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HIGHER DIMENSIONAL CONTINUED FRACTIONS

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The higher-dimensional analogue of a continuous fraction is the polyhedral surface, bounding the convex hull of the semigroup of the integer points in a simplicial cone of the euclidian space. The article describes some conjectures and theorems, extending to such higher-dimensional continouos fraction the Lagrange theorem on quadratic irrationals and the Gauss–Kuzmin statistics.

To Jürgen Moser admiringly

1. Classifications of graded algebras

While canoeing at the Onega lake and at the Svjatukha river in the summer of 1988, I became so nostalgic for mathematics that decided one rainy day to study, following the example of the algebraists, a problem having no relations to any serious mathematics.

This attempt to produce something useless ended with a failure. The present article is an account of the theories emerging from the investigation of the following unmotivated definition.

Consider a commutative associative graded algebra over the field $\mathbb{C}$ (or over $\mathbb{R}$).

Definition 1. The algebra is called *A-algebra* if its Poincaré series equals $1 + t + t^2 + \ldots$ (that is, if each homogeneous component is one-dimensional.

The problem was to classify the A-algebras. Such an algebra has a natural system of homogeneous multiplicative generators. If there is only one generator, the algebra is isomorphic to the algebra of polynomials in one variable. The case of two multiplicative generators is also easy to settle. I had started to investigate the algebras with three multiplicative generators.

One of generators is of degree one. Denote by $u < v$ the degrees of the two others. My result in this case (see [1]) was:

Theorem 1. *The number of the nonisomorphic A-algebras with three multiplicative homogeneous generators of degrees $1 < u < v$ is finite and is equal to $2a + 1$, where $a = a_1 + a_2 + \ldots$ is the sum of the elements of the continued fraction $v/u = a_0 + \dfrac{1}{a_1 + \ldots}$.*

Example 1. In the simplest case the degrees are $1 < 2 < 3$, the continued fraction is $3/2 = 1 + 1/2$, $a = 2$. Therefore the number of nonisomorphic A-algebras having 3 multiplicative generators of degrees 1, 2 and 3 is equal to five.

The defining relations of these are also described in terms of continued fraction (see [1]).

Therefore, it hapens that the original useless algebraic problem has unexpected relations to serious mathematics — to the continued fractions and hence to diophantine approximations, dynamical

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systems and Hamilton mechanics, resonances, small denominators — from one side, and to number theory from another.

Moreover, the original algebraic problem, leading to the continued fractions in the case of three generators, has a natural algebraic extension with an arbitrary number of generators. We get thus a natural algebraic problem, whose solution could be used to generalize the continued fraction notion.

The ordinary continued fractions being related to the geometry of the 2-lattice $\mathbb{Z}^2$, the problem of classification of A-algebras with n generators is related to the lattice $\mathbb{Z}^{n-1}$.

Denote by $(u_1, \dots, u_n)$ the degrees of the generators $(x_1, \dots, x_n)$. The monomial x^m is of degree $(m, u) = m_1 u_1 + \dots m_n u_n$. The relations of our algebra have the form $x^a = cx^b$ and correspond to Laurent monomials of degree zero ($x^{a-b} = c$). The exponents $a - b$ of these monomials belong to sublattice $\mathbb{Z}^{n-1}$ defined by equation $(m, u) = 0$ in the lattice $\mathbb{Z}^n$ of the exponents m.

The lattice $\mathbb{Z}^{n-1}$ contains n hyperplanes $m_i = 0$. The role of the continued fraction is played, for $n > 3$, by "star of sails" defined by the following construction.

Consider one of the $(n-1)$-dimensional (simplicial) piramides, bounded in the hyperplane $\mathbb{R}^{n-1} = \{m : (m, u) = 0\}$ by the n coordinate hyperolanes $m_i = 0$ (for $n = 3$ this plane $\mathbb{R}^2$ intersects 6 from 8 octants and is subdivided into 6 angles by the 3 coordinate planes).

The integer points, different from 0, which belong to the piramide that we have choosen, form a semigroup in $\mathbb{Z}^{n-1}$. The convex hull of this set of integer points is bounded by a polyhedral surface (of dimension $n - 2$), called the sail.

The sails, corresponding to all the piramides, form the star of sails that we have mentioned. In the case $n = 3$ this star consists of 6 sails, each of the sails belong an integer broken line in $\mathbb{Z}^2$ (corresponding to the usual continued fraction u_i/u_j).

The attempt to extend the A-algebras classification to the case of more than 3 generators by E. Korkina and her collaborators (see [2, 3, 4]) was based on computer experimentation. Korkina [4] has published the detailed proof of the classification of the A-algebras having 3 multiplicative generators.

However the further investigations showed that for more than 3 multiplicative generators the classification is no longer discrete: moduli (continious invariants) appear and the set of equivalence classes of A-algebras becomes a continuum.

The first example of this phenomenon was suggested by D.Eisenbud. In his example there were $n = 9$ generators. Later B. Sturmfels has discovered examples with $n = 4$ generators (of degrees 1, 3, 4, 7), using Gröbner bases and computer algebra (see [5]).

The description of the simple (having no moduli) A-algebras with $n > 3$ generators is open (even for $n = 4$). The experience shows that the classification of simple objects in all branches of mathematics always leads to natural and important results (including some mysterious relations between the simple objects of different theories).

One could guess that the classifications of simple A-algebras might lead to new discoveries of such relations. Even the list of the quadriles $1 < a < b < c$ (such that all the A-algebras whose multiplicative generators have such a quadriple of degrees are simple) is unknown.

2. Lagrange theorem and its generalizations

The classical theorem of Lagrange claims that a continued fraction is periodic if and only if it represents a quadratic irrational number.

The construction described above suggests an immediate generalization of this theorem to algebraic numbers of higher degrees n.

We have already seen, that the higher-dimensional continued fraction should be a sail (the boundary of a convex infinite polyhedron with integer vertices of dimension $n - 1$ in $\mathbb{Z}^n$ or eventualy the star of such sails).

It happens, that the periodicity of the usual continued fraction (corresponding to $n = 2$) is in the case $n > 2$ replaced by $(n - 1)$-periodicity of the sail (the periods forming the group $\mathbb{Z}^{n-1}$).

Consider an integer matrix $A \in SL(n, \mathbb{Z})$ of determinant one. Suppose that its eigenvalues λ_i are positive, pairwise differnt and irrational (*it* is covinient to suppose that characteristic equation of A is irreducible).

The eigenhyperplanes of A subdivide the space $\mathbb{R}^n$ into 2^n ortants. The nonzero integer points belonging to an ortant form a semigroup. Consider its convex hull. The boundary of this convex hull is an (infinite) polyhedral hypersurface — the sail of operator A in the ortant.

The sail is invariant under the action of operator A (which preserves both the integer points lattice and the ortant).

In the case of matrix of order $n = 2$ this implies the periodicity of continued fraction, corresponding to the tangent of the eigenline angle with the coordinate axis.

Indeed, the elements of this continued fraction are (alternating) the integral lengths of the segment and the integral angles of the broken line, which represents the sail.

The integral lengths of a segment between two lattice points is the number of parts into which it is subdivided by the other lattice points. The integral angle between two segments of integral lengths one, joining the origin 0 with two "neighbours", is the area of the parallelogram, generated by this segments, measured in the units of the fundamental parallelogram of the lattice of all integer points of the plane spanned by the two segments.

In other words this angle is equal to the index of the subgroup, generated by the point of the segments,in the group of integer points of the plane spanned by these two segments.

The integral volume of paralleliped (and the integral solid angle of simplicial piramide) are defined similary (and provide the integral volumes and solid angles for any integer polyhedron and polyhedral cone).

In the case $n > 2$ the symmetry group of the sail is larger then the cyclical group generated by operator A. Indeed, the "Cartan" subgroup of the integer operators having the same basis of eigenvectors as A (and having the same sails) is isomorphic to $\mathbb{Z}^{n-1}$ (it follows from the Dirichlet theorem on the units).

The remark above, due to H. Tsuchikashi [6], proves the $\mathbb{Z}^{n-1}$-periodicity of the n–dimensional sail), corresponds to a "completly real" algebraic number of degree n.

For instance, such an algebraic number defines a polyhedral subdivision of the torus T^{n-1} (which is the quotient space of the sail by the symmetry group).

The simplest cubical irrational number (which might be called the "cubical Fibonacci number") corresponding to the subdivision of the square (whose opposite sides are identified) by a diagonal (see [11]). The corresponding matrix is

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

(the usual Fibonacci number $(\sqrt{5}+1)/2$ corresponding to the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$).

To formulate the higher-dimensional version of the Lagrange statement that periodicity of a continuous fraction of a number implies that the number is a quadratic irrational algebraic number, I have suggested the following construction.

Consider n hyperplanes containing the origin O of the space $\mathbb{R}^n$ containing the lattice $\mathbb{Z}^n$.

Fix an integer r and consider the r-star of a vertex of the sail (consisting of all the vertex by a broken line consisting of at most r segment belonging to the faces of the sail). Call the *r-color* of a vertex the integer type of its r-star.

EXAMPLE 2. For $n = 2, r = 1$ the 1-color of a vertex of the broken line include the integral lengths of the adjoint segments and the integral angle between them (but is not reducible to this triple of integer).

The symmetries of the sails generated by a completely real operator $A \in SL(n, \mathbb{Z})$ preserve the colors of the vertices. Therefore the sails of such an operator have $n - 1$ periods both as simply an abstract infinite polyhedral surface in the sense of pure combinatorics, or taking into account the integer points on its faces or even taking into account the r-colors (whatever r we choose). In 1991 I formulated the following

Conjecture 1. *The periodicity (with $n-1$ periods) of the sail in $\mathbb{R}^n$ defined by n irrational hyperplanes containing the origin O of $\mathbb{R}^n$ equiped with the r-colors of the vertices implies the existence of an integer operator A, whose invariant hyperplanes are our hyperplanes, provided that r is sufficiently large.*

E. Korkina [7] had announced that the conjecture is true already for $r = 1$ (that is, the periodicity of the $\mathbb{Z}$-affine type of the union of the faces containing a vertex suffices), but the details of the proof have not yet been published.

The periodicity of the sails (considered as combinatorial polyhedral with distinguished integer points) could also suffice. For the case $n = 2$ this condition is exactly the periodicity of the continued fraction condition.

Different generalization of the Lagrange theorem and of Fibonacci numbers were discussed in many paper. G. Lachaud ([8] and [9]) and Iglesias (see [10]) have started the investigation of these problems together with Korkina when I had explained them my conjecture (the preprint of [11] already appeared in 1993).

The idea that the higher–dimensional generalisation of continued fraction is the geometry of the sails in higher–dimmensional space was well known already to Klein (see [12], [13], [14]), Minkowski (see [15]), Voronoy (see [16]) and others.

The attempts to generalise to higher dimensions the *algorithm* of continued fraction lead to complicate and ugly theories. For instance the sail corresponding to a cubical irrational number is a doubl–periodic polyhedral surface. However the algorithms define instead of this surface a *path* on it. Even in the case when the path visits all the faces, and provides all the very good diophantine approximation in the corresponding problem, the path is not periodic at all and looks as a rather chaotic object: it is unclear how to describe the cubical irrationals in terms of the combinatorics of this path.

Conversely, the study of the triangulation of the torus, corresponding to different cubic irrational, seems a natural geometrial approach to algebraic number theory.

3. Statistics of higher-dimensional continued fraction

The elements a_i of the continued fraction development of a random number are distributed nonuniformly: the probability of the value k of a_i decrease like k^{-2} and is equal to

$$p_k = \left(\ln\left(1 + \frac{1}{k(k+2)}\right)\right) \Big/ (\ln 2)$$

for almost every irrational number $a = a_0 + 1/a_1 + \dots$ (the Gauss–Kuzmin theorem, see [17]).

The "probability" p_k is defined here as the limit for $N \to \infty$ of the ratio of the number of the elements $a_i = k$ among the first N elements to N.

The Gauss–Kuzmin theorem is a particular case of the ergodic theorem in the following sense.

Consider the shift of the sequence of the elements of a continued fraction $(a_1, a_2, a_3, \dots) \mapsto (a_2, a_3, \dots)$. In terms of the number $x = 1/(a_1 + 1/a_2 + \dots))$ this shift is the mapping of the segment (0,1) into itself, defined by the formula $x \mapsto (\text{the fractionalpartof} 1/x)$.Gauss has remarked that this mapping T has an invariant measure μ (with density $\rho(x) = 1/((1+x)\ln 2)$): the measure of the preimage of any segment Δ is equal to the measure of Δ, $\mu(T^{-1}(\Delta)) = \mu(\Delta)$.

The derivative dT/dx is greater than 1 on the interval (0,1), hence mapping T is ergodic. Therefore the probability p_k can be calculated as the measure of the segment $1/(k+1) < x < 1/k$ where $a_1 = k$, according to the ergodic theorem:

$$p_k = \int\limits_{1/(k+1)}^{1/k} \rho(x)\,dx = (\ln{(k+1)^2}/k(k+2))/(\ln 2).$$

REMARK 1. In 1990 I have formulated the following conjectural generalisations of the Gauss–Kuzmin theorem: *a). The case of finite continued fractions.* Consider the continued fractions developments of all rationals p/q for which $p^2+q^2 \leq N^2$.

Conjecture 2. *The elements $a_i = k$ form among all the elements of all these continued fractions a part, asymptotically equal to p_k for $N \to \infty$.*

b). The case of periodic continued fractions. Consider the periods of the continued fractions of all real quadratic irrationals, satisfying the integer coefficients equations $x^2+px+q=0$ with $p^2+q^2 \leq N^2$.

Conjecture 3. *The elements $a_i = k$ form among all the elements of all these periods a part, asymptotically equal to p_k for $N \to \infty$.*

Conjecturally these limits remain the same if one replace the discs of radius N by the squares $|p|+|q| \leq N$ or by some other similar domains. One could also replace the abstract quadratic equations by the characteristic equations of the integer matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (say, with $a^2+b^2+c^2+d^2 \leq N^2$).

The computer experiments (by E. Korkina and by J.-O. Moussafir) seem to confirm these conjectures.

One is naturally lead to the question: how to extend all this to the higher-dimensional continued fractions? Already in 1990 I have formulated the following (slightly nonprecise)

Conjecture 4. *For almost every simplicial cone whose vertex is the origin O of the space $\mathbb{R}^n \supset \mathbb{Z}^n$, the statistic of any reasonable characteristic of the sail of this cone (that is of the boundary of the convex hull of the set of the integer points lying insidde the cone) is universal (independent of the cone).*

This universal statistic coincides with the asymptotic statistic for the rational cones and for the invariant cones of the integer matrices (which one obtains pushing to infinity the parameter N, bounding the matrix elements, of the coefficients of the integer equations of the faces of the cone).

As *reasonable* characteristics I have ment, for instance, the distribution of the integral lengths of the edges, the distribution of the 2-faces according to the number of their edges or by the integral areas and by the affine integer equivalence classes, the distribution of the vertices according to the number of edges containing them and the like.

In 1997 I have proposed the problem of calculation of these statistics to Ju. Sukhov (Cambridge). It resulted in a remarquable work by Sukhov and M. Kontsevich. Among other things they proved the following

Theorem 2. *All the statistical asymptotic characteristics mentioned above (as well as many others) do exist for almost every simplicial cone and are universal (independent of the cone).*

Unfortunatly, these distributions are still not calculated (while their calculation is reduced to the summation of some series in polylogarithms). For instance, it is unknown, whether the distribution of the integral lenghts of the edges of the sails is shifted toward the longer or toward the shorter lenghts (with respect to the distribution of the integral lenghts of the random integer segments and with respect to the elements of the continued fraction of a random real number).

It is also unknown, how fast is decreasing the probability p_k to have integral lenghts k for an edge of the sail of a random cone in $\mathbb{R}^n$ (conjecturally — as C/k^n).

For a random integral segment, p_k decreases like c_1/k^n; for the elements of the continued fraction of a random real number–like c_2/k^2. For $n=2$ one has $c_1 = 6/\pi^2$, $c_2 = 1/(\ln 2) > c_1$. In this case the distribution of the integral lenghts of the segments of a random sail is shifted (with respect to the distribution of the integral lenghts of the random segments) toward the longer segments. One could suppose it is also true for $n > 2$.

The main new idea of Kontsevich and Sukhov consists in the following inversion of the viewpoint. I suggested to fix the lattice and to move the simplical cone. They are fixing the simplicial cone (i. e. the «Cartan» commutative group H of matrices from $SL(n,\mathbb{R})$ diagonal in the same basis), moving the lattice.

The lattices with the fundamental parallelepiped of volume 1 naturally form the manifold

$$M^{n^2-1} = SL(n,\mathbb{R})/SL(n,\mathbb{Z})$$

(of the fundamental parallelepipeds quotioned by the choice of a basis in the lattice).

The commutative group H (of dimension $n-1$) acts on this manifold (since the coordinates rescalings act on the parallelepipeds). This action is ergodic.

The properties of the sail we are interested in are defined by the configuration of the orbit of this action with respect to some special hypersurface in M. The calculation of the asymptotical statistics is reducible to the calculation of the volumes of the corresponding domains in M, and the existence of these asymptotics — to the measureability of these domains.

Indeed, the decomposition of the sail into its faces is isomorphic to the decomposition of the orbits into parts by the intersections of the orbit with the following hypersurface in the lattices manifold M. A lattice is called *special* (belongs to the above hypersurface) if the diagonal $\{t,t,\ldots,t\}$ in $\mathbb{R}^n$ intersects the sail (defined by the lattice in the positive ortant) in a common point of several faces of the sail.

The structure of the hypersurface of the special lattices in M is not clear even at the homological level (how many components has its complement in M?).

Kontsevich has also proved, that the statistics of the segments of the sail defined by the lattice $\mathbb{Z}^2$ inside a random angle with a random vertex in $\mathbb{R}^2$ are almost always universal (independent of the vertex and of the angle). However these statistics are different from those corresponding to the vertice at a lattice point (that is, to continued fractions): for a random vertex

$$p_k = \left(\ln \frac{(k-1)^{(k-1)}(k+1)^{3(k+1)}}{k^{3k}(k+2)^{(k+2)}}\right) \Big/ (2\ln 2).$$

Rational vertices with denominator q define different universal statstics (for almost all angles having this vertex), which depend on q. These statistics are explicitly computable. It would be interesting to understand the behavior of the statistics of the integral lenghts of the segments of the sail defined by a random angle whose vertex is an algebraic point.

The number of edges in the part of the sail at distance at most t from the vertex grows (with probability 1) in all these cases similarly, namely as $c\ln t, c = (\ln 2)/(\zeta(2))$.

It seems that the statistics of this type are more stable also for higher dimensions n.

4. Uncertainty relations

The two conjectures formulated below were invented as byproducts of the theories described above. I had proved them only with superflans logarithms in the asymptotic, but later Kontsevich and Korkina have proved them completly (the Korkina's proof providing also the values of the constants).

Consider a closed subgroup G of the Euclidean space $\mathbb{R}^n$, such that the quotient space is a compact torus $T^m, m \leq n$ — for instance, a lattice Γ in $\mathbb{R}^n$ is a relevant example of G.

Suppose, that the complement to G contains a ball of radius r.

Conjecture 5. *The dual Euclidean space contains a nonzero «wavevector» k, such that $(k,x) \in \mathbb{Z}$ for all points of G, the length of k being at most C/r (where the constant C depends only on the dimension n).*

Conjecture 6. *Denote by $r(\Gamma)$ the maximal radius of the open balls centred at the points of lattice Γ which do not intersect each other. Denote by $R(\Gamma)$ the minimal radius of closed balls centred at the points of lattice Γ which cover all the space $\mathbb{R}^n$. Then the product*

$$r(\text{lattice } \Gamma) \cdot \mathrm{R}(\text{dual lattice } \Gamma^{\mathrm{V}})$$

is bounded from below and from above by two constants which depend only on the dimension n.

REMARK 2. The estimation from above is close to Conjecture 5. The estimation from below is proposed by M. Kontsevich. All the three estimations ought to be known to Minkovski, but it seems that they were not formulated even by his followers (see however [18, 19]). The bounds from below and from above, provided by Korkina, are $1/4$ and $\sqrt{3}\frac{\sqrt{(4/3)^n-1}}{4}$. The bound from below is the best possible, but that from above is not.

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МНОГОМЕРНЫЕ ЦЕПНЫЕ ДРОБИ

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Многомерным аналогом цепной дроби является полиэдральная поверхность, ограничивающая выпуклую оболочку полугруппы целых точек в симплициальном конусе евклидова пространства. В статье описаны гипотезы и теоремы, переносящие на такие многомерные цепные дроби теорему Лагранжа о квадратичных иррациональностях и статистику Гаусса–Кузьмина.