Topological problems of the theory of asymptotic curves

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to S.P. Novikov admiringly

I. Introduction.

An asymptotic curve on a surface in the euclidean or projective 3-space is an integral curve of the field of asymptotic directions (directions along which the second quadratic form vanishes). It is proved below that (typical) asymptotic curves on hyperbolic surfaces are exactly the (typical) space curves having no flattening points. These curves might also be defined as the smooth curves whose dual curves are also smooth. I call them rotating curves. A rotating curve may have inflection points (where the curve curvature vanishes).

The rotating curves in the three-dimensional projective space have remarkable topological properties. These properties imply, for instance, that an asymptotic curve of a surface, which is a graph of a smooth function, can't be a star-like curve (this result find by D. Panov was the starting point of the present work). In spite of this, there exist closed asymptotic curves on some hyperbolic surfaces which are graphs of functions of two variables. Some examples are presented below (to construct them is an unexpectedly complicated task). I do not know what is the minimal number of the inflections points on such a curve. (Using the method of the present paper, F.Aicardi has recently constructed curves with only two inflections).

The dynamical systems, defined by the asymptotic direction fields on the (doubled) hyperbolic domain, have not been studied yet, as far as I know. On a cubic surface there are 27 asymptotic straight lines (some of which may be real). But even the integrability of the asymptotic lines equations in the victinity of these lines has not been studied.

All the selfdual smooth curves in the three-dimensional space are rotating curves (the simplest example being the curve $x = t$, $y = t^a$, $z = t^{a+1}$). An immersion of of a circle into a projective space is called selfdual, if the dual immersion sends the circle onto a curve in the dual projective space, which is projectively equivalent to the image of the original immersion. Thus a selfduality defines a diffeomorphism of the immersed circle onto itself. The case where this diffeomorphism (or its power) is the identity mapping deserves a special study.

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The projective topology, that is the study of the discrete invariants of smooth objects (curves, surfaces, ...) in projective spaces, has been founded already by Möbius and Jacobi, but its development was slow and for a long time it was considered as a dull and unfashionable domain of differential geometry. However it has been understood recently that all this domain belongs rather to symplectic and contact topology. The projective topology provides a way to see the simplest manifestations of the complicated symplectic and contact topology phenomena, whose general theory is yet to be constructed.

II. Finite type curves and rotating curves

Consider a curve, smoothly immersed into the euclidean space. The curve is *of finite type* at a point, if some derivatives of the immersion at this point generate all the tangent space at it. The parametric equation of the curve can then be written (near this point) in suitable affine coordinates in $\mathbb{R}^n$ in the form

$$x_k = t^{a_k} + o(t^{a_k}), \quad 1 = a_1 < a_2 < \ldots < a_n \ . \tag{1}$$

The set of the degrees a_k is called the *symbol* of the point. Formula (1) implies that a finite type curve has a continuously rotating osculating flag of osculating spaces of all dimensions.

The property to have a finite type (and also the symbol itself) are invariant under the projective transformations of the space (and under the diffeomorpisms of the axis of the parametre t). Hence the finite type curves, as well as the symbols of their points, are well defined for the curves smoothly immersed into the projective spaces (real or complex).

All the curves, having a point of infinite type, form a codimension infinity set in the space of curves. For example, an analytic curve has a finite type at every point if it does not lie in a hyperplane.

We shall study below the properties of the asymptotic curves on hyperbolic surfaces (surfaces of negative curvature). All such surfaces, having an asymptotic curve carrying an infinite type point, form a codimension infinity set in the space of surfaces.

Hence we shall suppose everywhere below (sometimes implicitly) that all our curves are of finite type.

The *dual curve* of a finite type curve immersed into the projective space lives in the dual projective space and is formed by the osculating hyperplanes of the original curve. The dual curve is not always immersed. Its symbol (at the point corresponding to the point (1)) is dual to the original one :

$$a^\vee \ = \ (a_n - a_{n-1} < \ldots < a_n - a_1 < a_n) \ .$$

For instance, a typical curve, immersed into the 3-space, has symbol $(1 < 2 < 3)$ at the generic points and symbol $(1 < 2 < 4)$ at some isolated *flattening points* (where the curvature remains nonzero, but the torsion vanishes).

The dual curve has the usual symbol $(1 < 2 < 3)$ at the ordinary points, but the flattening points of the original curve correspond to the semicubical *cusps* (with symbol $(2 < 3 < 4)$) of the dual curve.

Definition. A finite type curve in three-dimensional projective space is called *a rotating* *curve* if this curve as well as the dual curve are smoothly immersed (that is, if their symbols at every point have the selfdual form $(1 < a < a+1)$).

The simplest degeneration of the symbol of a rotating curve is not the flattening (which is impossible for a rotating curve) but the inflection (whose symbol $(1 < 3 < 4)$ is selfdual) which is the zero of the curvature and not of the torsion. The set of the curves having an inflection point is a hypersurface (a codimension one set) in the space of all curves. It means that the (simplest) inflection points occur for isolated parametre values in typical one-parametre families of curves. But in the space of the rotating curves the inflection is a common event.

The theorems proved below show that the rotating curves (called this way for the following reason : the helical rotation orientation does not change along such a curve, the accompanying Frenet frame might be continued smoothly across the inflection point, provided that the curvature changes the sign at every generic inflection point) have interesting topological properties and do deserve a special study.

III. Asymptotic curves on hyperbolic surfaces.

We shall show below that the asymptotic curves on the hyperbolic surfaces form the same class of curves as the rotating curves in the ambient projective 3-space.

A *hyperbolic surface* in the real projective three-dimensional space is a surface whose second quadratic form is hyperbolic. At the neighbourhood of any of its point the hyperbolic surface may be defined (in suitable affine coordinates) by the equation $z = xy + \ldots$, where the dots denote the terms of higher order of the Taylor series.

The asymptotic directions of a hyperbolic surface are the directions along which the second quadratic form of the surface vanish. In the above coordinate system the asymptotic directions (at point 0) are the directions of the coordinate axis x and y. The asymptotic directions depend on the projective structure of the space rather than on its metric : they are the directions of the branches of the intersection curve of the surface with the projective plane tangent to it (and are hence invariant under the projective transformations of the space).

An *asymptotic curve* on a hyperbolic surface is a curve whose direction at every point is an asymptotic direction.

We shall see that the asymptotic curves on a hyperbolic surface are essentially those curve, whose osculating planes are tangent to the surface. More precisely, the following simple theorem holds :

Theorem 1. *(i) If the osculating plane of a curve (lying on a hyperbolic surface) is tangent to the surface at some point of the curve, then the direction of the curve at this point is an asymptotic direction.*

(ii) if a nonzero curvature curve, lying on a hyperbolic surface, has an asymptotic direction at a point, then the osculating plane of the surface at this point is tangent to the surface.

(iii) The osculating planes of a finite type asymptotic curve on a hyperbolic surface are everywhere tangent to the surface.

Proof. Choose affine coordinates, in which the surface is defined in a neighbourhood of the point of study by the equation of the form $z = xy + f_3 + \ldots$ (f_3 is the cubical part of the

Taylor series, the higher order terms being denoted by the dots).

(i). A curve of a nonasymptotical direction at the origin has the equation of the form $y = Ax + ...$, $A \neq 0$. Along this curve $z = Ax^2 + ...$, hence the tangent plane of the surface ($z = 0$) is not the osculating plane of the curve (which has with its osculating plane at least three infinitesimaly-close common points).

(ii). A curve of asymptotic direction at the origin, whose space curvature is not vanishing, has the equation of the form $y = Ax^2 + ...$, $A \neq 0$. Along this curve $z = Ax^3 + f_3(x, 0) + ...$ Hence the tangent plane ($z = 0$) of the surface is the osculating plane of the curve.

Remark. An osculating plane of an asymptotic curve of zero curvature may be transversal to the surface.

(iii). The curvature of a finite type curve is different from zero almost everywhere. The osculating plane of a finite type asymptotic curve is therefore tangent to the plane almost everywhere (see (ii)). Both planes — the one tangent to the surface ans the osculating plane of the asymptotic line— depend continuously on the point of the finite type curve. Hence they coincide everywhere and the Theorem is proved.

Theorem 2. *An asymptotic curve of finite type on a hyperbolic surface is a rotating curve.*

Proof. Use the affine coordinates, such that the surface equation takes the form $z = cxy + f(x, y)$ (where $c \neq 0$ and f contains the cubic and higher order terms of the Taylor series). According to Theorem 1, (iii), the asymptotic curve with symbol $(1 < a < b)$ at the origin has the equations

$$y = Ax^a + ..., \quad z = Bx^b + ..., \quad A \neq 0 \ , \quad B \neq 0.$$

Theorem 2 states that $b = a + 1$. We shall now prove it. Consider the derivative $h(x) = dy/dx = Aax^{a-1} + ...$ The asymptotic direction definition

$$F_{xx}(dx)^2 + 2F_{xy}\ dxdy + F_{yy}(dy)^2 \ = \ 0$$

takes for $F = cxy + f$, $dy = h\ dx$, the form

$$f_{xx} + 2(c + f_{xy})h + (f_{yy})h^2 \ = \ 0.$$

The terms of order smaller than a in x (along the curve $y = Ax^a + ...$) are only provided by the terms $f_{xx}(x, 0) + 2ch$. We deduce that $f(x, 0) = -(2cA/(a + 1))\ x^{a+1} + ...$ Hence, along the asymptotic line the variable z has the Taylor series

$$z = (cAx^{a+1} + ...) + f(x, Ax^a + ...) = ((a - 1)/(a + 1))Acx^{a+1} + ... \ ,$$

whence $b = a + 1 \Big(B = ((a - 1)/(a + 1))\ Ac \neq 0 \Big)$, which proves the Theorem.

Remark. Theorem 2 shows, that rotating curves form an interesting space: one should study the topological invariants of this space, the natural strati- fication of its boundary, the corresponding Vassiliev invariants and so on. Indeed, the following simple complement to Theorem 2 holds :

Theorem 3. *Every rotating space curve of finite type is an asymptotic curve on a suitable hyperbolic surface.*

Proof. Choose a straight line, transversal to the tangent line of the curve, in the osculating plane of the curve at each point (it should depend smoothly of the point, it suffice to take the normal line in some metric). The union of these straight lines form a smooth surface in a neighbourhood of the curve. The osculating planes of the curve are tangent to the surface by the construction. We shall prove that the surface thus constructed is hyperbolic. Theorem 1(i) would then imply that the original rotating curve is its asymptotic line.

To prove the hyperbolicity of the surface at a point of the curve choose an affine coordinate system centred at the point, whose x axis is tangent to the curve and whose y axis belongs to the surface, whose equation takes the form

$$z = H(x, y), \quad H = cxy + f(x, y), \quad f = O\Big(|x|^3 + |y|^3\Big) \ .$$

We shall prove that $c \neq 0$ (which means hyperbolicity). Write the equation of the original curve (lying on the surface) in the parametrical form

$$x = t, \quad y = g(t), \quad z = h(t), \quad \text{where } \ h(t) = H(t, g(t)) \ .$$

The tangency condition (of the osculating plane of the curve to the surface) claims that the vector of the second derivative $(0, g^{\cdot\cdot}, h^{\cdot\cdot})$ (generating the osculating plane together with the first derivative vector at a generic point of a finite type curve) should verify the tangent plane equation, $dz = H_x dx + H_y dy$. We get thus the relation

$$h^{\cdot\cdot} \ = \ H_y(t, g(t))\, g^{\cdot\cdot}(t) \tag{2}$$

identically along the curve.

Suppose that point $t = 0$ is of type $(1 < a < a+1)$, whence one has

$$g(t) = At^a + O(t^{a+1}), \ h(t) = Bt^b + O(t^{b+1}), \ b = a+1, \ AB \neq 0.$$

We shall see that these formulae imply $c \neq 0$. From the definitions of g and of H we get the relations

$$g^{\cdot\cdot} = pt^{a-2} + O(t^{a-1}), \quad H_y = cx + O(|x|^2 + |y|^2) = ct + O(t^2)$$

where $p = Aa(a-1)$. For h we get the expression $h(t) = cAt^{a+1} = f(t,0) + O(t^{a+2})$, implying that $h^{\cdot\cdot} = qct^{a-1} + f^{\cdot\cdot}(t,0) + O(t^a)$, where $q = Aa(a+1)$. Finally, we have reduced equation (2) to the form

$$qct^{a-1} + f^{\cdot\cdot}(t,0) + O(t^a) = (ct + O(t^2))(pt^{a-1} + O(t^{a-1})) \ .$$

If c were equal to zero, we would have $f^{\cdot\cdot}(t,0) = O(t^a)$. Since $f = O(t^3)$, this relation would imply that $f(t,0) = O(t^{a+2})$. But then one would have $h(t) = O(t^{a+2})$, and the curve would not be a rotating curve (since for a rotating curve $B \neq 0$).

Thus $c \neq 0$ and hence the surface is hyperbolic. The curve is then asymptotic (by Theorem 1(i)). Theorem 3 is proved.

Remark. If a finite type curve is defined in some affine coordinates by the equations $y = g(x)$, $z = h(x)$, the rotating conditions takes the form $(h''/g'')' \neq 0$ (the ratio of the second derivatives is continued to the point where $g'' = 0$ as a smooth function).

Indeed, choose affine coordinates for which the point we are studying is the origin, the x axing being directed along the curve and the osculating plane at the origin being the plane $z = 0$. This change of variables will transform functions g and h into functions

$$\begin{aligned} \tilde{g}(x) &= g(x_0 + x) - y_0 - v_0 x \;=\; Ax^a + ..., \\ \tilde{h}(x) &= h(x_0 + x) - z_0 - w_0 x - u_0 \tilde{g}(x) \;=\; Bx^{a+1} + ... \,. \end{aligned}$$

Hence the derivatives of the ratios of the second derivaties coincide in the corresponding points :

$$(\tilde{h}''/\tilde{g}'')' \;=\; (h''/g'')' \,.$$

But in the new coordinates we have

$$\tilde{h}''/\tilde{g}'' \;=\; (B(a+1)/(A(a-1)))x + ... \,.$$

Hence function h''/g'' has a smooth continuation to the points where g'' vanishes. The rotation condition, $B \neq 0$ takes hence the form $(h''/g'')' \neq 0$ both at the ordinary points, where $g'' \neq 0$, and at the inflection points, where $g'' = 0$.

Corollary. *An asymptotic line on a periodic surface*

$$z \;=\; F(x,y),\;\; F(x+1,y) \equiv F(x,y)$$

cannot be defined globally by the periodic equation

$$y = g(x),\;\; g(x+1) \equiv g(x) \,.$$

Proof. The ratio h''/g'' is a smooth periodic function along such a curve (it is periodic even at the zeros of the denominator). If its derivative were sign-preserving, it would violate the Morse inequality.

Remark. Strictly speaking, this is only proved for the finite type asymptotic curves. But the type finiteness condition is probably irrelevant, since any neighbourhood of a closed asymptotic curve on a periodic surface contains a finite type periodic asymptotic curve on a neighbouring periodic surface.

IV. The unavoidable bends of asymptotic and rotating curves

The union of the tangent straight lines of a space curve is called its *tangent developable* surface.

Theorem 4. *Suppose that the osculating planes of a closed rotating curve in $\mathbb{R}P^3$ do not contain some point O. Then the tangent developable surface of this curve intersects every straight line containg point O.*

Proof. If a straight line l, containing O, were not intersecting the tangent developable surface, then every osculating plane of the curve would intersects $l \backslash O$. The intersection point would move along l (when the osculation point moves positively along the curve) always in the same direction.
Indeed, the momentary rotation axis of the osculating plane is the tangent line of the original curve. This line does not intersect l. The infinitesimal rotation direction is constant along the rotating curve (by the very definition of the rotating curve). This implies the constancy of the direction of the motion of the intersection point of the osculating plane with l.

But a circle can't be mapped monotonically on the interval $l \backslash O$. This proves the Theorem.

Example. Call point O the *vertical direction infinite point* and call the straight lines containing point O *vertical lines*. The vertical lines project the complement to point O onto the complementary *horizontal plane* $\mathbb{R}P^2$. The Theorem implies the following

Corollary. *Suppose that a closed rotating curve has no vertical osculating planes. Then the tangent lines of the projection of this curve to the horizontal plane cover this projective plane completly.*

Indeed, the tangent lines of the original curve are projected onto the tangent lines of the projected curve. If a point of the horizontal plane were uncovered by the projected tangent lines, the vertical line, connecting this point to O, would not intersect the tangent developable surface, in contradiction with Theorem 4.

Corollary. *A closed asymptotic finite type curve on a hyperbolic surface, immersed into $\mathbb{R}P^3$ and transversal to the straight lines containing point O, is projected from O to $\mathbb{R}P^2$ as a curve, whose projections are covering $\mathbb{R}P^2$.*

Proof. Theorem 2 implies that this curve is rotating and that its osculating planes are tangent to the surface. Since the surface is nowhere vertical, these osculating planes do not contain point O. According to Theorem 4, every vertical straight line intersects one of the tangent lines of the asymptotic curve. The projection of this tangent line contains the point of $\mathbb{R}P^2$ corresponding to the vertical line and is tangent to the projection of the asymptotic curve to the horizontal plane.

Corollary (D. Panov). *The projection of a closed asymptotic line of a surface $z = f(x, y)$ to the plane $\{(x, y)\}$ can not be a star-line curve (in particular, this projection can't be a convex curve).*

Remark. The type finiteness condition is perhaps irrelevant for these corollaries, since one might approximate the given closed asymptotic curve by a finite type asymptotic curve on a neighbouring surface.

V. Examples of closed asymptotic curves on the surfaces z = f(x,y).

To construct such a curve it suffices (according to Theorem 3) to construct a closed rotating curve whose osculating plane is nowhere vertical and whose projection into the horizontal plane is an embedding.

I shall construct first a periodic curve with these properties (it is closed rather in the solid torus $\{(x \bmod 1, y, z)\}$ than in the Euclidean space).

To transform it into a rotating curve in the Euclidean space, we bend slightly a long segment of the periodic curve (say, of length N) to obtain a rotating curve, whose projection to the (x, y)-plane, seen from far, would look like a circular arc of length N with a small central angle $\theta = 2\pi/M$. The union of M such bended arcs forms a closed curve embedded into the plane (looking from far as a circle). This large plane curve is the projection of the required rotating space curve (having no vertical osculating planes) to the horizontal plane.

The periodic rotating curve will be constructed from the elements of two types : the regular helical curves (whose projection to the horizontal plane will be almost circular arcs) and from the small inflection segments, attached to the helical curves at the curvature maxima points on the segments. The smallness of the inflection segments will be controlled by a parametre ϵ. Thus, the projection of the rotating curve to the horizontal plane will consist of the alternating long arcs of approximate circles of radius 1 and short arcs of approximate sinusoids of length of order ϵ. The rotating curve will be constructed as a space perturbation of this horizontal curve (of the form $z = \mu h$, where μ is a small parametre whose value will be choosen after the choice of the small value of ϵ). The space perturbation will shift slightly the horizontal projection of the curve, but this shift will be of order μ^2 (and hence we may neglect it in the calculations of the first order terms in μ of perturbation theory).

I am starting from the general formulae of the space perturbation theory of any plane curve

$$\Big(x,\ y = g(x), \quad z = \mu h(x)\Big), \quad \mu \to 0 \ .$$

Direct calculations lead to the following

Lemma 1. *The curvature and the torsion of an almost plane curve are equal, in the first approximation, to*

$$k = |g''|/v^3 + O(\mu^2),\ \kappa = \mu(h'''g'' - h''g''')(g'')^{-2} + O(\mu^3)\ ,$$

where $v^2 = 1 + (g')^2$.

The Frenet accompagnying frame of an almost plane curve is given by the formulae

$$\tau = (A, B, \mu C), \quad n = s(P, Q, \mu R), \quad b = s(\mu U, \mu V, W)\ ,$$

where the coefficients are given, with an error at most $O(\mu^2)$, by the formulae $s = sign\ (g'')$,

$$\begin{array}{lll} A = 1/v, & B = g'/v, & C = h'/v\ ; \\ P = -g'/v, & Q = 1/v, & R = (h''v^2 - h'g'g''/(|g''|v) \\ U = (h''g' - h'g'')/(g''), & V = -h''/(g''), & W = 1\ . \end{array}$$

<u>Remark</u>. In these formulae $g'' \neq 0$, and the frame (τ, n, b) orients the space as the coordinate system (x, y, z), the normal direction being choosen by the condition that the curvature k should be positive. The Frenet formulae have the usual form

$$\dot\tau = kn, \quad \dot n = -k\tau + \kappa b, \quad \dot b = -\kappa n\ .$$

At the crossing of a simplest inflection point (of type $(1 < 3 < 4)$) the frame jumps, turning at an angle π around the tangent direction τ (this jump is reflected by the appearance of the sign s in the formulae for the unit vectors n and b).

To everse the rotating curve I shall use the following special curve having a point of the ordinary inflection $X = 0$:

$$(3) \qquad (X, \quad Y = \sin X, \quad Z = \xi^2 - \sin^2\xi)\ ,$$

where $\xi = \lambda X, 0 < \lambda < 1, 0 \leq X < \pi/2$.

<u>**Lemma 2**</u> . *This curve is a rotating curve.* At the point $X = 0$ the curve has an ordinary inflection point (described by the symbol $(1 < 3 < 4)$). At the points where $X \neq 0$ the derivative of the ratio of the second derivatives of the components with respect to X does not change the sign :

$$g''/g''' \neq H''/H'''\ ,$$

where $g = \sin X$, $H(X) = \xi^2 - \sin^2\xi$. Indeed,

$$g''/g''' = \text{tg}\ X, \quad H'' = 4\lambda^2 \sin^2\xi, \quad H''' = 4\lambda^3 \sin 2\xi\ ,$$

whence $H''/H''' = (\text{tg}\ \lambda X)/(2\lambda)$. But for the values of X belonging to the interval $(0, \pi/2)$ and for the values of λ , belonging to the interval $0 < \lambda < 1$, we have $\text{tg}\ \lambda X < \lambda\ \text{tg}\ X$. Hence on our interval $\text{tg}\ X > \text{tg}\ (\lambda X)/(2\lambda)$ and therefore curve (3) is a rotating curve.

Now we shall translate curve (3) and shall shrink it to get a curve whose projection on the (x, y) plane is tangent to the circle $x = \sin\varphi$, $y = \cos\varphi$ at the point $\varphi = 0$ having there the same curvature as the circle :

$$(4) \qquad x = \epsilon(X - \pi/2), \quad y = \epsilon^2 Y + 1 - \epsilon^2, \quad z = \mu Z\ .$$

Lemma 1 implies that this translated curve (4) has a curvature maximum point (whose critical value differs at most by $O(\mu^2)$ from 1) in a small neighbourhood of the point $X = \pi/2$ (that is $x = 0$, $y = 1$, $z = 0$), provided that μ is small.

Choose a helical line having at this point the same 3-jet (that is, the same acompagnying frame, curvature and torsion) as the curve (4). To choose such a helical curve is possible just because we have choosen the curvature maximum point : the 3-jet takes also into account the curvature derivative, which is vanishing for the helical curve as well as for curve (4) at the curvature maximum point.

Lemma 1 provides the formulae for the parameters of this helical line in first approximation with respect to μ. For $\mu = 0$ curve (4) is a plane curve. Its curvature centre (at point $(0, 1)$ on the curve) is $(0, 0)$. Therefore the required helical line lies on a cylinder which is close to the cylinder $x^2 + y^2 = 1$ whose axis forms an angle of order μ with the z axis direction and whose radius is close to 1 (its difference from 1 being at most $O(\mu^2)$ according to Lemma 1). The torsion of curve (4) at the connection point is of order μ. Hence the helical line, 3-connected to curve (4) at the curvature maximum point, may be obtained (up to the perturbations of order μ^2) from the circle ($x^2 + y^2 = 1$, $z = 0$) by the following three perturbations of order μ : 1) a vertical shift along the z-axis ; 2) an inclination of the cylinder axis at an angle of order μ to the z-axis ; 3) the replacement of the circle $z = 0$ by the helical curve $z = \kappa\varphi$, the constant κ being proportional to μ and being the torsion of curve (4) at point $x = 0$.

These three operations may be performed independently, and the resulting perturbations of order μ should be simply added to each other (since we neglect the nonlinear effects of order μ^2). The inclination of the cylinder axis transforms the circle into an ellipse, but the shortening of the smaller axis is of the order of μ^2 only. Therefore we may neglect this inclination calculating the corrections of order μ to the coordinates x and y.

It results that the helical line having the third order contact with curve (4) at the curvature maximum point has the equations of the form (neglecting $O(\mu^2)$)

$$x = \sin\varphi, \quad y = \cos\varphi, \quad z = \mu(u_1 + u_2 + u_3) ,$$

where μu_1 — the influence of the shift of the contact point along the z axis — is a constant, independent of φ, μu_2 denotes the effect of the cylinder axis inclination and $\mu u_3 = \kappa\varphi$ being the effect of the torsion of curve (4) at the contact point.

To calculate u_2, write the equation of the plane, containing the origin and inclinated at an angle of order μ to the horizontal plane $z = 0$, in the form $z = \mu(ax + by)$. We obtain, substituing $x = \sin\varphi$ and $y = \cos\varphi$, the perturbation of the z coordinate due to the cylinder axis inclination, to be

$$u_2 = a \sin\varphi + b \cos\varphi$$

(up to the corrections of order μ^2). Taking all the three perturbations into account, we get finally the third order contact conditions in the following form :

<u>Lemma 3</u>. *The helical curve, which has a third order contact with curve (4) at its curvature maximum point, close to ($x = 0$, $y = 1$, $z = 0$), is defined (up to the corrections of order μ^2) by the parametric equations*

$$x = \sin\varphi, \quad y = \cos\varphi, \quad z = \mu F(\varphi) ,$$

where $F(\varphi) = c + a\sin\varphi + b\cos\varphi + d\varphi$, the coefficients being related to the values of function $Z = h(x)$ in (4) and of its derivatives at point $x = 0$ by the formulae

$$c = h_0 + h_0'', \quad a = h_0' - h_0''', \quad b = -h_0'', \quad d = h_0''' .$$

Proof. The values of the coefficients result from the comparision of the Taylor series of $h(x)$ and of F (arc $\sin x$) at point 0 taking into account that $\varphi = x + x^3/6 + ...$).

The normal to the perturbed helical curve is generated by the vector of the second derivatives with respect to φ (up to correction of order μ^2), $(-\sin\varphi,\ -\cos\varphi,\ \mu F^{\cdot\cdot})$. We deduce that on the helical curve there exists a point, where this normal is horizontal (perpendicular to the z-axis). This point is defined (in first approximation) by the equation $F^{\cdot\cdot} = 0$, that is $a\sin\varphi + b\cos\varphi = 0$. Substituting the values of the coefficients a and b from Lemma 3 and returning from function $h(x)$ to function $H(X) = h(\epsilon(X - \pi/2))$, we get the normal horizontality condition in the form

$$\text{tg } \varphi_0 \ = \ -\epsilon h_*'' \left(H_*''' - \epsilon^2 H_*' \right) , \tag{5}$$

where the derivatives are derivatives with respect to the variable X and the stars mean the evaluation at $X = \pi/2$. (The author is greatfull to F. Aicardi who has pointed an error in the original version of formula (5)).

Lemma 4. *Equation $H_*''' = \epsilon^2 H_*'$ has a smooth solution $\lambda = \lambda_*(\epsilon)$, $\lambda_*(0) = 1$, $\lambda_*(\epsilon) < 1$ for small $\epsilon > 0$.*

Indeed, $H_*''' = 4\lambda^3\ \sin(\pi\lambda) > 0$ and $H_*' = \lambda^2\pi - \lambda\sin(\pi\lambda) > 0$, while $(d/d\lambda)(H_*''') < 0$ at $\lambda = 1$.

Thus choosing $\lambda(\epsilon)$ we may get $\varphi_0 = \pi/2$.

The fact we really need is the inequality $\varphi_0 < 5\pi/6$ for sufficiently small values of ϵ. We fix φ_0 and $\lambda(\epsilon)$.

Now we are ready to construct the periodic space curve. We start from a segment Γ of this curve, consisting of curve (4) and of the helical curve third order tangent to it at the curvature maximum point, close to the point $x = 0$. We include only the arc of this helical curve bounded by the point on the horizontal normal N (close to $\varphi = \varphi_0$). The value of the parametre ϵ is choosen (after the choice of λ) to be so small, that the angle φ_0 is close to $\pi/2$. Then the value of parametre μ is choosen so small (even with respect to ϵ^3, in fact) that the error of our first approximation in μ will be small.

The helical curve is strictly symmetric with respect to any of its normals, for instance with respect to N (the reflection A in this normal N maps the helical curve onto itself, reversing its orientation). Consider the binormal B to curve (4) at the inflection point $X = 0$ (this binormal is the vertical line $x = -\epsilon\pi/2$, $y = 1 - \epsilon^2$). Curve (4) ($|X| < \pi/2$) is sent exactly onto itself by the reflection C in the binormal B (reversing the curve orientation). Lines N and B are strictly orthogonal.

Add to the segment Γ the symmetric segments $A\Gamma$ and $C\Gamma$ reflecting Γ in the bounding normal N and binormal B. The segment $A\Gamma$ is bounded by an inflection point with a vertical binormal AB. The reflection C' in this binormal sends the segment $A\Gamma$ onto the segment $C'A\Gamma$, bounded by a point on the normal $C'N$. This normal is horizontal and is a symmetry axis of the helical curve contained in the segment $C'A\Gamma$. The reflection A' in the axis $C'N$ sends the segment $C'A\Gamma$ onto the segment $A'C'A\Gamma$, bounded by a point on the vertical binormal $A'AB$.

The four segments ($\Gamma,$, $A\Gamma$, $C'A\Gamma$, $A'C'A\Gamma$) form together one period of the periodic curve that we are constructing. The minimal period is the translation $A'A$ along the horizontal (x, y)-plane, sending B onto $A'AB$.

The projection of the resulting curve to the horizontal plane is formed by a sequence of arcs, which are close to the arcs of the circles of radius 1, supporting the angles of equal magnitude $2\varphi_0$ and almost tangent to each other at a point, close to the place where the binormal $(B, AB, ...)$ is vertical. More precisely, the almost circular arcs are connected by small inflection segments (of lengths of order ϵ). Our rotating periodic curve is a chain of such tangent arcs, whose projections are convex alternatively toward the directions of positive and of negative x. The minimal period is close to the pair of tangent circular arcs supporting the angles $2\varphi_0$.

If $\varphi_0 < 5\pi/6$, all the circular arcs in such a chain are not intersecting each other. Hence for the sufficiently small values of parametre ϵ the periodic curve that we have constructed has a smoothly embedded projection to the horizontal plane.

Our construction provides a curve of class C^3 (we were controlling only the continuity of the derivatives up to order 3). But a standard smoothening allow us to make our rotating curve as smooth as we wish (even analytic). We conclude therefore that the following results holds :

Theorem 5. *There exists a smooth closed nonselfintersecting curve in the plane (x, y), over which on a suitable hyperbolic surface $z = f(x, y)$ lives a closed asymptotic curve (a limit cycle of the asymptotic directions field of the surface).*

Theorem 6. *The set of the embedded (immersed) rotating curves, equipped with their osculating planes fields, is C^0-dense in the space of the embedded (immersed) smooth curves in $\mathbb{R}P^3$, equipped with continuous fields of tangent planes.*

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