Functional Analysis and Its Applications, Vol. 32, No. ?, 1998

Towards the Legendrian Sturm Theory of Space Curves*

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§1. Introduction

Sturm theory extends the Morse inequality (minorating the number of critical points of functions on a circle) to the higher derivatives. The Legendrian Morse theory (created by Yu. V. Chekanov in 1986) provides the Morse inequality for the multivalued functions (corresponding to the unknoted Legendrian submanifolds of the space of 1-jets of functions). It is a generalization of the Lagrangian intersection theory due to Conley, Zehnder, Chaperon, Floer, Sikorav, Laudenbach, Hofer, Gromov, and others. Below an attempt to extend the Legendrian Morse theory to higher derivatives is presented. It extends the Legendrian Morse theory in the same sense in which the ordinary Sturm theory extends Morse theory. Sturm theory implies the existence of at least four flattening points on every curve in projective 3-space having a convex projection. The present paper results imply, for instance, the preservation of at least four flattening points under (not necessary small) deformations of the standard curve of Sturm theory ($x = \cos t$, $y = \sin t$, $z = \cos 2t$), provided that the curve has no inflection (zero curvature) points at any moment and that the Legendrian knot type remains unchanged under the deformation. The Legendrian knot is a singular surface in the five-dimensional manifold of the contact elements of the space, associated to the curve. In terms of the front of the curve (which is a surface in the dual projective space) the restrictions on the deformation forbid the selftangency of the front, the selfintersections of its cuspidal edge and the tangencies of the cuspidal edge to the front, whose cuspidal edge should remain irreducible during the deformation. The main idea of the present paper is the attempt to isolate in the topological study of the evolution of the fronts of the space curve the part which is independent on the fact that the front is related to a curve, making it applicable to the more general fronts of arbitrary Legendrian varieties. In a similar way, the theory by A. V. Alexeev and V. A. Vassiliev of the earlier 80ths (described for instance in [5] and [7]) already contained in a more general form many results of [6].

§2. The Legendrian Hulls of Curves in Projective Space P^N and Their Fronts

I first recall some basic facts of the contact geometry of curve γ in the N-dimensional projective space P^N (due to O. P. Shcherbak [1]). We shall deduce below global topological corollaries of this local theory. The *Legendrian hull* L of curve γ is the manifold formed by all the contact elements of P^N, tangent to γ. It is a smooth Legendrian submanifold of the manifold $E = PT^*P^N = PT^*(P^N)^\vee$ of contact elements on P^N.

* Partially supported by the Russian Foundation for Basic Research, project 96-01-01104.

V. A. Steklov Mathematical Institute and CEREMADE, Université Paris-Dauphine. Translated from Funktsional′nyi Analiz i Ego Prilozheniya, Vol. 32, No. 2, pp. ??–??, April–June, 1998. Original article submitted December 29, 1997.

A point of E is a pair (a,A) where a is a point of P^N and A is a hyperplane containing this point. The manifold E of contact elements of P^N is the total space of two Legendrian fibrations (over the initial and over the dual projective spaces):

$$\rho\colon E \to P^N, \quad \rho^\vee\colon E \to (P^N)^\vee \qquad (\text{where } \rho(,A) = a\,,\ \rho^\vee(,A) = A\,).$$

The *front* F of curve γ is the projection $F = \rho^\vee L$ of the Legendrian hull of the curve into the dual projective space. The dimension of the front equals $N-1$. The front of a generic curve has only the standard Legendrian singularities of type A_k (in the three space they are just the semicubical cuspidal edges A_2 and the swallowtails A_3). The front is the "*tangents developable*" (the union of the $(N-2)$-dimensional osculating spaces) of the dual curve $\gamma^\vee \subset (P^N)^\vee$ (in the three-space-the union of its tangent lines). The *dual curve* $\gamma^\vee$ is formed by the osculating hyperplanes of the original curve. The singularities of the curve, dual to a generic curve of the three-space, are the semicubical cusps (corresponding to the flattening points of the original curve, to the swallowtail vertices of the front and to the folded umbrellas vertices on the "tangents developable" surface of the original curve). In a neighborhood of such a point the original curve admits the parametric equations ($x = t\,,\ y = t^2 + \dots\,,\ z = t^4 + \dots$) (where the points mean higher order terms) in suitable affine coordinates (x,y,z). The dual curve is described by the equations ($p = t^2 + \dots\,,\ q = t^3 + \dots\,,\ r = t^4 + \dots$) in suitable local affine coordinates of the dual space. In the generic one-parameter families of curves in P^3 there may occur (for some isolated values of the parameter) two more events: the *biflattening* of the curve (the degrees (1,2,5) replacing (1,2,4) in the original space and the degrees (2,3,5) replacing (2,3,4) in the dual space) and the *inflections* of the curve (degrees (1,3,4) in both spaces). The corresponding perestroikas of the fronts are the standard Legendrian perestroikas of type "butterfly" (A_4) in the first case and "bec to bec" (A_3) in the second. We shall give below the minoration of the number of the topologically necessary flattening points on a curve, obtained from a fixed curve of the projective space by a generic regular homotopy (in terms of the selfintersection lines of the front of the original curve or of the selfintersection lines of its "tangent developable" and of their perestroikas under the homotopy).

§3. The Sturmianity of a Curve of Projective Space

Orient the projective space P^N, $N = 2n+1$. In a neighborhood of a flattening curve a generic parametrized curve is defined by the equations ($x_1 = t\,,\ x_2 = t^2 + \dots\,,$ $x_{N-1} = t^{N-1} + \dots\,,\ x_N = t^{N+1} + \dots$) in an affine coordinate system whose origin is the flattening point.

Definition. A flattening point is *positive*, if this coordinate system coorient the space P^N positively (and *negative* in the opposite case).

The signs of the consecutive flattening points on a generic curve alternate.

Remark. To define the sign one might also use the dual curve. The reversal of the curve direction multiplies the signs of all its flattening points by $(-1)^n$.

The number of the flattening points of a closed curve in an odd-dimensional projective space is even, since the sign of the osculating frame changes under each transition through a flattening point. There is an involution with no fixed points acting on the set of the flattening points of a generic closed curve in the odd-dimensional projective space: each flattening point has its *joint point*. The joining is due to the fact, that each swallowtail vertices is an endpoint of a selfintersection line of the front, starting at some other swallowtail vertices.

DEFINITION. A *pair of joint vertices* of swallowtails of a generic front in a three-manifold is a pair of end points of a segment of a front selfintersection line (this line may contain cusps and intersection points of two nontangent branches of the selfintersection line).

REMARK. Similarly, one might define the pairs of the joint swallowtails vertices on the fronts in $(2n+1)$-manifolds, having just A_k-singularities.

In a neighborhood of the vertices of a "swallowtail A_{2n+1}" the front is diffeomorphic to the variety of the polynomials $x^{2n+2} + \lambda_1 x^{2n} + \cdots + \lambda_{2n+1}$ having a multiple real root. The vertices $(\lambda = 0)$ is the endpoint of the curve of the polynomials $2A_n$ of the type $(x^2 - a^2)^{n+1}$.

DEFINITION. Two flattening points of a generic curve in an odd-dimensional projective space form a *joint pair*, if the corresponding vertices of the front swallowtails from a joint pair.

REMARK. In the original-three space the flattening points forming a joint pair are joint by a segment of the selfintersection line of the "tangent developable" curve. This follows from the fact, that the intersection point $q \in P^3$ of the tangent lines l_a and l_b of the original curve γ at a and at b belongs to the plane Q, generated by l_a and l_b. This plane is a point $Q \in (P^3)^\vee$ of the dual space, contained in the lines L_A and L_B of the dual space, tangent to the dual curve $\gamma^\vee$ at the points A and B corresponding to a and b. Namely, the line L_A consists of those planes of the original space, which are tangent to γ at a, and L_B at B.

Thus the point q of the selfintersection of the "tangent developable" of the curve γ (which is the front surface of the dual curve $\gamma^\vee$) corresponds to a selfintersection point Q of the front F of the curve γ (which is the selfintersection point of the "tangent developable" of the curve $\gamma^\vee$ dual to γ). The point Q sweeps the segment of the selfintersection line of the front F of the curve γ joining two front's swallowtail vertices while q sweeps the segment of the selfintersection line of the "tangent developable" of the original curve γ joining two flattening points of γ.

MAIN DEFINITION. The *sturmianity* s of a generic curve in an odd-dimensional projective space is the number of joined pairs of flattening points of the curve having equal signs.

REMARK. The sturmianity of a closed curve is even, since the number of positive pairs is equal to the number of negative ones.

EXAMPLE. The sturmianity of the standard curve of the Sturm theory in P^{2n+1}

$$x_1 = \cos t, \quad y_1 = \sin t, \quad \dots, \quad x_n = \cos nt, \quad y_n = \sin nt, \quad z = \cos(n+1)t,$$

where n is odd, equals $s = n + 1$ (the joined points being opposite).

The sturmianity of the curve ($x = \cos t$, $y = \sin t$, $z = \cos 2nt$) in the 3-space is $s = 2n$ (the joined points being opposite).

§4. The Minoration of the Number off Flattening Points

Consider a smooth closed generic curve γ, embedded into the projective 3-space (genericity means that we suppose that the curve torsion is vanishing at most at finite set of first order roots).

DEFINITION. A regular homotopy of curve γ is called *admissible*, if it is a generic regular homotopy (containing a finite set of transversal biflattening but no inflections) whose front experiences no (i) selftangency moments; (ii) moments of selfintersection of the cuspidal edge (iii) moments of tangency of the cuspidal edge of the front with the front.

REMARK. A selftangency of the front means a selfintersection of the original curve. A selfintersection of the cuspidal edge of the front means a selfintersection of the dual curve (and hence a selftangency of the "tangent developable" of the original curve). A tangency of the cuspidal edge of the front with the front means a tangency of the original curve with one of its osculating planes (and hence a tangency of the original curve with its "tangent developable" surface).

At the inflection moment the cuspidal edge would be reducible (would contain a straight line).

THEOREM. *The sturmianity of a curve remains constant under the admissible homotopies.*

COROLLARY. *The number of flattening points of a curve experiencing an admissible homotopy remains bounded from below by the doubled sturmianity of the original curve.*

REMARK. The above lower bound of the number of the flattening points is a multiple of 4.

PROOF. The list of the perestroikas in the typical one-parameter families of the fronts (in the spaces of not two big dimension) is discovered in [2]. O. P. Shcherbak has proved in [1] that the perestroikas of the fronts of typical one-parameter families of curves in the projective 3-space form the following part of this list:

I. *The perestroikas changing the number of the swallowtails.* 1. A_4 (the Thom's "butterfly"): the birth or death of a joined pair of two close vertices of the swallowtails on the cuspidal edge of the front (connected by a short segment of the front selfintersection line, containing two cusps). 2. A_3 (the Thom's "bec to bec"): the birth or the death of two close vertices of the swallowtails on the cuspidal edge of the front (which are separated by a long segment of the swallowtail). II. *The perestroikas preserving the* *swallowtails number.* 3. The front selftangency (the Morse perestroika of the trace, formed by the intersection with a different smooth branch of the front on a given branch). 4. The tangency of the cuspidal edge of the front with a smooth branch of the front (accompanied with the birth of the death of two cusps on the front selfintersection line). 5. The selfintersection of the cuspidal edge of the front (moving two cusps of the front selfintersection line from one branch of the cuspidal edge to another). 6. The passing of a smooth branch of the front through the vertex of the swallowtail (accompanied with the birth or the death of two cusps on the front selfintersection line). 7. The transition of the cuspidal edge of the front through the intersection line of two smooth branches of the front (accompanied with the transition of this intersection line through a cusp of another branch of the selfintersection line). 8. The transition of four smooth branches of the front through the same point (all the six lines of the pairs of branches intersection moving smoothly under the perestroika). 9. The tangency of the front selfintersection line with the smooth part of the front (all the three lines of the pairs of branches intersection moving smoothly under the perestroika). We should prove that the sturmianity remains unchanged under the admissible perestroikas. This is obvious for the perestroikas 6,7,8,9 which are not changing neither the joined pairs nor the signs of the swallowtails vertices. The perestroikas 3, 4, 5 are forbidden by the admissible deformations definition. The perestroika 1 happens at the biflattening moment. This is the moment of the birth or of the death of a joined pair of neighboring flattening points of the curve. They have opposite signs and they are joined by a selfintersection segment on the front (or on the "tangent developable" surface). Hence this birth or death does not change the sturmianity. The perestroika 2 happens only at the inflection moment. Since admissible homotopies have no inflection moments, the theorem is proven. The proof shows also what is happening to the sturmianity under the nonadmissible perestroikas 2, 3, 4, 5. Thus we might evaluate the sturmianity (and hence the number of the flattening points) of the deformed curve in terms of the sturmianity of the original curve and of the number of the nonadmissible perestroikas.

Remark. At the transition of an inflection moment two close flattening points are born or killed on the curve. These points have opposite signs, but they are not necessarily joint. The two points, joint with them, might have the same signs as the joint vanishing flattening points respectively. In this case the sturmianity of the curve is decreased by 2 at the death of the vanishing flattenings.

If at least one of the joint flattening points has a sign different from the sign of the vanishing flattening point joint to it, the sturmianity of the curve does not change under the inflection transition. The transition through such inflection perestroikas might be included into the list of the admissible perestroikas. The same way, one might admit the "elliptical" perestroikas 3, 4, and 5 (accompanied with the birth or the death of small front selfintersection lines). They do not change the curve sturmianity. One also might get better minorations distinguishing the joining segments of the selfintersection line of the "tangent developable" surface which are contractible to γ and which are not.

§5. Examples

Example 1. Consider the space curve described by the equations ($r = 1 - a\cos\varphi$,

$z = \sin 2\varphi$) in cylindrical coordinates. While the parameter a is growing from 0 to $1/2$, this curve has four flattening points. Two of them vanish (at the inflection point $\varphi = 0$) for $a = 1/2$.

While a is smaller than $1/2$, the curve has a quadratically convex projection and two alternating joint pairs of flattening points of signs $(++)$ and $(--)$. The sturmianity of this curve is hence equal to 2. At the transition of the parameter a through the value $1/2$ the perestroika 2 happens. The sturmianity is diminished by 2 and the resulting curve has only 2 flattening points. For instance, the resulting curve has no convex projection (from any point of the projective 3-space), since a curve having a convex projection has at least 4 flattening points (see [3, 4]). This example shows that one should not accept the inflections transitions, at least in the case where the inflections joint to the vanishing ones have the same signs as the corresponding vanishing inflections.

Example 2. Consider the space curve, defined by the equations ($x = \cos\varphi$, $y = \sin\varphi$, $z = \cos 3\varphi$). This curve has 6 flattening points. The selfintersection line of the "tangent developable" surface consists of 3 straight lines segments joining the opposite flattening points intersecting the infinite line and of the infinite line $z = 0$. At the intersection points of the radial lines with the infinite line both smooth branches of the "tangent developable" surface are Morse tangent to each other.

A small deformation of the original curve ($z = \cos 3\varphi + \varepsilon f(\varphi)$) modifies the selfintersection line into a smooth curve, and the joint pairs may be $(+-)(+-)(+-)$ along the curve (say, for $f = \sin 6t$). The sturmianity of the deformed curve is then equal to zero. For the deformations of this curve our theorem does not, guarantee any flattening points, while the curve has a convex projection. The "tangent developable" surface of a curve having a convex projection may be considered as a two-valued section of a one-dimensional line bundle over the Möbius band (which is the complement in the projective plane to the convex domain, bounded by the projection of the curve). The projections of the selfintersection lines of the "tangent developable" surface divide the Möbius band into connected parts. These parts may be colored into two colors (the color being changed at each generic crossing of the projection of the intersection line). If the parts are orientable, the sturmianity of the original curve cannot be smaller than 2. In this case the number of the flattening points is at least 4 (and remains at least equal to 4 under the admissible homotopies). Example 2 shows that one of the parts may be nonorientable and the sturmianity may vanish even for the curves having a convex projection.

Example 3. The immersed curve ($x = \sin 2t$, $y = \cos 2t$, $z = \cos 4t$) has 8 flattening points. A small deformation, making it an embedded curve, transforms this curve into a generic curve of sturmianity 4. This curve has no convex projections (since the planes containing the axis intersecting the curve more than twice sweep all the projective space).

Remark. One might probably extend the theorem to the more general case of the fronts of the Legendrian varieties which are not related to the curves. What is important is the distinction between the swallowtails of different signs. The Legendrian variety associated to the curve is the union of the Legendrian hulls of the original curve and of its dual curve. Even the local normal forms of the singularities of these varieties have not been studied as they should be in contact geometry (as far as I know).

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Translated by V. I. Arnold