t^a + t^b$ in one of the components, the element a_i of the symbol will be replaced by (a, b). For instance, the symbol $(4, (6, 7))$ denotes the plane curve $x_1 = t^4$, $x_2 = t^6 + t^7$.

Theorem 1 *The stably simple singularities of curves are stably equivalent to the singularities of the following list (which are all simple):*

1. *almost all the curves with a nonzero quadratic part of the Taylor series;*

2. *almost all the curves with zero quadratic and nonzero cubic part of the Taylor series;*

3. *almost all the curves whose 6-jet is* $(t^4, t^6, 0, \dots, 0)$ *;*

4. *thirty two sporadic curves, listed below.*

Remark. The infinitely degenerate curves, eliminated by the words "almost all" are not embedded topologically (the curve "covers itself", as, for instance, the curve with the symbol $(2,6)$). For the formal series and for real smooth curves the infinite degeneracy is defined by the same infinite system of equations on the Taylor coefficients, as in the holomorphic case, but the toplogical description is not that simple.

The list of sporadic curves.

This list contains five parts.

1. *Seven curves whose symbol starts with* $(4,5)$*:*

$$\begin{array}{ccccccc}
(4,5,6,7)^{3N} & & & & & & \\
\uparrow & & & & & & \\
(4,5,6)^{4N-3} & \leftarrow & (4,5,7)^{4N-2} & \leftarrow & (4,(5,7),11)^{5N-4} & \leftarrow & (4,5,11)^{5N-3} \\
 & & & & \uparrow & & \uparrow \\
 & & & & (4,(5,7))^{6N-6} & \leftarrow & (4,5)^{6N-5}
\end{array}$$

This part contains all the curves whose symbol starts with $(4,5)$*. All curves with this property are simple.*

The upper index means the codimension, and arrows – the adjunctions.

2. *Fifteen curves whose symbol starts with* $(4,7)$*:*

$$\begin{array}{ccccc}
(4,7,9,10)^{5N-2} & & & & \\
\uparrow & & & & \\
(4,7,(9,10))^{6N-5} & \leftarrow & (4,7,9)^{6N-4} & & \\
\uparrow & & & & \\
(4,(7,9),10,13)^{6N-4} & \leftarrow & (4,7,10,13)^{6N-3} & & \\
\uparrow & & \uparrow & & \\
(4,(7,9),10)^{7N-7} & \leftarrow & (4,7,10)^{7N-6} & & \\
\uparrow & & \uparrow & & \\
(4,(7,9),13)^{7N-6} & \leftarrow & (4,7,13)^{7N-5} & & \\
\uparrow & & & & \\
(4,(7,9),17)^{8N-8} & \leftarrow & (4,(7,13),17)^{8N-7} & \leftarrow & (4,7,17)^{8N-6} \\
\uparrow & & \uparrow & & \uparrow \\
(4,(7,9))^{9N-10} & \leftarrow & (4,(7,13))^{9N-9} & \leftarrow & (4,7)^{9N-8}.
\end{array}$$

This part contains all the curves whose symbol starts with $(4,7)$*. All the curves with this property are simple.*

3. *The curve* $(4,9,10,11)^{6N-3}$.

4. *Six curves whose symbol starts with* (5)*:*

$$\begin{array}{ccccc}
(5,6,7,8,9)^{4N} & \leftarrow & (5,6,7,8)^{5N-4} & \leftarrow & (5,6,7,9)^{5N-3} \\
 & & & & \uparrow \\
(5,7,8,9)^{6N-5} & \rightarrow & (5,7,8,9,11)^{5N-1} & \rightarrow & (5,6,8,9)^{5N-2}
\end{array}$$

5. *Three curves whose symbol starts with* (6)*:*

$$(6,7,8,9,10,11)^{5N} \leftarrow (6,7,8,9,10)^{6N-5} \leftarrow (6,7,8,9,11)^{6N-4}$$

Theorem 2 *Each nonsimple singularity of curve either belongs to one of the three infinitely degenerate limit classes of the families 1,2 and 3 of theorem 1, or is adjacent to one of the following three classes of fencing curves:*

$$\begin{gathered}
(t^4+t^5+at^6, t^9, t^{10}, t^{15}), \\
(t^5, t^6+t^8+at^9, t^7), \\
(t^6, t^7+t^{10}+at^{11}, t^8+bt^{10}, t^9).
\end{gathered}$$

The codimensions of these three classes are equal to $7N-6, \quad 6N-6, \quad 7N-8$ *respectively.*

For instance, the codimension of the set of nonsimple singularities of germs of curves in $\mathbf{C}^N$ equals $6N-6$, if $N \geq 3$. The coefficient 6 defines the slope $7/6$ of the upper boundary of the domain of the good dimensions of J.Mather at infinity.

3 Classification of curves having nonzero 3-jet

All such curves, except those which are infinitely degenerate, are simple and are stably equivalent either to plane curves or to curves of the 3-space.

Theorem 3 *The singular curves whose 2-jet is nonzero (except the infinitely degenerate ones) form one infinite series* A_{2k} *of stable plane curves:*

$$(2,3)^N \leftarrow (2,5)^{2N-1} \leftarrow \ldots \leftarrow (2,2k+1)^{kN-(k-1)} \leftarrow \ldots$$

Suppose now that the quadratic part of the Taylor series vanishes but the cubic part does not vanish.

Theorem 4 *The singular curves having zero 2-jet and nonzero 3-jet (except those which are infinitely degenerate) form two 3-indices series of curves, stably equivalent to the following curves in the three–space:*

$$\begin{aligned}
E_{6k,p,i} &= (3,(3k+1,3k+2+3i),3k+2+3p)^{(2k+p)N-(2k+2p-2-i)}, \\
E_{6k+2,p,i} &= (3,(3k+2,3k+4+3i),3k+4+3p)^{(2k+p+1)N-(2k+2p-1-i)},
\end{aligned}$$

where $k \geq 1,\ 0 \leq p \leq k,\ 0 \leq i \leq k-1$.

Remark. For the boundary values of the indices one can simplify the normal form, choosing

$$\begin{array}{lcl} E_{6k,k,i} & = & (3,(3k+1,3k+2+3i))^{3kN-(4k-2-i)}, \\ E_{6k+2,k,i} & = & (3,(3k+2,3k+4+3i))^{(3k+1)N-(4k-1-i)}, \\ E_{6k,p,p} & = & (3,3k+1,3k+2+3p)^{(2k+p)N-(2k+p-2)}, \\ E_{6k+2,p,p} & = & (3,3k+2,3k+4+3p)^{(2k+p+1)N-(2k+p-1)}, \\ E_{6k,k,k-1} & = & (3,3k+1)^{3kN-(3k-1)}, \\ E_{6k+2,k,k-1} & = & (3,3k+2)^{(3k+1)N-3k}. \end{array}$$

One has the adjacencies

$$E_{.,p,i} \ \to \ E_{.,p,i-1} \ \to \ E_{.,p-1,i-1}.$$

The hierarchy of these E – singularities starts with:

$$\begin{array}{ccccccccc} (3,4,5)^{2N} & \leftarrow & (3,4)^{3N-2} & \leftarrow & (3,5,7)^{3N-1} & \leftarrow & (3,5)^{4N-3} & & \\ & & & & \uparrow & & & & \\ & & & & (3,7,8)^{4N-2} & \leftarrow & (3,(7,8),11)^{5N-4} & \leftarrow & \cdots \\ & & & & \uparrow & & \uparrow & & \\ & & & & (3,8,10)^{5N-3} & & (3,7,11)^{5N-3} & \leftarrow & \cdots \\ & & & & \uparrow & & & & \\ & & & & \vdots & & & & \end{array}$$

4 Classification of curves having the 6-jet $(t^4, t^6, \ldots)$

All such curves, except those which are infinitely degenerate, are simple.

Theorem 5 *Curves having the 6-jet $(t^4, t^6, \ldots)$, which are not infinitely degenerate, form seven one-indice families:*

$$\begin{array}{lll} a_k = & (4,(6,2k+1))^{(5+k)N-(6+k)}, & k \geq 3; \\ b_k = & (4,(6,2k+3),2k+11)^{(5+k)N-(5+k)}, & k \geq 2; \\ c_k = & (4,(6,2k+5),2k+9)^{(5+k)N-(4+k)}, & k \geq 1; \\ d_k = & (4,(6,2k+7),2k+9,2k+11)^{(5+k)N-(3+k)}, & k \geq 0; \\ e_k = & (4,(6,2k+9),2k+11)^{(5+k)N-(2+k)}, & k \geq -1; \\ f_k = & (4,6,2k+5,2k+7)^{(5+k)N-(5+k)}, & k \geq 1; \\ g_k = & (4,6,2k+7)^{(5+k)N-(4+k)}, & k \geq 0. \end{array}$$

These seven series adjacencies are:

$$
\begin{array}{ccccccccc}
b_{k-1} & \leftarrow & c_{k-1} & \leftarrow & d_{k-1} & \leftarrow & e_{k-1} & & \\
 & & & & \uparrow & & \uparrow & & \\
\uparrow & & \uparrow & & f_k & \leftarrow & g_k & & \\
 & & & & \uparrow & & \uparrow & & \\
a_k & \leftarrow & b_k & \leftarrow & c_k & \leftarrow & d_k & \leftarrow & e_k \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

The hierarchy of these singularities starts with

$$
\begin{array}{ccccccccccc}
e_{-1} & & & & & & & & & & \\
\uparrow & & & & & & & & & & \\
g_0 & & & & & & & & & & \\
\uparrow & & & & & & & & & & \\
d_0 & \leftarrow & e_0 & & & & & & & & \\
\uparrow & & \uparrow & & & & & & & & \\
f_1 & \leftarrow & g_1 & & & & & & & & \\
\uparrow & & \uparrow & & & & & & & & \\
c_1 & \leftarrow & d_1 & \leftarrow & e_1 & & & & & & \\
 & & \uparrow & & \uparrow & & & & & & \\
\uparrow & & f_2 & \leftarrow & g_2 & & & & & & \\
 & & \uparrow & & \uparrow & & & & & & \\
b_2 & \leftarrow & c_2 & \leftarrow & d_2 & \leftarrow & e_2 & & & & \\
 & & & & \uparrow & & \uparrow & & & & \\
\uparrow & & \uparrow & & f_3 & \leftarrow & g_3 & & & & \\
 & & & & \uparrow & & \uparrow & & & & \\
a_3 & \leftarrow & b_3 & \leftarrow & c_3 & \leftarrow & d_3 & \leftarrow & e_3 & & \\
 & & & & & & \uparrow & & \uparrow & & \\
 & & \uparrow & & \uparrow & & f_4 & \leftarrow & g_4 & & \\
 & & & & & & \uparrow & & \uparrow & & \\
 & & a_4 & \leftarrow & b_4 & \leftarrow & c_4 & \leftarrow & d_4 & \leftarrow & e_4 \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

5 The coartinian algebra and the semigroup of the curve singularity.

The classification of the curve singularities can be interpreted in dual terms as the listing of "coartinian" subalgebras of finite codimension in the algebra of power series in one variable (considering the subalgebras up to the automorphisms of the ambient algebra of the series, that is up to the formal diffeomorphisms of the line).

Definition. *The local algebra* A_f of the curve $x_i = f_i(t)$ in $\mathbf{C}^N$ is the subalgebra of the algebra of the power series in t, generated (as the closure with respect to the powers of the maximal ideal) by the multiplicative generators f_i. In other terms it is the algebra

$$A_f = f^*\mathcal{M} \subset \mathbf{C}[[t]],$$

where $\mathcal{M}$ is the maximal ideal of the ring $\mathbf{C}[[x_1, \ldots, x_N]]$.

The filtrations of the algebras of the series in t and in x according to the order of the zero at the origin generate in the algebra A_f two filtrations. Denote by M^i the ideal of the algebra $\mathbf{C}[[t]]$, formed by the series, divisible by t^i. The filtration

$$A_f = {A_f}^0 \supset {A_f}^1 \supset \cdots, \quad {A_f}^s = A_f \cap M^s,$$

coincides with $M^i \supset M^{i+1} \supset \cdots$ from some place i, since the codimension of A_f in $\mathbf{C}[[t]]$ is finite (provided that the singularity is not infinitely degenerate).

We obtain a finite-dimensional commutative algebra A_f/M^i (equiped with a filtration), represented by commuting nilpotent linear operators acting in the finite-dimensional vector-space M^1/A_f.

The dimension $a = \dim_{\mathbf{C}} M^1/A_f$ of this vector space appears in the expression $aN - b$ for the codimension of the class of the simple singularity f of a curve belonging to $\mathbf{C}^N$. The correction b equals to the dimension of the quotient space $(M^1 df/dt + {A_f}^N)/{A_f}^N$, as it follows from the formula for the tangent space to the orbit of the germ f.

Definition. *The semigroup* P_f of the curve singularity $x_i = f_i(t)$ is the subsemigroup of the additive semigroup of natural numbers formed by the orders of zero at the origin of all the linear combinations of the products of the series f_i.

In other terms, the semigroup P_f consists of the degrees of the one (nonzero) dimensional homogeneous components of the graded algebra G_f, associated with the filtred algebra $A_f \supset {A_f}^1 \supset \ldots$:

$${G_f}^i = {A_f}^i/{A_f}^{i+1}, \quad G_f = \bigoplus {G_f}^i.$$

Using this terminology we can say that *the graded algebra G_f, corresponding to the curve f, is the semigroup algebra of the semigroup P_f of this curve* (the semigroup algebra consists of the linear combinations of the monomials t^i, whose exponents i belong to the semigroup).

Example. The semigroup of the plane curve germ $x = t^3$, $y = \sin^3 t$ is described by the line

$$P_f = ([1], [2], \mathbf{3}, [4], \mathbf{5}, 6, [7], 8.).$$

Here the square brackets enclose the *gaps* $[i]$, (for which $\dim {G_f}^i = 0$), and the bold entries represent the degrees of the multiplicative generators of the algebra G_f. The point following 8 means that all the larger integers $(9, 10, \ldots)$ do belong to the semigroup.

The hierarchy of the singularities, described in our theorems, reflects the semigroups arithmetics. For the semigroups generated by two elements one has a very useful *Sylvester duality* between the semigroup and its complement. Let u and v be two relatively prime natural numbers. Denote by K the even number $(u-1)(v-1)$.

Sylvester [3] has published in a nonmathematical journal the following

Proposition. *An integer c is representable as a linear combination $mu + nv$ with integer nonnegative coefficients if and only if the number $K - 1 - c$, (symmetric to c with respect to the middle of the segment $[0, K-1]$), is not representable in this form.*

For instance, all the integers greater or equal to K are representable, while $K - 1$ is not representable in the form $mu + nv$, $m \geq 0$, $n \geq 0$. The representable numbers form exactly one half of the segment $[0, K-1]$.

Example. For $u = 3$, $v = 5$ the semigroup of combinations is

$$(0, [1], [2], \mathbf{3}, [4], \mathbf{5}, 6, [7], 8.).$$

The sequence of the lengths of the gaps $(2, 1, 1)$ is dual to the sequence $(1, 1, 2)$ of the lengths of the antigaps. Among the eight integral points of the segment $[0, 7]$ four do and four do not belong to the semigroup of the representable points.

6 Technics of calculations using semigroups

The proofs of the classification theorems formulated above consist of long calculations, which can be made rather short in terms of the semigroups. I shall show how the semigroup technics works in a particular example proving the sufficiency of the symbol $(3, 7, 8)$ providing a polynomial normal form for any curve $(t^3 + \cdots, t^7 + \cdots, t^8 + \cdots)$. We shall kill all the higher order terms by the consequtive changes of the independent and of the dependent variables.

The change of the dependent variables x_i is called *left*, the change of the independent variable t is called the *right* change of variables.

The right change $t = T(1 + aT^r)$ transforms the monomial t^m to its sum with the increment of the form $t^m \sum c_k t^{kr}$, $\quad k \geq 1$.

The left change $X_i = x_i + g(x)$ adds to x_i a correction series, whose support belongs to the semigroup P_f. Choosing g one can obtain any initial term t^j in the correction series, provided that j belongs to the semigroup. One should only control, that the correction would provide a genuine change of variables, verifying the implicite function theorem condition. It suffices that j be larger than the lowest term of the series f_i.

We shall denote the series like $t^3 + at^5 + o(t^5)$ by the symbol like (3&5.). The dot inside the brackets is important – it means that the addition of the series $o(t^5)$ is permitted.

Proposition. *Every curve* $(3., 7., 8.)$ *is equivalent to the monomial curve* $(3, 7, 8)$.

Proof. The semigroup has the description

$$([1], [2], 3, [4], [5], 6.).$$

Killing one by one all the summands whose degrees belong to the semigroup in each component, we get the left equivalence

$$(3., 7., 8.) \sim (3\&4\&5, 7, 8)$$

of the initial curve to the curve, each component of which has as its support the union of the gaps of the semigroup which follow the exponent of the leading term of this component (thus, the gaps [4] and [5] follow the exponent 3 of the leading term of the first component).

Now we kill the t^4 in the first component using the right equivalence with $r = 1$. Next we kill by the left equivalence the newborn increment terms (of degree 6 or larger in the first, 8 or larger in the second, 9 or larger in the third component). We thus get the equivalence

$$(3\&4\&5, 7, 8) \sim (3\&5, 7, 8).$$

The remaining perturbation t^5 in the first component can be killed by the right equivalence with $r = 2$. The newborn increment terms have degrees 7 or higher in the first, 9 or higher in the second, 10 or higher in the third component. All their exponents do belong to the semigroup. Therefore all these newborn terms can be killed by a left change of variables :

$$(3\&5, 7, 8) \sim (3, 7, 8).$$

Thus we get the required equivalence to the normal form.

Remark. Strictly speaking, all these blood-thirsty reasonings are valid only at the formal series level. However in the case of the holomorphic (or even real smooth) curves the changes of variables might be made holomorphic (or smooth, respectively), as it is well known from the Weierstrass–Malgrange preparation theorem of singularity theory.

The proofs of the remaining theorems follow the same pattern, but the calculation of the semigroup is sometimes less obvious (the semigroup of a curve is **not** defined by the exponents of the leading terms of the components). For instance, the semigroup of the curve defined by the symbol $(4, (6, 7 + 2k))$ contains $13 + 2k$.

To study the curves whose 6-jet is $(t^4, t^6, 0, \dots, 0)$ one should use the stationary group of the plane curve $x^3 = y^2$. The Lie algebra of this group is generated (as a module over the algebra of functions) by the Euler quasihomogeneous field $2x\partial_x + 3y\partial_y$ and by the Hamiltonian field $2y\partial_x + 3x^2\partial_y$.

The spectral sequence of the corresponding quasihomogeneous filtration provides immediately the equivalence

$$(t^4, t^6 + t^{7+2k} + o(t^{7+2k})) \sim (t^4, t^6 + t^{7+2k}),$$

and reduces to the normal forms (described in theorem 5) the other components of the corresponding space curve. Thus one gets, for instance, the nonimmediate equivalence

$$(t^4, t^6 + t^{7+2k}, t^{9+2k} + at^{11+2k}) \sim (t^4, t^6 + t^{7+2k}, t^{9+2k}).$$

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