FIRST STEPS OF LOCAL SYMPLECTIC ALGEBRA

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Dedicated to D.B.Fuchs

1. Introduction

The goal of this paper is the solution of a very special problem of symplectic singularity theory – that of the classification of the symplest singularities of curves in a symplectic manifold. The interest of this problem consists in the following observation: there exist nonobvious discrete symplectic invariants of such singularities. These invariants should be expressed in terms of the local algebra's interaction with symplectic structure. The study of this interaction, which I propose to call *local symplectic algebra*, is necessary for the understanding of the results of the present paper, which at present seem to me to be rather misterious.

The classical Darboux–Givental theorem claims, that the germ of a smooth submanifold of a symplectic manifold is defined (up to a symplectomorphism) by the restriction of the symplectic form to the tangent space of the submanifold.

In the case of a smooth curve this restriction vanishes. The results of the present paper suggest that something nontrivial remains from the symplectic structure at the singular points of the curve. It would be interesting to describe this ghost of the symplectic structure in terms of the local algebra of the singularirty. In this paper such a formula is missing: I just provide the classification of the curves with simplest singularities in a symplectic manifold.

A *singularity* of a curve is a germ of a smooth (holomorphic, formal, ...) mapping of a line into a smooth (holomorphic, formal, ...) manifold at a singular point (where the derivative of the mapping vanishes). The singularities are considered up to the diffeomorphisms of the images in the target manifold (that is, up to the right-left equivalence in terms of the singularity theory).

A singularity is callled *simple*, if all the singularites of the neighbouring mappings belong to a finite set of equivalence classes. The classification of simple singularities of curves is described in [1].

The codimension of the set of nonsimple singularities of mappings of a line to $\mathbb{C}^N$ equals to $6N-6$ (nonsimle singularites first occur at some points of some curves of generic families, if the number of the parameters reaches $6N-7$).

The list of the simple singularities of curves starts with the series

$$A_2 \leftarrow A_4 \leftarrow \dots, \qquad A_{2k} = \left(t^2,\ t^{2k+1},\ 0, \dots,\ 0\right).$$

These are all the singularitis, whose Taylor series contains a nonzero quadratic term (neglecting the infinitely degenerate ones). The list of simple singularities of curves contains also all the (finitely degenerate) singularities, whose Taylor expansion starts from the nonzero cubic term (they form two 3-indices series) and seven (one-index) series of finitely degenerate singularities, whose 6–jet has the form $\left(t^4,\ t^6,\ 0, \dots,\ 0\right)$.

The Taylor series of the remaining "sporadic" simple singularities of curves start with $\left(t^4,\ t^5,\ 0, \dots,\ 0\right)$ (7 curves), $\left(t^4,\ t^7,\ 0, \dots,\ 0\right)$ (15 curves), $\left(t^5,\ 0, \dots,\ 0\right)$ (6 curves) and $\left(t^6,\ 0, \dots,\ 0\right)$ (3 curves). They live in spaces of dimension at least 2 (5 curves), at least 3 (13 curves), at least 4 (9 curves), at least 5 (4 curves) and at least 6 (one curve).

The principal result of the present paper is the classification of the singularities A_{2k} in the symplectic space (up to the symplectomorphisms). All such singularities are simple. It means, that each curve singularity A_{2k} has only a finite number of symplectic forms.

The fact, that these singularities should have several symplectic forms is evident. Indeed, the flag, formed by the tangent line and by osculating two-plane of the curve at the singular point is intrinsically defined as a flag in the tangent space to the ambient manifold. The symplectic 2-form vanish or not vanish at the 2-plane of the flag, providing at least two symplectically different A_{2k} singularities.

The calculations below show that the number of the symplectically different versions of the curve singularity A_{2k} equals $2k+1$. This shows that the restriction of the symplectic form to the curve is highly nontrivial at the singular points of the curve (vanishing outside the singular point).

2. Classification of the symplectic singularities of the curves of the series A_{2k}

Consider the symplectic space $\mathbb{C}^{2n}$ equipped with the standard Darboux coordinates $(p_i,\ q_i)$ (the symplectic structure being $\sum dp_i \wedge dq_i$).

The following curves provide all the symplectic versions of the A_{2k} singularities :

$$A_{2k,0}: \quad \left(q_1 = t^2,\ p_1 = t^a,\ p_i = q_i = 0 \text{ for } i > 1\right);$$

$$A_{2k,r}: \quad \left(q_1 = t^2,\ q_2 = t^a,\ p_1 = t^b,\ p_2 = 0,\ p_i = q_i = 0 \ \text{ for } i > 2\right),$$

where $a = 2k+1,\ b = a + 2r, \quad r > 0$.

Theorem 1. *Almost every curve in a symplectic space of dimension $2n \geq 4$, whose Taylor series at the singular point has a nonzero quadratic term, is simple and symplectomorphic to one of the curves $A_{2k,r}$, $\quad 0 \leq r \leq 2k$.*

The exceptional curves are nonsimple and form a subset of codimension infinity in the space of curves. In the holomorphic case such an infinitely degenerate curve "covers itself more than once" (the mapping of the complex line to the complex space being not a topological immersion). In the formal or real smooth case the infinitely degenerate germs verify the same infinite system of the equations on the coefficients of the Taylor series, as in the holomorphic case, but their topological description is not so simple.

Remark. In the case $r = 2k$ one might replace the monomial t^b in the normal form by zero. The resulting "plane" curve is symplectically equivalent to the original one. It belongs to the Lagrange plane $p = 0$. The other curves do not belong to any smooth Lagrange manifold. The invariant b mesures the maximal order of tangency of the curve with a smooth Lagrange (or isotropic) surface at the singular point.

Denote the symplectic curves $A_{2k,0}$ and $A_{2k,r}$ by the symbols $(2;a)$ and $(2,a;b)$, where $a = 2k+1, \quad b = 2k+r, \quad 0 < r < 2k$. Denote by $(2,a)$ the curve

$$\left(q_1 = t^2,\ q_2 = t^a,\ q_i = 0 \text{ for } i > 2,\ p_j = 0\right).$$

This curve is symplectomorphic (as we sall see later) to $A_{2k,2k}$.

The symplectic classification of the singular curves in $\mathbb{C}^4$ starts from

$$\begin{tikzcd}
(2;3)^0 & (2,3;5)^1 \arrow[l] & (2,3)^2 \arrow[l] & & & & \\
& (2;5)^3 \arrow[u] & (2,5;7)^4 \arrow[l] \arrow[u] & (2,5;9)^5 \arrow[l] & (2,5;11)^6 \arrow[l] & (2,5)^7 \arrow[l] & \\
& & (2;7)^6 \arrow[u] & (2,7;9)^7 \arrow[l] \arrow[u] & (2,7;11)^8 \arrow[l] \arrow[u] & (2,7;13)^9 \arrow[l] \arrow[u] & \ldots \arrow[l] \\
& & & (2;9)^9 \arrow[u] & (2,9;11)^{10} \arrow[l] \arrow[u] & (2,9;13)^{11} \arrow[l] \arrow[u] & \ldots \arrow[l] \\
& & & & \ldots \arrow[u] & \ldots, \arrow[u] &
\end{tikzcd}$$

where the upper index is the codimension in the space of singular curves in $\mathbb{C}^4$. I hope there are no further adjacencies. I have verified that there is no adjacency from $(2;5)^3$ to $(2,3)^2$ (permitted by the codimension).

3. Reduction to four-dimensional problem

A finitely degenerated curve, defined by a mapping of a line, whose quadratic term at the critical point is not vanishing, belongs to a smoooth 2-surface. The image curve is defined on this surface by an equation of the form $x^a = y^2, \ a = 2k+1$ (in some local coordinates on the surface, see [1]).

Consider the restriction of the symplectic form of the ambient space to this surface's tangent space at the singulat point of the curve. This 2-dimensional subspace is defined unambiguously (does not depend on the choice of the surface). The restriction of the symplectic form to this 2-space might be a nonzero or zero bilinear form.

1°. *Let the restriction be a nonzero bilinear form.*

Lemma 1. *In this case the curve is symplectomorphic to* $A_{2k,0}: \ q_1 = t^2, \ p_1 = t^a \ (a = 2k+1)$.

Proof. By the Darboux-Givental theorem the 2-surface is symplectomorphic to the coordinate plane (q_1, p_1). Lemma 1 follows now from the 2-dimensional version of it :

Lemma 2. *The curve* $x^a = y^2$ *on a symplectic surface is symplectomorphic to the curve* $q_1^a = p_1^2$, *where* (p_1, q_1) *are some Darboux coordinates.*

Proof. Let $\omega = dp_1 \wedge dq_1 = f dx \wedge dy, \ f(0) > 0$. Consider the quasihomogeneous Euler vector field $v = 2x\partial_x + ay\partial_y$. This field is tangent to our curve.

For any function g the field gv is also tangent to it. To reduce f to 1 (using the homotopy method) it suffices to solve (regularly) the homological equation $L_{gv}\omega = h\omega$ with respect to the unknown function g. Since $d\omega = 0$ this equation takes the form

$$\operatorname{div}(gv) = h.$$

Denote by $g_{l,m}$ and $h_{l,m}$ the Taylor coefficients of g and h. The solution of the homological equation in the holomorphic (or formal) case is provided by the formula $g_{l,m} = h_{l,m}/\left(2(b+1) + a(m+1)\right)$. In the smooth case one solves the linear inhomogeneous first order partial derivatives equation for g integrating along the characteristics (starting at the origin with $g = 0$).

Lemma 2 and hence Lemma 1 is thus proved.

2°. *Let the restriction of the symplectic form to the tangent plane of the surface vanish.*

Lemma 3. *In this case the mapping of the line to the symplectic space can be written in the form*

$$\left(q_1 = t^3, \quad q_2 = t^a, \quad p_1 = f(t), \quad p_2 = g(t), \quad p_{>2} = q_{>2} = 0\right),$$

where $f, g = o(t^a)$ *and* p, q *are some Darboux coordinates,* $\omega = \sum dp_i \wedge dq_i$.

Proof. The tangent plane of the surface at the singular point of the curve is symplectically reducible to the coordinate plane (q_1, q_2). Hence in some Darboux coordinates the mapping takes the form $\left(q_1 = t^2, \quad q_2 = t^a, \quad q_{>2} = 0, \ p = 0\right)$, up to a perturbation, which is $o(t^a)$.

The projection to the q-plane along the p-plane can be therefore reduced to the form $\left(q_1 = t^3, \quad q_2 = t^a, \quad q_{>2} = 0\right)$ by a diffeomorphism of the q-plane. This diffeomorphism induces a symplectomorphism, preserving the Lagrangian plane $p = 0$. Therefore in the new coordinates $p = o(t^a)$.

But every smooth function of t, which is $o(t^a)$, can be represented as the restriction of a smooth function of $(q_1, \ q_2)$ to the curve $q_1 = t^2, \quad q_2 = t^a$ according to the preparation theorem.

Hence our curve belongs to some 4-submanifold, defined by the equations $q_{>2} = 0$, $p_{>2} = F(q_1, \ q_2)$ and parametrised by the coordinates (q_1, q_2, p_1, p_2).

The restriction of the symplectic structure to this submanifold takes the form $dp_1 \wedge dq_1 + dp_2 \wedge dq_2$ (since $dq_{>2} = 0$).

According to Darboux-Givental theorem, all the symplectic 4-submanifolds are locally symplectomorphic. Hence our manifold can be symplectically sent to the coordinate 4-space (p_1, q_1, p_2, q_2), $p_{>2} = q_{>2} = 0$. Reducing the curve inside this space to the above form by its symplectomorphism, we get Lemma 3.

4. Lagrange approximating surface.

Now we study curves in symplectic 4-space, equipped with Darboux coordinates, $\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$.

Lemma 4. *The mapping of a line to the symplectic space*

$$q_1 = t^2, \quad q_2 = t^a, \quad p_1 = f(t), \quad p_2 = g(t),$$

where $f, g = o(t^a)$, $a = 2k+1$, *can be transformed to the form*

$$Q_1 = t^2, \quad Q_2 = t^a, \quad P_1 = \sum_{r=1}^{2k-1} c_r t^{a+2r} \ , \quad P_2 = 0$$

by a symplectomorphism of the form $P = p - \frac{\partial S}{\partial q}$, $Q = q$.

Proof . For simplicity denote $q_1 = Q_1 = x$, $q_2 = Q_2 = y$. Choose

$$S(x,y) = A(x) + yB(x) + y^2C(x) + y^3D(x)$$

(we shall see in section 6 that the higher degree terms in y are not helpful.)

We can represent f and g along our curve as

$$f = u(x) + yz(x), \quad g = v(x) + yw(x),$$

where u and v are divisible by x^{k+1}, while z and w are divisible by x.

We get the following shifts in the p-direction :

$$\frac{\partial S}{\partial x} = A' + yB' + y^2C' + y^3D',$$
$$\frac{\partial S}{\partial y} = B + 2yC + 3y^2D.$$

Along our curve $y^2 = x^a$. Therefore along it we have

$$\frac{\partial S}{\partial x} = (A' + x^aC') + y(B' + x^aD'),$$
$$\frac{\partial S}{\partial y} = (B + 3x^aD) + 2y(2C).$$

Choose now C, A and B from the equations $2C = w$, $A' + x^aC' = u$, $B + 3x^aD = v$, where D will be choosen later. Along our curve we get $g - \frac{\partial S}{\partial y} = 0$ (hence $P_2 = 0$) and $f - \frac{\partial S}{\partial x} = y\left(z - (B' + x^aD')\right) = y\left(z - v' + 2x^aD' + 3aDx^{a-1}\right)$. Choose now D to kill all the terms of degree $a-1$ and higher in $z-v'$. We thus reduce the bracket to the polynomial in x of degree smaller than $a-1=2k$

$$z - v' + 2x^a D' + 3aDx^{a-1} = \sum_{r=1}^{2k-1} c_r x^r .$$

Here $r>0$, since $z(0)=v'(0)=0$. Along our curve $P_1 = y\sum c_r x^r = \sum c_r t^{a+2r}$, and Lemma 4 is proved.

Remark. The meaning of these calculation is the choice of the Lagrange surface $p=\frac{\partial S}{\partial q}$, which have the maximal tangency order with our curve at the singular point. The distance to this surface is of order t^b, where $b = a + 2\min\{r|\ c_r \neq 0\}$. If all the coefficients c_r are vanishing, the distance is zero (the curve belong to a smooth Lagrange surface).

5. Proof of the normal form theorem.

Suppose that c_r is the first nonzero coefficient of the polynomial of Lemma 4. Taking into account Lemmata 1–4, we see, that it remains to prove the following

Lemma 5. *The curve, defined in terms of the Darboux coordinates in symplectic $\mathbb{C}^4$ as the image of the mapping*

$$(q_1 = t^2,\quad q_2 = t^a,\quad p_1 = t^b,\quad p_2 = 0)$$

$(a = 2k+1,\quad b = a+2r,\quad 0<r<2k)$, *is symplectomorphic to the curve, defined as the image of the mapping*

$$\left(Q_1 = T^2,\quad Q_2 = T^a,\quad P_1 = T^b F(Q_1),\quad P_2 = 0\right)$$

whatever be the function F, provided that $F(0)\neq 0$.

Proof. We use the symplectomorphism, induced by the change of the q-variables $q_1 = E^2 Q_1,\ \ q_2 = E^a Q_2$, where $E = E(Q_1),\ \ E(0)\neq 0$.

The curve $q_1^a = q_2^2$ is sent to itself: $Q_1^a = Q_2^2$. Since $dq_1 = \left(2EE'Q_1 + E^2\right) dQ_1$, $dq_2 = aE^{a-1}Q_2 dQ_1 + E^a dQ_2$, we find from $p_1 dq_1 + p_2 dq_2 = P_1 dQ_1 + P_2 dQ_2$ that

$$P_1 = p_1\left(2EE'Q_1 + E^2\right) + p_2 a E^{a-1} Q_2,\quad P_2 = p_2 E^a .$$

Along our curve $p_1 = q_2 q_1^r = E^{a+2r} Q_2 Q_1^r,\ \ p_2 = 0$. Therefore along this curve

$$P_2 = 0,\quad P_1 = Q_2 Q_1^r E^{a+2r}\left(2EE'Q_1 + E^2\right) = T^b E^{a+2r}\left(2EE'Q_1 + E^2\right).$$

To get the required value of P_1 it suffices to solve (with respect to the unknown smooth function $E(Q_1)$) the singular differential equation $2E'Q_1 = FE^c - E$, where $c = -a-2r-1$.

Consider the system defined by the regular vector field on the (E,Q)-plane (we denote $Q_1 = Q$):

$$\frac{dE}{ds} = FE^c - E,\quad \frac{dQ}{ds} = 2Q.$$

The vector field has a stationnary point $(E_0, 0):\quad Q=0,\ \ E_0^{c-1} = F(0)^{-1}\neq 0$. This point is a saddle point with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = F(0)cE_0^{c-1} - 1 = c-1 = -a-2r-2$. The separatrice, corresponding to $\lambda_1 = 2$, is analytic (smooth if F is smooth) at the origin. It is the graph of the required regular solution $E = E(Q),\ \ E(0) = E_0 \neq 0$. Lemma 5 (and therefore the theorem) is proved.

6. Symplectic invariants of singularities.

We shall now prove that all the curves $A_{2k,r}$ of Theorem 1 are symplectically different. The difference between $A_{2k,0}$ and al the others is obvious, since k is the invariant of the diffeomorphisms and $A_{2k,0}$ is the only case, where the corresponding tangent two-space is not isotropic.

We shall now show, that r is a symplectic invariant of the singularity $A_{2k,r}$, $0 < r < 2k$ in $\mathbb{C}^4$.

Theorem 2. *The curve* $(q_1 = t^2,\ q_2 = t^a,\ p_1 = t^b,\ p_2 = 0)$, $a = 2k+1$, $b = a+2r$, $0 < r < 2k$ *has no smooth Lagrange surface, whose tangency with the curve at the origin is higher than that of the plane* $p = 0$.

Proof. Suppose there exists such a better approximating surface. First we prove, that its tangent plane at the singular point should be the plane $p = 0$.

Lemma 6. *If a smooth surface has the property that the distance from the point of our curve to this surface is vanishing for* $t = 0$ *with a higher order than* t^a*, then the tangent plan of the surface at the singular point of the curve is the plane* $p = 0$.

Proof. Write the regular equation of the surface in the form

$$Aq_1 + Bq_2 + Cp_1 + Dp_2 + R(p, q) = 0,$$

where R is the remaining part of the Taylor series (containing quadratic and higher order terms). Substituing for p and q their expressions, we get

$$At^2 + Bt^a + Ct^b + R_2t^4 + R_3t^6 + \dots + O(t^{a+2}) + \dots\ .$$

Since $b > a > 2$, we get $A = 0$ (otherwise the vanishing order would be $2 < a$). Since $a < b$ is odd, we get $B = 0$ (otherwise the vanishing order would be equal to a or would be even smaller). Hence $A = B = 0$ and hence the tangent plane is the plane $p = 0$. Lemma 6 is thus proved.

According to Lemma 6, the better approximating Lagrange surface would have the equation $p = \frac{\partial S}{\partial q}$.

We shall denote $q_1 = x$, $q_2 = y$. The terms, containing y^4 will give the contribution to p, which is $O(y^3) = O(t^{3a})$ along our curve. For $r < 2k$ we have $3a > b$ (since $b = a + 2r < a + 4k = 6k + 1 < 6k + 3$). Therefore these terms are small with respect to t^b and can't change the tangency order. Thus it suffices to consider the Lagrange surface, defined by

$$S = A + yB + y^2C + y^3D.$$

Along the curve $x = t^2$, $y = t^a$ we get

$$\frac{\partial S}{\partial x} = (A' + x^aC') + y\,(B' + x^aD'),$$

$$\frac{\partial S}{\partial y} = (B + 3x^aD) + y\,(2C)\,.$$

To have a better tangency we need have along the curve

$$\frac{\partial S}{\partial y} = o(t^b), \quad \frac{\partial S}{\partial x} - t^b = o(t^b).$$

We shall see that these requirements are contradictory.

The only term with t^b in $\frac{\partial S}{\partial x}$ is proportional to the term containing x^r in B'. Indeed, $x^a y = t^{3a}$, $3a > b$, and therefore D' contributes only $o(t^b)$. Therefore B contains x^{r+1} with a nonzero coefficient. Since $r+1 < 2k+1 = a$, $\frac{\partial S}{\partial y}$ contains the term $x^{r+1} = t^{2r+2}$ with nonzero coefficient. The condition $\frac{\partial S}{\partial y} = o(t^b)$ therefore requires the inequality $2r+2 > b = a+2r$, which is impossible since $a \geq 3$.

The contradiction shows that the required Lagrange surface (approximating the curve better than the plane $p=0$) does not exist.

Theorem 2 is thus proved. The quantity r is therefore a symplectic invariant of the curve $A_{2k,r}$ in $\mathbb{C}^4$. It remains an invariant in higher-dimensional symplectic manifold (it follows from calculations similar to the calculations above).

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