Arnold's Problems — 2001

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Problem 1. (V. I. Arnold, A. Ortiz) *Betti numbers of parabolic sets.*

Let $f(x, y)$ be a real polynomial in two variables. Denote by $P(f)$ the set of parabolic points on the surface $\{z = f(x, y)\}$, i.e. the zero set of the Hessian $H[f] = f_{xx}f_{yy} - f_{xy}^2$. Determine the maximal number of compact connected components of the set $P(f)$ for all polynomials f of given degree n.

This problem can be viewed as a specialization of the classical *oval counting problem* for polynomials representable in the form of a Hessian.

The first case when the answer is unknown is $n = 4$. Then $m = \deg H[f] = \deg f = 4$, and the Harnack inequality ensures that $b_0(P(f)) \leq (m-1)(m-2)/2 + 1 = 4$. There is a well-known construction of a polynomial ($uv - \varepsilon$, where $u = 0$, $v = 0$ are equations of ellipses that intersect in 4 points and ε is a small number) for which this estimate is attained. It is not known if it can be attained for polynomials of the form $H[f]$.

Problem 2. (V. I. Arnold) *Caustics of periodic functions.*

Let $g : S^1 \to \mathbb{R}$ be a smooth function and u, v two real parameters. The plane curve

$$C_g = \{(u, v) \mid \text{ function } G_{u,v}(\varphi) = g(\varphi) + a\cos\varphi + b\sin\varphi \text{ is not Morse}\}$$

is called the *caustic* of the function g. The condition that $G_{u,v}(\varphi)$ is not Morse means that there exists a value $\varphi \in S^1$ such that $G'_{u,v}(\varphi) = G''_{u,v}(\varphi) = 0$ (derivatives over φ).

Example. The caustic of the function $g(\varphi) = \cos 2\varphi$ is the astroid $u = -4\cos^3\varphi$, $v = 4\sin^3\varphi$.

For generic (Morse) functions g caustics are fronts (smooth curves with generic singularities) that satisfy a number of specific conditions:

1. A caustic has at least 4 cusps.
2. The number of cusps is even.
3. If P_1, P_2, ..., P_{2n} are cusps, then the barycentres of the sets P_1, P_3, ..., P_{2n-1} and P_2, P_4, ..., P_{2n} coincide. In particular, if $n = 2$, they form a parallelogram.
4. The alternating length of a caustic (we change sign after each cusp) is 0.
5. From every point of the plane one can draw at least two tangents to the caustic.
6. Caustics do not have inflexion points.

Problem. Describe all curves that are caustics of periodic functions, i.e. give a necessary and sufficient condition for a front to be a caustic.

These are two out of five problems announced on September 25, 2001 at Arnold's seminar.

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