Smooth functions statistics

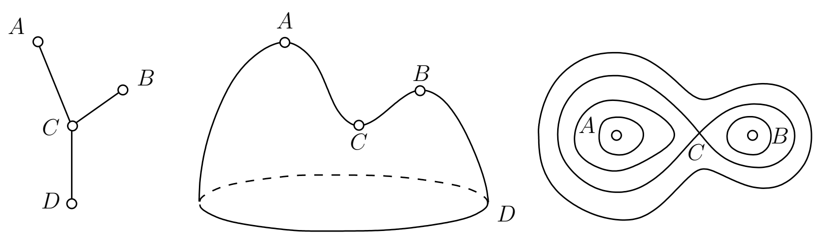
V. I. Arnold

To describe the topological structure of a real smooth function one associates to it the graph, formed by the topological variety, whose points are the connected components of the level hypersurface of the function.

For a Morse function $f : S^n \to \mathbb{R}$, $n > 1$, such a graph is a tree. Generically, it has T triple vertices, $T + 2$ endpoints, $2T + 2$ vertices and $2T + 1$ arrows.

Example 1. For the Elbrous mountain, with two maxima A and B, separated by the saddle point C, the tree is

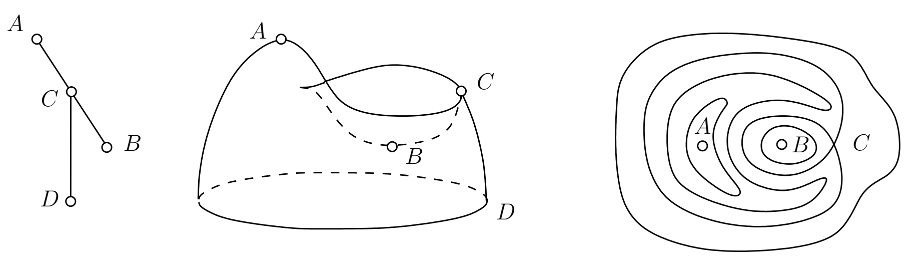
$T = 1$,



We shall consider functions on $\mathbb{R}^n$, behaving as $-r$ far from the origin, as continued to S^n with a Morse minimum at $\infty = S^n \setminus \mathbb{R}^n$ (represented by the vertex D for the above mountain).

Example 2. For the Vesuvius mountain, with crater B, the tree is

$T = 1$,



We shall include in the graph structure the ordering of the critical values, $f(A) > f(B) > f(C) > f(D)$ in Example 1, $f(A) > f(C) > f(B) > f(D)$ in Example 2.

For simplicity we will suppose below, that all the $2T+2$ critical values are different, T being the number of the saddle-points of a function on S^2 (the graphs of Morse functions on S^n are similar, and the cases of other domains, like $T^n = (S^1)^n$, can be studied by a similar technique).

The main goal of the present paper is to study the statistics of the graphs, corresponding to T triple points: what is the growth rate of the number $\varphi(T)$ of different graphs? Which part of these graphs is representable by the polynomial functions of corresponding degree? A generic polynomial of degree n has at most $(n-1)^2$ critical points on $\mathbb{R}^2$, corresponding to $2T+2 = (n-1)^2 + 1$, that is to $T = 2k(k-1)$ saddle-points for degree $n = 2k$.

Theorem 1. *The first numbers $\varphi(T)$ of the graphs with T triple points are*

T	1	2	3	4
$\varphi(T)$	2	19	428	17746

The growth rate of these numbers is bounded by the following two results.

Theorem 2. *The lower bound for the number $\varphi(T)$ of the graphs with T triple points is*

$$\varphi(T) \geq (T^2 + 5T + 5)\ \frac{(2T+2)!}{(T+4)!}\ .$$

For $2 \leq T \leq 4$ the right hand side values are 19, 232, 3690, and their "Stirling" growth rate is $4(4/e)^T T^T > T^T$.

Theorem 3. *The upper bound for the number $\varphi(T)$ of the graphs with T triple points is*

$$\varphi(T) \leq T^{2T} \qquad \text{(for } T > 2\text{)}\ .$$

The main technical tool for the proof of Theorem 2 is provided by the explicit calculation of the number of those graphs with T triple points, whose triple points form a monotone A-chain (with the critical values $f(A_1) > f(A_2) > \cdots > f(A_T)$ of adjacent points A_1—A_2—…—A_T).

Theorem 4. *The number of the graphs with $T \geq 2$ triple points, forming monotone A-chains, is equal to*

$$\psi(T) = (T^2 + 5T + 5)\, \frac{(2T+2)!}{(T+4)!}\ .$$

Theorem 2 follows from Theorem 4, the number of all graphs $\varphi(T)$ being at least equal to the number of graphs, whose triple points form A-chains.

Remark. Some of our graphs are corresponding to the generic Morse polynomials (of degree $n = 2k$ for $T = 2k(k-1)$), and some does not. It would be interesting to know, whether the representable graphs (or the nonrepresentable ones) do form an asymptotically small part of the totality of graphs with T triple points (for $T \to \infty$). The numbers of the topologically different realizations of realizable graphs are also unknown.

Proof of Theorem 4. Let us denote by a the critical point, adjacent to A_1 which has the highest critical value, $f(a) > f(A_1)$.

Similarly, denote by z the critical point, adjacent to A_T, which has the lowest critical value, $f(z) < f(A_T)$.

Denote by α the third critical point, adjacent to A_1 (different from a and A_2). Denote by ω the third critical point, adjacent to A_T (different from z and from A_{T-1}).

To classify the framings of the triple points $A_1, \ldots, A_T$ of the graphs by the ending segments, we first observe, that the critical value $f(\alpha)$ belongs to the complement of the following set of $T+1$ real numbers, smaller than $f(a)$:

$$\{f(A_1), \ldots, f(A_T), f(z)\}\ .$$

Therefore, there are $T+1$ cases, for which $f(a) > f(z)$ and one more, where $f(a) < f(z)$.

Knowing the situation of the critical value $f(\alpha)$, consider the critical value $f(\omega) > f(z)$. It must be different from the $T+2$ values

$$\{f(A_1), \ldots, f(A_T), f(a), f(\alpha)\}$$

on the ray $\{t > f(z)\}$ if $f(a) > f(z)$, differing from the $T+1$ values

$$\{f(A_1), \ldots, f(A_T), f(a)\}$$

on the ray $\{t > f(z)\}$ if $f(a) < f(z)$.

Therefore, we get the total number

$$(T+1)(T+3)+1(T+2)=T^2+5T+5$$

of the variable framings $\{\alpha, \omega\}$ of the triple endpoints, A_1 and A_T.

At the endpoint a_2, adjacent to A_2, the critical value should differ from the $T+4$ already choosen values

$$\{f(A_1), \ldots, f(A_T), f(a), f(\alpha), f(z), f(\omega)\} ,$$

which subdivides every one of the previous cases into $T+5$ subcases.

Choosing $f(a_2)$, we get for $f(a_3)$ the necessity to avoid $T+5$ places, for $f(a_i)$ – to avoid the $T+2+i$ values

$$\{f(A_1), \ldots, f(A_T); f(a), f(z); f(\alpha), f(\omega); f(a_2), \ldots, f(a_{i-1})\} .$$

These values subdivide the real line into $T+3+i$ intervals, multiplying the number of different graphs by $T+3+i$. Using this reasoning $T-2$ times (for $i=2,3,\ldots,T-1$), we subdivide each of the T^2+5T+5 classes of the framings of the endvertices A_1 and A_T of the chain of triple points into the subcases, whose number equals to the product of the intervals numbers of all the steps,

$$(T+5)(T+6)\ldots(T+3+T-1)=\frac{(2T+2)!}{(T+4)!} .$$

All these subclasses correspond to different graphs, proving Theorem 4. $\square$

To prove Theorem 3, we start from the

Lemma. *The inequality*

$$\varphi(T) \leq 4T^2\varphi(T-1)$$

holds for any $T \geq 2$.

Example. For $T=2,3$ and 4 one has

$$(\varphi(2)=19) < (16 \cdot 2 = 32) ,$$
$$(\varphi(3)=428) < (36 \cdot 19 = 684) ,$$
$$(\varphi(4)=17746) < (64 \cdot 428 = 27392) .$$

Proof of the Lemma. The maximal critical value is attained at an end-point A of the connected graph with T triple points. This end-point is a neighbour of one of the triple-points, B. Deleting the edge AB, we reduce the original graph with T triple points to a new connected graph with $T-1$ triple points.

There are $\varphi(T-1)$ such graphs. To reconstruct the initial larger graph one should choose an arrow, where to insert the new triple point B (there are $2(T-1)+1 = 2T-1$ arrows) and choose the value at B (there are $2T$ values at the $2(T-1)-2$ vertices of the smaller graph, leaving $2T+1$ different possibilities).

The total number of the choices is $(2T-1)(2T+1) < 4T^2$, proving the Lemma. $\square$

These reasonings prove in fact more, than stated: one evaluates this way the larger number, including those ordered graphs, for which the triple point value is higher, than all its three neighbouring vertices values, and those, where it is smaller, than all the three (which never happens for the graphs of functions, whose number is therefore even smaller, than our Lemma inequality proves).

Proof of Theorem 3. For $T = 3$ we have

$$\bigl(\varphi(3) = 428\bigr) < (3^6 = 729)\ .$$

If Theorem 3 holds for $T = S-1$, we get by the Lemma

$$\varphi(S) \leq 4S^2(S-1)^{2S-2}\ . \qquad (*)$$

The obvious inequality

$$\left(1 - \frac{1}{S}\right)^S < \frac{1}{e}$$

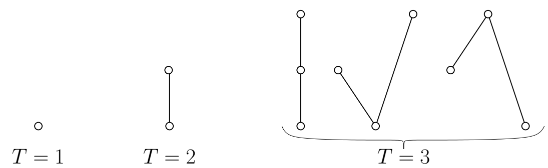
implies, that

$$\begin{aligned} 4S^2(S-1)^{2S-2} &\leq \frac{4S^2}{(S-1)^2}\ \left(\frac{S-1}{S}\right)^{2S}\ S^{2S} \leq \\ &< \frac{4}{e^2}\ \frac{S^2}{(S-1)^2}\ S^{2S}\ . \end{aligned}$$

The coefficient $4S^2/\left(e^2(S-1)^2\right)$ is smaller, than 1, for $2S \leq e(S-1)$, which holds for $S \geq 4$.

Thus, if $S \geq 4$ Theorem 3 for $T = S - 1$ together with inequality $(*)$, imply that $\varphi(S) \leq S^{2S}$. Theorem 3 is therefore proved, since it is true for $T = 3$, being hence true for $T = 4, 5, \ldots$. $\square$

Proof of Theorem 1. Consider the T triple points of the graph (with $2T + 2$ vertices). They form the vertices of a smaller connected graph, which is obtained for the initial big graph with T triple points deleting its $T + 2$ end-points (together with the $T + 2$ edges, leading to them from the T triple points). For $T = 1, 2$ and 3 the resulting (ordered) graphs are of the forms



$T = 1$ $T = 2$ $T = 3$

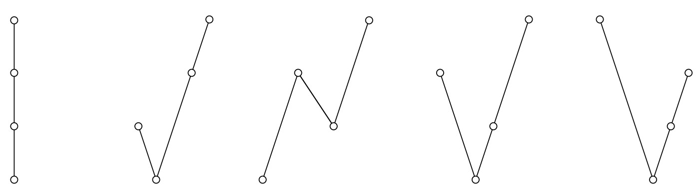
and the framings are counted by Theorem 4 in first 3 cases, providing

$$\varphi(1) = 2 \ , \quad \varphi(2) = 4 + 5 \cdot 2 + 5 = 19$$

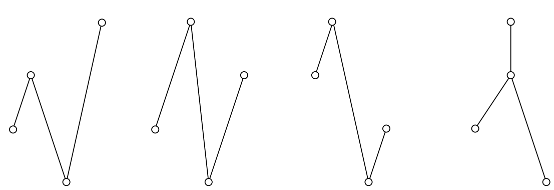
and $\psi(3) = (9 + 5 \cdot 3 + 5)\ 8 = 232$ framings.

Each of the last two cases is studied similarly to the proof of Theorem 4, providing 98 framing each (the equality of both numbers is an evident corollary of the symmetry, transforming each of the two last diagrams to the other).

For $T = 4$ triple points, there are IX essentially different cases to study,



I II III IV V



VI VII VIII IX

The cases II, IV, V, VIl, IX, are representing two (symmetric) graphs each (similarly to the two last graphs of the $T = 3$ case). These graphs are different, but their numbers of framings are equal (by the symmetry bijection between the graphs sets).

These numbers of framings, ψ, are

case	I	II	III	IV	V	VI	VII	VIII	IX
ψ(case)	3690	1680	586	1360	1360	534	486	756	1180

The largest case I is described in the proof of Theorem 2 above. The framings counting in the other 8 cases is similar, but the details are too long to be reproduced here.

The total number of graphs, taking the symmetrical cases into account, is

$$\begin{aligned}\varphi(4) = \psi(\mathrm{I}) + \psi(\mathrm{III}) + \psi(\mathrm{VII}) + \psi(\mathrm{VIII}) +& \\ + 2\,(\psi(\mathrm{II}) + \psi(\mathrm{IV}) + \psi(\mathrm{V}) + \psi(\mathrm{VI}) + \psi(\mathrm{IX})) =& \\ = 5518 + 2(6114) = 17746\ ,&\end{aligned}$$

proving Theorem 1. $\square$

Remark. It would be interesting to study, what is the true growth rate of $\varphi(T)$ for large T (perhaps T^{cT}, neglecting the "logarithmical corrections", for some $1 \le c \le 2$, and with a constant c which looks to be closer to 2).

The preceding statistics for $T = 4$ shows some uniformity of the distribution of the graphs among the 9 triple vertices subgraphs types. For the special subgraph of T triple vertices with critical values

$$\mathrm{II}(T) = \{a_1 > a_2 > \cdots > a_{T-1}\ ,\ a_1 > b > a_2\}$$

the asymptotics of the numbers of framings is

$$\frac{\psi\big(\mathrm{II}(T)\big)}{\psi\big(\mathrm{I}(T)\big)} \longrightarrow \frac{1}{2} \quad \text{for } T \to \infty\ ,$$

namely,

$$\psi\big(\mathrm{II}(T)\big) \sim \frac{T^2(2T+2)!}{2(T+4)!}\ .$$

It would be interesting to study such asymptotical interrelations between the numbers of framings of different diagrams.

I hope, that similar reasonings might provide the lower bound

$$\varphi(T) > BT^2\varphi(T-1)$$

for some constant B, perhaps even for $B = 2$.

Indeed, among the $4T^2 - 1$ ways of adding a new edge, leading to the maximum critical value point, most ways provide genuine larger graphs with one more triple point, choosen at an arbitrary one of the $2T-1$ edges of the given graph with $T-1$ triple vertices. The only difficulty is the choice of the value at this newly created triple point. There are $2T+1$ intervals where the value ought to be choosen, but if the values at the boundary points of the choosen edge, where the new triple point is situated, are u and v, the new value should not be smaller, than both u and v.

The third neighbour of the added point being the maximum point, the choice of the new value $w < u$, $w < v$, would produce in the bigger graph a vertice, where the value is smaller, than all its three neighbours, which never happens for the graphs of the Morse functions.

An euristical (nonrigorous) estimation of the part of bad situations is about $1/4$ of the total number of the $4T^2 - 1$ attempts to construct a bigger graph : the "probabilities" of the inequalities $w < u$ and of $w < v$ for a "random choice" of w seem to be $1/2$ each, and these two events looking "independent", one gets $1/4$ bad cases among the total number $4T^2 - 1$ proposals of the bigger graph, provided that statistically the distribution of the values at the vertices behaves similarly to the random independent choices of the values. This asymptotical "ergodicity" is a difficult conjecture, if one desires a rigorous proof, but one might check this conjecture experimentally (say, for $T = 5, 6, 7, 8$) in few hours of the time of a computer.

Assuming the probability $1/4$, we would replace the coefficient $B = 4$ by $B = 3$, but the Theorem 1 numbers suggest for the limit of the ratio

$$\frac{\varphi(T)}{T^2\varphi(T-1)} \qquad (T \text{ tending to infinity})$$

a smaller value, closer to 2, than to 3. This might be explained by the next obstacle, reducing the number of the successful constructions once more. Suppose, that we choose the new value in between the two neighbours,

$$u < w < v\ .$$

This choice would spoil the vertex , where the value is u, provided, that its other two neighbours have higher values, than u : for the new graph there will be a triple vertex, where the value is smaller, then all the three neighbours, which never happens for a Morse function. Calculating the "probability" of this new desaster by the previous "ergodic" conjectures, one reduces the value $B = 3$ to $3/4$ of it, which is $B = 9/4$. These semiempirical suggestions are not too far from the observed values:

$$(\varphi(T = 4) = 17746)/(\varphi(3) = 428)$$

is not far from

$$(B = 9/4)(T = 4)^2 = 36 \ .$$

Anyway, whatever be the constant B, the inequality

$$\varphi(T) > BT^2\varphi(T - 1)$$

would imply the essentially T^{2T} behaviour of $\varphi(T)$, the value of the positive constant B influencing only some "logarithmical" factor (the exponents const^T being "logarithmically small" with respect to the main term T^{2T}).

In fact the upper bound of Theorem 3 is proved above for the larger number, than $\Phi(T)$, counting among other graphs those oriented graphs, which contain some vertices, where the value is greater, than all the three values at the neighbouring vertices. This event would be impossible for the graph of a function, but the upper bound for the larger number bounds also the smaller number φ.

To get by these reasonings the lower bounds one needs to know the "probability" of the impossible situations, described above, and the proofs depend, therefore, on difficult "ergodicity" statements for the random graphs, which I had not proved.

For the functions on the torus T^2 (needed for the trigonometrical polynomials study) the graphs are no longer trees, they have $g = 1$ cycles for the surfaces of genus $g = 1$. The numbers of such graphs with T triple points for $T = 2, 3, 4$ are $1, 16, 550$.