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Topological Problems in Wave Propagation Theory and Topological Economy Principle in Algebraic Geometry

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Abstract. This is the third of the series of three lectures given by Vladimir Arnold in June 1997 at the meeting in the Fields Institute dedicated to his 60th birthday.

> From the most skillful definition, free as it might be of any inner contradictions, one can never deduce a new fact.
>
> *M. Planck. Thermodynamics.*

I wish to explain one general principle. In some cases it leads to theorems and I shall also discuss these theorems. But in my opinion the principle itself, which is not rigorously formulated, is more important because it contains more possibilities. I call this general principle the *economy principle* in algebraic geometry. It says that if you have a geometrical or topological phenomenon, which you can realize by algebraic objects, then *the simplest algebraic realizations are topologically as simple as possible*. So replacing algebraic objects by topological objects you are unable to gain in simplicity. If you have something complicated in the algebraic case, it means that you *need* it for topological reasons. You cannot avoid it. You cannot make the object more simple. I shall give some examples of this in the theory of wave propagation, in symplectic geometry, in contact geometry, in projective geometry and so on.

I shall start from a simple example, where the principle is by now proved and well known. This is *the Thom conjecture*. Take $\mathbb{C}P^2$ and choose a cycle of dimension two in this four dimensional manifold. Of course, you have one generator $[\mathbb{C}P^1]$ for the homology group $H_2(\mathbb{C}P^2, \mathbb{Z})$, so the chosen cycle is n times the generator. You can realize it by an algebraic object which is an algebraic curve of degree n. If it is smooth, its genus is equal to $g = (n-1)(n-2)/2$ by the Riemann formula. You

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ask whether it is possible to realize the cycle by a smooth surface with less handles (not necessarily an algebraic curve). The Thom conjecture claims that you cannot do this. *All those g handles are necessary for topological reasons.* You cannot avoid them whatever you do. What you have in the simplest algebraic model of the realization of this object is really needed for topological reasons. That is the principle. The Thom conjecture is recently proved by Kronheimer and Mrowka and so it is a theorem.

The second example is *the Milnor conjecture*. You start with a singularity $\{f(x,y)=0\}$ of a function f in two complex variables x,y. Suppose that it is an irreducible curve. You may think of $\{x^2+y^3=0\}$. It has a local singularity at the origin. Intersect it with a small sphere S^3 of dimension 3 centered at the singular point. The intersection you obtain is a curve of real dimension one. And so it is a knot $K\subset S^3$ in the sphere, *the knot of the singularity.* For the particular example of $\{x^2+y^3=0\}$ it is easy to see that you get just the trefoil knot.

For a knot there is an invariant, called the *Gordian number*, which is the number of necessary crossings of the discriminant during the unknotting. In the space of all loops consider the discriminant formed by the singular knots (loops which are not embeddings). The Gordian number measures the distance between your knot and the unknot. It is the minimal number of crossings of the discriminant which are necessary to unknot the given knot. In the case of the trefoil knot this number is equal to one. Everyone knows that by one crossing you can unknot the trefoil knot. We have thus defined the topological Gordian number of the knot.

On the other side, there is an algebraic way to unknot an algebraic knot. This is defined by the following construction. Consider a map from t-axis to $\mathbb{C}^2$ parametrizing the curve $\{f(x,y)=0\}$. In our example $x=t^3$, $y=t^2$. Deform slightly this map. In the example you can add εt to x: $x=t^3+\varepsilon t$, $y=t^2$. Then you get a curve which is a deformation of the initial one. This curve might have double points. Define Milnor's number δ as the number of double points of this curve. Since we are working in the complex domain, for almost all deformations you will get the same number. There are algebraic formulas for δ in terms of the singularity ideal but I shall not use them here. From this construction you can get an unknotting of the initial knot. For each double point you have to cross the discriminant and then unknot your knot. And so there is an algebraic unknotting procedure with this algebraic number of crossings.

The Milnor conjecture was that *there are no simpler topological unknottings.* That is the Gordian number of an algebraic knot is equal to δ. You cannot unknot the knot faster even if you use arbitrary constructions of unknotting. The simplest algebraic construction to unknot your knot gives you the correct answer for the minimal crossing number. And this is also now proved by Kronheimer and Mrowka.

Of course, there are many other examples of this principle. Maybe the simplest reason for events of this kind is the following fact. A polynomial of degree n has exactly n roots. You can think on this in a more general setting. In a compact complex manifold, something like $\mathbb{C}P^2$ or $\mathbb{C}P^m$, there is the intersection theory. For given homology classes x and y you can consider their *intersection number* $a=x\cdot y$ which is an integer. Suppose you have some algebraic realizations of these classes, for instance, complex algebraic curves or something like this. Then you count the intersection number and get a intersection points. If you fix the homology classes and try to perturb your algebraic representatives by some smooth deformations then, of course, you can get more intersection points. But you cannot get less points because in the complex case every intersection always contributes plus one to the total intersection number, because complex varieties are oriented. So a is the sum of those plus ones. And if you perturb the realizations in the smooth category you get some pluses and minuses. But their sum remains the same. Therefore, the number of summands can increase but it cannot decrease. In the case of polynomials the representatives of our classes are the graph of the polynomial and the axis x of its variable in the plane. To preserve the topology of our classes under the deformation means to preserve the boundary condition at infinity. With such a deformation you can, of course, get more roots — but you cannot get less because of the intersection number with the axis x. So all n roots of a polynomial of degree n are really needed for topological reasons. If you replace the polynomial by another mapping of S^2 to S^2 which fixes the point at infinity, then the number of preimages of a generic point may be larger than the degree n of this mapping but it cannot be smaller. This is also the simplest manifestation of the same principle.

Now I shall give you more examples of this principle that are less trivial. Consider the following problem from the theory of wave propagation. Suppose you have a wave propagating from a circle in the plane inside it with velocity one. Then at some moment of time, which is equal of the radius of the circle, the wave collapses to a point. And after that it reappears on the other side of the circle. So you get an *eversion* of the circle. Of course you can do the eversion of a spherical front in any space $\mathbb{R}^n$ (in our case $n = 2$).

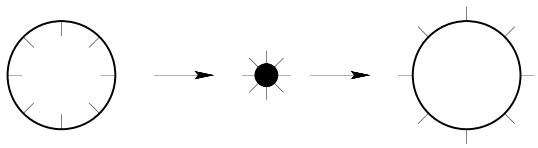


Figure 1: A spherical front eversion

The collapsing point is a very exceptional case because we know that in general the singularities of wave fronts are classified by the Coxeter groups. In particular, in the plane case they should normally have cusps but not isolated points. So we have an extremely nongeneric situation. Replace the sphere by a generic front, say deform it slightly to make it elliptical. Consider the wave propagating inside it, i. e. a family of equidistant curves of the initial ellipse. In the beginning of the process we have a deformation of what we had initially, therefore, according to the Hamilton–Jacobi theory, in the end we will also get a deformation of what we had initially, namely, a smooth wave front propagating outside. But what would be in the middle?

Generically you have to go through wave fronts that have no singularities other than the simplest one. In the plane case these singularities are semicubical cusps and, at some exceptional moments, the singularity of order 3/4 which corresponds to the birth or the death of two cusps at the same point. Studying this for the case of an ellipse you get a movie of what is happening.

Initially the smooth front is propagating inside. Then (at the same moment, because of the symmetry of the ellipse) you observe the births of a pair of cusps at two places. The front continues to propagate inside to become a front without self-intersections. After that, two new triangles appear. Then both triangles die at some moment and finally you get the front propagating outside. This is a generic eversion which happens if you start with an ellipse.

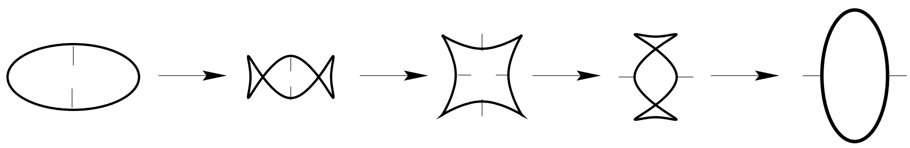


Figure 2: Equidistant curves of an ellipse

Now the deformation of a circle to an ellipse is certainly the simplest possible algebraic way to deform the circle. Hence, by the general principle we get the following conclusion. *Whatever happened in the propagation process above is topologically necessary*. The eversion cannot be simpler. This is a conjecture for which I have no proof and which implies a lot of interesting theorems. But the conjecture still has not been proved in a general form.

I will explain what does the eversion mean in general.

In contact geometry, there is a mathematical model of wave propagation defined by Hamilton-Jacobi type equations in all possible media. We consider the space $ST^*\mathbb{R}^2$ of cooriented contact elements of $\mathbb{R}^2$. A point in this space is a tangent line at a point of $\mathbb{R}^2$ equipped with a unit normal vector. So topologically this space is the solid torus $ST^*\mathbb{R}^2 \cong S^1 \times \mathbb{R}^2$ with a natural contact structure, which is given by the following *skate condition*.

The contact plane consists of all infinitesimal rotations of a contact element at a fixed point and all infinitesimal motions of the contact element in its own direction. It cannot move in the transversal direction. You can introduce the following coordinates in $ST^*\mathbb{R}^2$: x, y in the plane $\mathbb{R}^2$ and φ for the angle of the coorienting normal. The contact plane of the contact structure in the (x, y, φ)-space is defined by the equation $(\cos\varphi)dx + (\sin\varphi)dy = 0$.

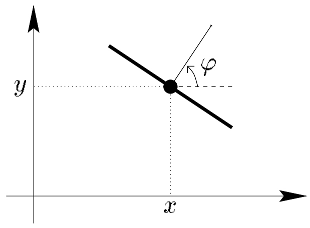


Figure 3: A cooriented contact element of the plane

A *wave front* is the image under a natural projection $S^1 \times \mathbb{R}^2 \to \mathbb{R}^2$ of a *Legendrian curve* which is an immersed integral curve of the distribution of planes of our contact structure. The front below may have cusps, but the corresponding Legendrian curve upstairs is always a smooth curve. It has no singularities. All singularities of the front are singularities of the projection $ST^*\mathbb{R}^2 \to \mathbb{R}^2$ (forgetting the direction of the contact element). Generically these singularities are semicubical cusps. When the front moves it might experience perestroikas which are the birth or the

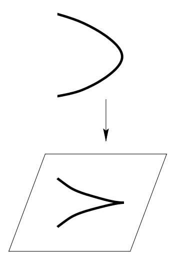


Figure 4: A cusped front of a smooth Legendrian curve

death of a pair of cusps. It follows from the Huygens principle that during these perestroikas the knot type of the corresponding Legendrian curves cannot change. The Legendrian curve is, by definition, an immersed curve but in the wave propagation theory it is embedded all the time. For a one-parameter family of immersed curves in three-space it might go through a singular knot at some isolated values of the parameter. In the case of wave propagation such self-crossings never happen for the following reason. In the projection a crossing of the Legendrian curve looks as a self-tangency of the front with coinciding coorientations of both branches. But the Huygens principle says that there is a duality between waves and particles. Wave propagation can be described in terms of rays, defined by the characteristics. An element of the front uniquely defines the corresponding ray. Therefore, if at some moment you have a coinciding coorientation tangency of two branches of the front, then there has always been such a self-tangency. You must have it already in the initial condition — because of the existence of the rays.

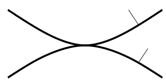


Figure 5: A criminal self-tangency of a cooriented front

In other terms, you can say that all these Legendrian curves are obtained from the initial curve, which was embedded, by a family of diffeomorphisms. This family is the phase flow of the corresponding Hamiltonian equation. Hence, all these Legendrian curves are equivalent knots. The initial Legendrian curve, the final one and all knots in the middle are equivalent. We shall call a self-tangency of a front with coinciding coorientations of the branches a *criminal self-tangency*, since it is forbidden by the Huygens principle. The fronts of propagating waves have no criminal self-tangencies. And now we arrive to the following

Circle Eversion Conjecture. *In any path connecting two Legendrian knots in the space of Legendrian knots (embedded circles), where the first curve is projected onto a circle with the inside coorientation, while the last curve is projected onto a circle with the outside coorientation, there is an intermediate Legendrian knot whose projection is a plane curve with at least four cusps.*

This conjecture is not proved. The following particular case of the eversion conjecture might be easier (but is perhaps as difficult as the eversion conjecture itself).

Any path, connecting the curves (1) and (2) in the space of plane curves having two semicubical cusps and otherwise immersed contains an intermediate curve with a criminal self-tangency.

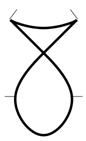
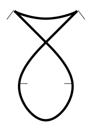
(1)  (2) 

Figure 6: The fish-like curve eversion problem

An example of a connecting path is

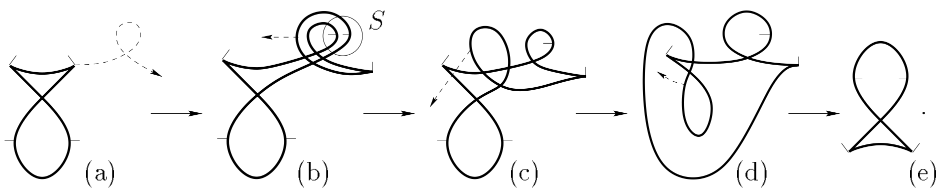


Figure 7: The fish-like curve eversion with two criminal self-tangencies

There is a criminal self-tangency in the region S between the moments (b) and (c). It is easy to see that the number of criminal self-tangencies on any generic path connecting the curves (1) and (2) is even. But it is not proved yet that it is nonzero, and this statement is very close to the eversion conjecture.

I shall describe some other consequences of the eversion conjecture. Some of these consequences are proved. Those are the theorems I shall discuss.

The four cusps first appeared in the following statement, which I call *The Last Geometric Theorem of Jacobi*. It is published in his posthumous "Lectures on Dynamics". The last volume of Jacobi's "Collected Works" contains one more, previously unpublished, paper on this subject.

Consider a convex surface in the Euclidean space $\mathbb{R}^3$, say, a sphere or an ellipsoid, and a point N on it, say, the North pole. Consider a geodesic line g starting from N. An infinitely close geodesic line g' starting at N intersect g at some point P called the *conjugate point* to N along the geodesic line g. For the sphere this point is the South pole. The conjugate points along all the geodesic lines issued from a point on a generic surface (say, on an ellipsoid) form a curve. This curve is called the *caustic* C of the initial point.

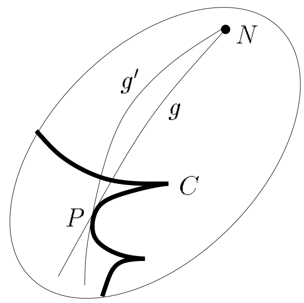


Figure 8: The caustic C of the point N on a surface

In fact, there is an infinite sequence of caustics: the first caustic is formed by the first conjugate points, the second — by the second conjugate points and so on. The first caustic of the North pole of the sphere is the South pole, the second — the North pole and so on. But after a generic small perturbation of the sphere the consequtive caustics will become small curves near the South pole, near the North pole and so on.

Jacobi observed that the caustic cannot be smooth — it must have singular points. Geometrically they are semicubical cusps. The number of cusps on each caustic is even (this was proved by Jacobi using topological arguments; this work of Jacobi is perhaps the first work on global variation calculus and one of the first works in global analysis in general).

At the end of his lecture on caustics Jacobi remarks that for the simplest perturbation of the sphere (making it an ellipsoid) the number of cusps is 4 (rather than 2 as one might expect). He also states that in the case of an ellipsoid a caustic has always four cusps.

I do not know whether this statement of Jacobi is true or not. It is a challenge both for the algebraic geometry and for the scientific computing. Indeed, Jacobi solved the equations of geodesic lines on ellipsoids (in theta-functions of two variables). Hence, the problem of the number of cusps on the caustics is a problem in algebraic geometry.

However this real problem is too difficult for the algebraic geometers, who are not interested in real problems, and thus the "Last Geometric Jacobi Statement" is rather a conjecture than a theorem.

It is known that the first caustic has at least four cusps (for any convex surface). The statement that the second, third and so on caustics have at least four cusps each is very close to the circle eversion conjecture above. Formulating these different geometric conjectures in terms of symplectic or contact topology (instead of the Riemannian geometry) we get essentially equivalent problems: all the occurences of the number four in the answers of these different geometrical problems are all manifestations of the same phenomenon in symplectic or contact topology.

Conjecturally every caustic of a generic point on a generic convex surface has at least four cusps, but it is unproved even for the surfaces close to a sphere. More precisely, for a given n the n-th caustic of a point has at least four cusps provided that the perturbation is sufficiently small (smaller than some $\varepsilon(n) > 0$) in the smooth metric. But whether it holds simultaneously for all the values of n provided that the perturbation ε is sufficiently small is still a conjecture.

The number four in the conjectures above is closely related to the four-vertex theorem in classical differential geometry and to the number four in the Sturm theorem that I shall decribe. This theorem is the infinitesimal version of the conjectures in symplectic and contact geometry whose global version ipmlies the preceeding geometrical corollaries. I think that this Sturm theorem is less known than it should be (it has been discovered independently at least four times with intervals of about fifty years by different people). It is a theorem about Fourier series and provides one of the manifestations of the general principle of economy in algebraic geometry.

But first I shall tell you about a corollary which is a part of this theorem and which is proved. Consider the above described circle eversion as a cylinder in the space $ST^*\mathbb{R}^2$. Now perturb the cylinder slightly in the class of cylinders consisting of Legendrian curves. Projecting these curves to the plane you will get a family of fronts which does not contain an isolated point as a front (as it was the case for the standard circle eversion). The statement is that some intermediate front necessarily has at least four cusps. There is a product structure in the neighborhood of the initial cylinder. And whatever you choose for a family of Legendrian curves in this neighborhood which is a section of the bundle of the normals to the cylinder you always have a projected front with at least four cusps. This is implied by the Sturm theorem that I shall formulate.

There is no evidence that the corollary becomes false if you have a larger deviation when the cylinder is no longer a section. In fact, this Sturm theory is an extension of Morse theory to higher derivatives. In Morse theory you have a cotangent bundle space and you have a section determined by the gradient of the function $p = df(q)$. The ordinary Morse theory is related to the critical points where $p = 0$. But in some cases the function f might be replaced by a multivalued function. The corresponding Lagrangian manifolds would be replaced by Lagrangian manifolds which are no longer sections of the bundle. Hence, Morse theory can be replaced by its version which I have suggested in 1965 and which is called the Arnold conjecture. It is related to the symplectification of topology.

These conjectures imply generalized Morse inequalities. So the Lagrangian intersection theory is essentially the set of Morse inequalities for multivalued functions. In the case of higher derivatives one should substitute Sturm theory for Morse theory. But it is only available for well defined functions. There is no Sturm theory for multivalued functions. This is the reason why you cannot solve the problem of eversion. The problem is essentially a particular case of the general theory of Sturm for multivalued functions for which we have no proofs in general. We have only some examples.

Now what is the Sturm theorem about? The paper by Sturm on Fourier series was published in the very first issue of Liouville's journal. He has considered the algebraic case, the case of trigonometric polynomials. But I shall formulate it in a more general form. In fact, there are theorems of this kind associated to any convex curve in higher dimensional space. But in the original Sturm theorem you use one particular trigonometric curve. Otherwise you get a generalization of the theorem to other curves or in other terms to other disconjugate linear differential equations. But I shall formulate the simplest case. Sturm theorem is related to the conjecture on the one-dimensional front eversion. A similar conjecture in higher dimensions was suggested by V. Zakalyukin (who is present here). If you start, instead of an ellipse in the plane, from an ellipsoid in three-space you also can study the fronts (the equidistant surfaces) and see what is happening. Indeed, this had been done by Arthur Cayley but I shall not describe the answer since it is too complicated. Zakalyukin's conjecture is that whatever you see in this example is topologically necessary. You have, for instance, three cuspidal edges and four ombilical points corresponding to D_4^- singularities related to the Coxeter–Dynkin diagram D_4. All these are topologically necessary.

A more general formulation is that if you start from a generic ellipsoid in a higher dimensional space, then whatever you see in the wave fronts eversion for the ellipsoid is necessary for topological reasons. You cannot avoid this in the same sense as I have explained in the simplest version. But the difference here is that in the simplest case we had this Sturm theory. And Sturm theory gives us the proof of the conjecture while the perturbations are small. In higher dimensional case even for infinitesimal perturbations of the standard sphere eversion we have no proof of the fact that what you see in the standard ellipsoidal eversion is topologically necessary. Even for infinitesimal or finite perturbations (such that the cylinder is still a section of the bundle) we have no theorems. Even the Sturm theory is missing in higher dimensions. This is an interesting phenomenon. All the attempts I know to extend Sturm theory to higher dimensions failed. For instance, you can find such an attempt in the Courant–Hilbert book, in chapter 6, but it is wrong. The topological theorems about zeroes of linear combinations of eigenfunctions for higher dimensions, which are attributed there to Herman, are wrong even for the standard spherical Laplacian.

I return now to the simplest case of Sturm theory. Consider a Fourier series

$$f(q) = \sum_{k \geq N} (a_k \cos kq + b_k \sin kq) \ .$$

I denoted the argument by q because of the traditions of classical mechanics. It is a point of the configuration space which is a circle: $q \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Note that in the Sturm theorem you start from high order harmonics, from the harmonics of order N. Now we can guess the statement from the general principle. It says that a topological object related to the situation attains its minimal value at the simplest algebraic realization of the situation. What is the simplest algebraic realization of such a function? Of course, it is just one harmonic. Moreover, sin and cos behave the same way, and you can shift from one to another. So the simplest function is $\cos Nq$. It is certainly algebraic because you can parametrize the circle in such a way that $\cos q$ and $\sin q$ will be Cartesian coordinates on the algebraic curve. According to the general principle *the number of zeroes of f is minorated by the number of zeroes of the simplest function*. This is the Sturm theorem:

$$\#\{q \in S^1 : f(q) = 0\} \geq 2N \ .$$

In fact, Sturm has proved this for trigonometric polynomials only. In 1903 Hurwitz has proved it in the general case. Then it has been rediscovered by Kellogg, by Tabachnikov and so on. I have learned it from Sergei Tabachnikov, a former student of D. Fuchs who has applied it to the four vertex theorem. But in fact, its relation to the four vertex problem was quoted already by Blaschke.

There are many proofs of this Sturm theorem but all of them are incomprehensible. Of course, I can reproduce them but you get no intuition from those proofs. One of them is by the argument principle in complex variables. You can write the statement for complex functions instead of the real ones and use the argument principle.

Another proof is by the heat equation. You can start with this function as an initial condition for the heat equation. Then the harmonics will die one after another. But certainly the lowest harmonic will die the last. So for large time you will see practically only the lowest harmonic, all the remaining harmonics will be very small. The lowest harmonic has $2N$ roots. These roots are simple, so the other small harmonics do not change the number of roots. Therefore at that time moment you will have exactly $2N$ roots. But it is easy to derive from the maximum principle that during the evolution defined by the heat equation you never create new roots. Hence, the initial number of roots can be only larger.

There is a proof by orthogonality arguments which is really a Sturm type proof. It is based on the fact that the function is orthogonal to trigonometric polynomials of degree smaller than N. So if f has fewer roots you would be able to construct the trigonometric polynomial with the same sign as f but orthogonal to it. This is, of course, impossible.

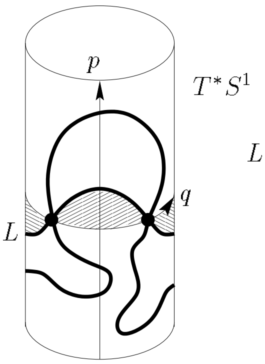
We return once again to the eversion problem. If you write the equation of the cusp points, then from the perturbation theory technique you will get the equation of the form $f(q) = 0$, where the function f is periodic (it is a function on the circle parametrizing our front). Moreover, the function f satisfies the orthogonality property with $N = 2$. Namely, if g is the function which defines the perturbation then the equation of the cusp can be written as $(\partial^3 + \partial)g = 0$. Denoting by f the left hand side function you immediately see that it is orthogonal to 1, to $\sin q$ and to $\cos q$. This is because expanding g in a Fourier series you see that ∂ kills the constant and $\partial^2 + 1$ kills $\sin q$ and $\cos q$. So f starts with the second harmonic. Then applying the Sturm theory you will get at least four cusps. This is a rather strange proof for a topological theorem and you need some technique from contact and symplectic geometry to prove all details. I shall not explain these details. You can find them in my articles.

I shall now speak about the relation of this Sturm theorem to Morse theory. Consider the case $N = 1$. The Sturm theorem says that if there is no constant term in the Fourier series of f, then there f has at least two roots:

$$(\bar{f} = 0) \Longrightarrow (\#\{q : f(q) = 0\} \geq 2) \ ,$$

where $\bar{f}$ is the mean value of f. This is, in fact, the Morse inequality since the condition $\bar{f} = 0$ implies that our function f is a derivative $f = g'$ of another function. And the zeroes of the derivative are the critical points of g. Hence, the statement is that a function on a circle has at least two critical points. And this is the Morse inequality.

In this case we can formulate the Morse inequality geometrically and we can prove it geometrically. Consider a circle S^1 with coordinate q and its cotangent bundle space T^*S^1 which is a cylinder. The initial function f gives you a section $L = \{p = f(q) = \frac{\partial g}{\partial q}\}$. It is called *the exact Lagrangian manifold* associated to our function g which is called *the potential* or *the generating function* for the Lagrangian manifold. Then the condition that p is the derivative of a function can be written in the form $\oint pdq = 0$, where the integral is taken along the section L. Geometrically this means that the areas above and below of the zero section of the cotangent bundle $B := \{p = 0\}$ are equal.



$$L := \left\{p = f(q) = \frac{\partial g}{\partial q}\right\}$$

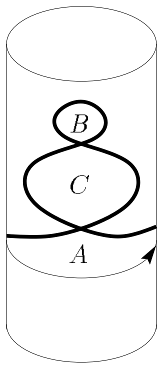
Figure 9: Morse inequality and Lagrangian intersection

Since the areas are equal you need two intersection points. If you have less than two intersection points then the graph will be higher or lower than you need. This implies the Morse inequalities which guarantee the existence of those two points. From these geometrical arguments you can immediately see that you do not need L to be a section. You can perturb it. And even if it is no longer a section you still need two intersection points since the area arguments are there until you have a self-intersection of L. So if a Lagrangian submanifold is exact, i.e. the integral of pdq is vanishing, and you can obtain it from the zero section of the cotangent bundle by a deformation in the space of embedded exact Lagrangian submanifolds, then the number of intersections can be minorated by the sum of Betti numbers

$$\#\{L \cap B\} \geq \sum b_i(B) \ ,$$

which is equal to two in our case $B = S^1$. Motivated by this, in 1965 I have formulated a conjecture that it is true for any dimension. The paper also contains a counterexample showing that for immersions the fact is not true. There is an exact Lagrangian immersion such that the areas A, B, C satisfy the condition

$A + B = C$, the integral of pdq is zero, but you have no intersections with the submanifold $\{p = 0\}$:



$A + B = C$

Figure 10: An exact Lagrangian immersion violating the Morse inequality

So if you admit self-intersections you no longer have the Lagrangian intersection theory. This was the starting point for a long series of works of many people that are mentioned in the second lecture. The most known of them is the Floer theory which was invented to prove that conjecture and its counterpart for fixed points of diffeomorphisms. By the way, I do not know today's situation: is the conjecture finally proved or not. Last year there were many people claiming that they have proved it but I really do not know if it is true.

Voice from the audience: It is proved for rational coefficients.

OK! I am very glad to know this. I hope that this conjecture is true. I recall that the minorations in my conjectures were by the numbers of critical points of (generic in one case and arbitrary in the other) functions on the manifold rather than by the sum of the Betti numbers or by the category. I wish to add something to this conjecture. The counterexample shows that it cannot be true if you replace a Lagrangian embedding by a Lagrangian immersion. However, Yu. Chekanov, who is now sitting here but at that time was an undergraduate student of Moscow State University, had observed the following. If you insist that the Legendrian knot corresponding to this Lagrangian submanifold L is unchanged during the deformation of the zero section then the inequality persists. This is so called Legendrian Morse theory by Chekanov which was announced in 1985 and now seems to be published (even with the proofs).

In this Morse theory we have the first order operator $f = g'$ while in our eversion problem instead of this operator ∂ we need the operator $\partial^3 + \partial$ containing the third derivative and in the general theory — higher order derivatives. So the theorem of Sturm is an extension of Morse inequality for the function on a circle to the operators of higher order replacing the first derivative. It is a non-holonomic version of the Morse inequalities because here we use a higher order operator to define the generalized critical points. So the non-holonomic version of this theory of Lagrangian embeddings and Legendrian knots should contain a proof of the eversion conjecture. I was not able to construct the proof because I have to rely on this Sturm theorem which is proved only for the sections. There is no version of it for manifolds which are no longer sections. However the attempts to find it provide some interesting conjectures and even theorems.

I shall formulate one of them. It does not solve our problem but it is a nice theorem. This is the so called *tennis ball theorem*. Consider a sphere S^2 and a curve on it which divides the whole area into two equal parts A and B. We call such a curve *an exact curve*.

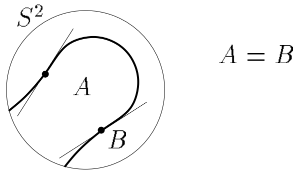


Figure 11: The tennis ball theorem for exact Lagrangian embeddings

Then the statement is that *an exact curve necessarily has at least four inflection points*.

On a tennis ball you can see such a curve and can easily detect the four inflection points. Now start with the sphere and its equator. Perturb it slightly to make it generic. Then you calculate the equation for inflection points and apply the Sturm theorem. You will get this number four. But in the theorem the curve does not need to be a section, it is OK as long as there are no self-intersections. If there are self-intersections there exists a counterexample. It is the same as in my 1965 paper.

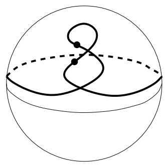


Figure 12: An exact Lagrangian immersion violating the tennis ball property

Here you have only two inflection points. But this exact curve has self-intersection points.

Conjecture. *If during the deformation the corresponding Legendrian knot remains unchanged (this means that there is no loop of this curve whose area equals one half of the area of the sphere), then there are at least four inflection points.*

This is not proved yet. Probably, the new theory of Legendrian contact homology might help.

Now I shall show you one more example where it is easy to understand the statements and still the theorems are difficult and the conjectures remain unproved. The following problem was first considered by Möbius. I think he has invented the Möbius band just solving this problem. Consider the projective plane $\mathbb{R}P^2$ and a projective line $\mathbb{R}P^1$ in it. Of course, the projective line is an extremely degenerate curve. All its point are inflection points. So perturb it a little to make it generic. The Möbius theorem is that *the perturbed curve has at least three inflection points*. The nonorientability of the Möbius band implies that the number is odd. The theorem claims that it cannot be one.

This theorem is a new manifestation of the economy principle. Indeed, the simplest algebraic perturbation of a straight line is a cubical curve. A generic cubical curve has three real inflection points. By the economy principle any perturbed curve should have at least three of them.

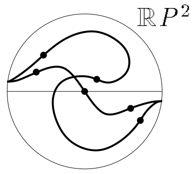


Figure 13: The 3 inflection points on a noncontractable circle embedded into the projective plane

For the infinitesimal perturbations it is just a Sturm theorem. You have to think how to apply the Sturm theorem to obtain an odd number while in the Sturm theorem the number of zeroes is always even. Of course, you have to use the double covering of the Möbius band to have a function. So you will have six points from the Sturm theorem which provide the three points you need.

Now if you work more you can see that you do not need your curve to be a section. Whenever you perturb $\mathbb{R}P^1$ in such a way that it is embedded you always have at least three inflection points. This is the Möbius theorem. I do not know who was the first who proved it but in any case it is true. Now if you perturb the curve more, then you can get a curve with just one inflection point which is in the same class of immersions.

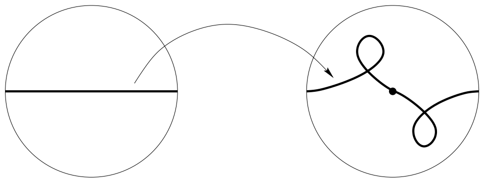


Figure 14: The immersed circle with only one inflection point

You can deform the initial curve $\mathbb{R}P^1$ to this curve by a regular homotopy.

Conjecture. *A regular homotopy between a generic embedded curve and an immersed curve with one inflection point is impossible unless one goes through a change of the knot, i.e. through an oriented self-tangency.*

This conjecture is not proved. An undergraduate student from Moscow Dima Panov has recently proved that it is true if you do not create more than seven inflections at any intermediate moment. So it is impossible to kill two of the three initial inflections unless you go through an oriented self-tangency but it might be possible if you first create more inflection points and then kill them. But if you create at most seven inflections, then you never can get only one inflection point at the end. I believe that the conjecture is true in general but it is still a challenge for undergraduates.

Now I shall formulate a conjectural higher dimensional version of the Sturm theory based on the same general topological economy principle. Consider the Möbius problem in higher dimension. We start from the plane $\mathbb{R}P^2$ in the space $\mathbb{R}P^3$. It is of course highly degenerate, all the points are flattening points. Deform it slightly. You get a surface in the projective space. Problems on flattening topology look like problems of projective geometry but in fact they belong to symplectic and contact topology (it would take a long time to explain why). So I formulate instead of general symplectic and contact topology conjectures just the corresponding geometric statements. In the case of surfaces instead of inflection points you have a classification of points on your surface into *elliptic points* (E), *hyperbolic points* (H) and *parabolic points* (P). The type of a point depends on the disposition of the surface with respect to the tangent plane at the point. You usually learn in courses of differential geometry that the type is defined by the second quadratic form which depends on the metric. But the metric has nothing to do with these types. The type of a point is a projective property and the metric is not preserved by projective transformations. Whether the second differential of the function measuring the deviation of the surface from its tangent plane is a positive (negative) defined, or a hyperbolic, or a degenerate quadratic form is independent of the metric. Parabolic points form lines P. A natural analog of the problem of inflections is the problem of parabolic lines.

In the hyperbolic domain the surface is equipped with two fields of asymptotic directions (where the quadratic form vanishes). At the parabolic points the two asymptotic directions coincide, forming a line field along the parabolic curves. At some isolated points of the parabolic curve the asymptotic direction is tangent to the parabolic curve. Such points are called *special*. We wish to minorate the number of parabolic curves and of their special points.

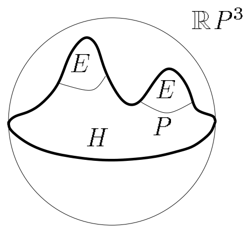


Figure 15: The parabolic curve separating the elliptic and hyperbolic domains on a surface in the projective space

We can apply the general economy principle. The general principle says that you have to try to find the simplest algebraic objects and then to count the number of parabolic lines and of special points on the simplest algebraic perturbation of the plane. So we have to consider an algebraic perturbed surface. The simplest algebraic surface, which is a perturbation of the plane, has degree three. Parabolic curves and special points on such surfaces have been studied in 1942 by Bengiamino Segre (who has also studied differential geometry of all possible real projective cubical surfaces). He has calculated the number of parabolic curves on the cubical surfaces diffeomorphic to $\mathbb{R}P^2$ and this number is equal to four, the number of special points being equal to 6. We arrive at the following conjecture.

Conjecture. *Whenever you perturb a projective plane in the projective space you get a surface with at least four parabolic curves and at least six special points.*

This conjecture has been formulated by F. Aicardi from Trieste. She was trying to find a surface with less parabolic curves using computers. After many hours of experiments, adding higher terms and so on she was not able to get a surface with less than four parabolic curves. To study this problem it is natural to use the following strategy. You construct some special perturbations. Say, start with a triangulation of the projective plane. Then, by hand, make a polyhedron from it and then smooth it out. Or you can do other things using PDEs and fundamental solutions of the heat equation and so on. Whenever you do you will obtain some special class of perturbations. For a special class of perturbations you can control the number of parabolic curves. So you get a combinatorial information on your method. I have tried a dozen of methods of this kind and every time I have obtained the result that the conjecture is true. For different constructive methods I obtained at least four parabolic curves by some strange reasons, different for different perturbations. It would be interesting to study what is a polyhedral version of the parabolic curves and special points and to minorate their numbers. But I have neither a good definition of parabolic curves nor a good formulation of the conjecture for polyhedra[1].

Thanks for your attention.

Question. Is there a theory of inflections and flattenings for curves with cusps and generic wave front surfaces?

Answer. I think the cusps are acceptable. There are many theorems where it is the case. For instance, there is a nice problem which looks like a problem in Euclidean geometry but which is in fact a problem in contact and symplectic geometry. That is the minoration of the number of diameters. Consider a surface, say a torus, in the Euclidean 3-space $\mathbb{R}^3$. *A diameter* is a chord which is orthogonal to the surface at both ends. What is the minimal number of diameters, say for an embedded torus?

The Morse theoretical study of this problem by F. Takens and J. White provides the answer that the number is at least eight. But if you look at this problem from the point of view of contact and symplectic geometry, then you can find that the number is at least ten. This was proved by a graduate student from Moscow P. Pushkar (present here). And in his theorem the torus does not need to be embedded. You might deform the torus in the space of fronts admitting cuspidal edges, swallow tails and whatever you wish but assuming that during the deformation you never change the Legendrian knot type. The really important point is not to change the Legendrian knot type. There are several examples of this kind so I think in the cases of Sturm and Möbius theorems you also can extend the results to the fronts with generic singularities in the projective spaces. But I am not formulating French style conjectures, I am formulating Russian style conjectures, that is conjectures about the simplest possible case.

Question. Do you have an explanation why Russian undergraduates are so brilliant?

Answer. First about the fact itself. Recently an Israeli mathematician told me that he had read in a review on some graduate student written by his professor in the United States: "He is the best student I have ever had. But you have not to evaluate him too much, since he had done his undergraduate studies in Moscow". I was already shocked by this discrimination when the professor — the greatest American combinatorialist I knew, able to find combinatorial problems even in the Plutarkh's "Table Talks" — had told me this story a month earlier at Berkeley. The name of the student is Sasha Postnikov. In Moscow he was a student of A. Khovansky. He was really brilliant already there in Moscow. In the Russian system mathematical researches start very early. We have an enormous pyramid and you see just the top. The distance between a teacher and a student is usually less than five years of age, undergraduate students are teaching highschool

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[1] In September 1997 D.Panov succeeded to construct examples of generic small perturbations of $\mathbb{R}P^2$ having only one parabolic curve. In his examples there are 12 special points. It is still possible that, say the perturbations with 6 special points should have at least 4 parabolic curves (the Aicardi – Panov conjecture).

children, highschool children are teaching kindergarten children and so on. Of course with such a wide culture you get brilliant students at the top. Also they have an experience that their teachers have started young. They are not afraid to start to think on difficult problems when they are undergraduate and even when they are in the highschool. And many of them have already published papers coming to the university as undergraduate students.

Maybe another reason is that they have wider mathematical interests and knowledge than the western students of the same age. In Russia you cannot say that someone is a good mathematician if he is an expert only in mathematics modulo five and does not know much in mathematics modulo seven.

While in the West, according to my experience, it is dangerous to say in a reference letter (for the tenure, or for the thesis, or for a graduate student, but especially for an undergraduate student) that the person have worked in several domains. This is the worst thing you can do to a student.

Question. Should the curve in the conjecture on the three inflection points be an one-sided curve?

Answer. Yes. The initial curve should be homotopical to the straight line $\mathbb{R}P^1$ in $\mathbb{R}P^2$ and the corresponding Legendrian curve (formed by the orienting unit tangent vectors in $ST\mathbb{R}P^2$ should be in the same knot type as the Legendrian curve constructed from the straight line. These are the conditions implying (conjecturally) the existence of at least three inflection points.