Problems to the Seminar : 15 january 2002

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1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial of degree D. Find the maximal number of the connected components and the maximal number of closed components of the *parabolic curve* of its graph (where $f_{xx}f_{yy} = f_{xy}^2$) :

$$b_0\ (Par(f))\ = ?, \quad b_1(Par(f))\ = ?$$

Even for $D = 4$, it is not known whether b_1 attaines the value 4, and the constants C in the lower and upper bounds for large degrees D, $b_1 \sim CD^2$, differ 4 times :

$$(D-1)(D-2)/2\ \leq b\ \leq (2D-5)(D-3)+1\ .$$

2. Let $M \subset \mathbb{R}P^3$ be a *a smooth algebraic surface* of degree D. Find the maximal number of the connectec components of its parabolic line.

The constants C in the lower and in the upper bounds CD^3 differ 20 times :

$$D(D-1)(D-2)/2 \leq b \leq 10D^3 - 28D^2 + 4D + 3\ .$$

The lower estimates in problems 1 and 2 mean the existence of the surfaces with many close parabolic curves.

*Partially supported by RFBR-02-01-00655

3. Let $F : S^1 \to \mathbb{R}$ be a smooth function; it is called D-hyperbolic if the second differential d^2f of homogeneous function $f(x, y) = r^D F(\varphi)$, (where $x = r\cos\varphi$, $y = r\sin\varphi$), is hyperbolic (has the signature $(+, -)$) everywhere in $\mathbb{R}^2 \backslash 0$.

Find the connected components of the space of D-hyperbolic functions : *is the index* (equal to the number of te rotations of the cross $d^2f = 0$ when the point (x, y) makes one turn around the origin) *the unique invariant of the connected component ?* The set of the values attained by the index is infinite and is unbounded from below (but bounded from above).

4. For the polynomial case (where F is a trigonometric polynomial and f an ordinary homogeneous polynomial of degree D) find the *number of the connected components of the D-hyperbolic polynomials.* Is it growing linearly in the degree D when it is high ?

More details on the problems 1-4 are described in my recent paper "Astroidan geometry of hypocicloids and hessian topology of hyperbolic polynimials", Russ. Math. Surveys 2001, Vol. **56** N^{o}6.

5. Consider a *controlled dynamical system* $\dot{x} = v(x, u)$, where x is a point of compact phase manifold M and u belongs to a compact controlling parameters manifold U. Let $f : M \to \mathbb{R}$ be a smooth *goal function.*

Study the *mean optimisation problem, maximising the time average* $\hat{f} = \lim T^{-1} \int_0^T f(x(t))dt$ by a clever choice of the control $u(t)$ (and virtually of the initial state $x(0)$).

If the problem (that is, the pair formed by v and f) depends generically on some exterior parameters, the optimisation strategy and the optimal average might have singularities ("phase transitions") at the points of a *hypersurface of phase transitions* in the manifold P of the values of the exterior parameter.

Find the *generic phase transitions in the mean optimisation problem*, at least when the dimensions of the manifolds M, U, P are not big. The problem is open even wen all these manifolds are 1-dimensional, where there are already some nontrivial stable singularities (see my paper in Functional Analsis and its Appl., 2002, Vol. **36** N^{o}2).

6. Let $f : M \to \mathbb{R}$ be a smooth function on a compact Riemann manifold, and $0 < r < R < \infty$ - two smooth functions on M. Study the *mean value* *optimisation problem* for the space average

$$\hat{f} \ = \ \Big(\int_M f(x)\rho(x)dx\Big)/\Big(\int_M \rho(x)dx\Big)$$

for the mass distribution defined by the density function ρ with respect to the euclidean volume element dx provided that this density is restricted by te inequalities $r \le \rho \le R$ everywhere on M.

Study the generic phase transitions for the case where f, r and R depend smoothly on exterior parameters.

It is known that the optimal strategy is $\{$ to choose $\rho = r$ where $f(x) < c$ and $\rho = R$ where $f(x) > c$ for some constant $c\}$, but the study of the phase transitions requiers the investigation of the influence of some strange logarithmic singularities and of their regularisations in the case of even-dimensional manifolds M, like in many physical problems. See the article "On a variational problem related to the phase transitions of the averages in controlled dynamical systems" by V. Arnold in the "International Mathematical Series. Nonlinear Problems in Mathematical Physics I, In Honor of Professor O.A. Ladyzhenskaya", Kluwer / Plenum Publishers (www.wkap.nl).

7. Let $u_0 : M^2 \to \mathbb{R}$ be a smooth "initial" function on a Riemann manifold M (the case of a 2-dimensional ball B^2 is already relevant). Study the *minimisation problem for the Dirichlet integral* $\int_M (\nabla u)^2 dx$, where the function u is obtained from the initial function u_0 by an area-preserving diffeomorphism of M onto itself ("*by an incompressible fluid motion*").

The extremal function u is smooth if the smoot initial mountain $u_0 : B^2 \to \mathbb{R}$, vanishing o the boundary of the ball, has just one nondegenerate (Morse) maximum inside the ball. In this case the extremal function u is the symmetrisation of u_0 (depending only on the distance from the centre of the ball).

But for the initial smooth mountain u_0 having (like the Elbrous mountain) two local maxima points, separated by a saddle point, the extremal function seems to have a $|x|$ type singularity along a curve with unknown extremal function singularities at its endpoints. The problem is to study such singularities for generic u_0.

The problem 7 is related to the *Euler equtions for the stationary states in two-dimensional incompressible hydrodynamics and magnetohydrodynamics*. The smooth version of this equation is the functional dependence of some function f on M^2 with its Laplacien (*their Poisson bracket should vanish identically).*

And the problem 7 is to extend this equation to the "generalized functions" case. This problem is also a simplified (2-dimensional) version of Sakharov's 3-dimensional problem of a *minimal energy magnetic field of a star* (in the 3-dimensional case the Euler equation means the vanishing of the Poisson bracket of a divergence-free vector field with its curl field). More details (and references to my initial, 1974, paper on this subject) are contained in the book with Khesin "Topological Methods in Hydrodyamics", (Applied Mathematical Sciences **125**, Springer, 1998, pp. 69-81 and pp. 112-193).

8. The (C, B, A)-*permutation of the set* $\{1, 2, \ldots, a\}$, transports to the last place the subset $A = \{1, 2, \ldots, a\}$, preceeded by the transported set $B = \{a + 1, \ldots, a + b\}$, while the starting position is occupied by $C = \{a + b + 1, \ldots, n\}$.

Some of these $(n - 1)(n - 2)/2$ permutations permute *cyclically* (like the addition of a constant to the residues mod n), and some of these cyclic permutations are *transitive* (like the addition of the constant 1).

Find *the proportions of the cyclic and the of the transitive cyclic permutations among the* (C, B, A)-*permutations for high values of* n.

Among the 45 (C, B, A)-permutations for $n = 11$, there are 22 cyclic permutations (which are all transitive), and for the preceeding values of n the cyclic and the transitive cyclic permutations numbers form two strange nonmonotone sequences.

Among all the $n!$ permutations of n elements these which are transitively cyclic form a small part (their number is $(n - 1)!$), and the number of all the cyclic ones seems to be asymptotically equivalent to the same $(1/n - th)$ part of the total number of permutations as that of the transitively cyclic.

Thus it seems, that *the statistics of the* (C, B, A)-*permutations is distinctively different from that of the generic ones* (and one might study the same difference for the other permutations of few letters, acting on a large set $\{1, \ldots, n\}$).

All these questions had been formulated at my 1958-th Moscow seminar as the simplified models for the segments interchange mapping problem in dynamical systems theory, which was later in my 1963 long paper (in Russian Mathematical Review, **18**, issue 6) on Hamiltonian dynamics. The present day state of this problem (studied especially by M. Kontsevich and A. Zorich) is described in the Zorich's recent paper "How do the leaves of a closed differential form wing around a manifold" in the book "Pseudoperiodic topology", AMS 1999 (V. Arnold, M. Kontsevich, A. Zorich Eds).

Kontsevich had announced recently, commenting the new edition of the Book "Arnold Problems", that the 3 segments interchange is always rotation equivalent. Problem 8 requiers to study the *frequency* of the counterexamples to this "equivalence" (the existence of such counterexamples, being, of course, known to me in the 1950-th, while I was inventing the segments interchange problem).

9. A mapping $\mathbb{C}^n \to \mathbb{C}^n$ (or $\mathbb{C}P^n \to \mathbb{C}P^n$) is called a *pseudocomplex* mapping, if it *sends the complex subspaces to complex subspaces* (one may consider separatly the cases of the linear affine or projective subspaces - all the 3 versions are intersting).

A real diffeomorphism $\mathbb{C}P^2 \to \mathbb{C}P^2$ is pseudocomplex if and only if it is either a complex projective mapping, or its product with the complex conjugation (and similarly for the other versions and for the other ns).

Do there exist other pseudocomplex homeomorphisms ? Other pseudocomplex bijections ?

These questions ought to be studied by Hilbert as a part of the axiomatic projective geometry, but his school missed, it seems, these foundational problems.

10. To formulate the quaternionic versions of the preceeding problem, one should distinguish the left-subspaces and the right-subspaces. I would suggest to study those mappings, which send the left and the right subspaces onto the left and right subspaces (permitting to send a left one onto a right one).

More details on te (real smooth) versions of problems 9 and 10 can be found in my 2 papers in Functional Analysis and Its Applications, 2001, Vol. **35** N4, pp. 1-7 and 2002, Vol. **36** N1, pp. 1-15. ("On the complexification of the tetrahedron" in the first one and "On the pseudo-quaternionic geometry" in the second).

11. The complexifications and quaternionizations paradigma had been used by me many times (see, for instance, "Polymathematics : is mathematics a single science or a set of arts" in "Mathematics : Frontiers and

Perspectives", IMU-AMS, 1999 (V. Arnold, M.F. Atiyah, P. Lax, B. Mazur - Editors) and my simplectizations and complexifications in the Vol. **24** of the Toronto Fields Institute "Communications" ("Arnold fest", 1997) pp. 23-27).

For instance, it is now proved, that the *complex version of the tetrahedron is the octahedron* : ${}^{\mathbb{C}}A_3 \; = \; B_3$.

The problem is now to prove my old conjecture that its quaternionic version is the icosahedron :

$$ {}^{\mathbb{H}}A_3 \; = \; H_3 \; , \quad {}^{\mathbb{C}}B_3 \; = \; H_3 \; . $$

Perhaps, one should start with the easier plane versions :

$$ {}^{\mathbb{C}}A_2 \; = \; B_2 \; , \; {}^{\mathbb{H}}A_2 \; = \; H_2 \; , \; {}^{\mathbb{C}}B_2 \; = \; H_2 \; , $$

relating the symmetries groups of the triangle, of the square and of the pentagon.

The difficulty of all this subject lies in its nonmathematical character : *the problem is to find the definition* or the informal quaternionization operation rather than to prove any ready mathematical statement.

12. The *caustic of a periodic function* $g : S^1 \to \mathbb{R}$ is the curve in the plane $\mathbb{R}^2$ of the functions

$$ G_{A,B} : S^1 \to \mathbb{R} \; , \; G_{A,B}(\varphi) = g(\varphi) + A\cos\varphi + B\sin\varphi \; , $$

consisting of those functions, which are not the Morse functions :

$$ \Big\{ (A,B) \in \mathbb{R}^2 : \exists\varphi : G'_{A,B}(\varphi) = G''_{A,B}(\varphi) = 0 \Big\} \; . $$

The caustics of generic periodic functions have many peculiar properties: for instance, *each caustic has at least four cusps*, and *its alternative lenght* (the alternating sum of the lenghts of its segments beween the cusps) vanishes. *The cusps* of a caustic having just 4 cusps *form a parallelogram* (and the baricenters of the odd and of the even cusps coincide if there are more than 4 cusps).

The problem is to replace in this theory the smooth periodic functions by the exact Lagrange submanifolds of the phase cylinger T^*S^1. Such a submanifold, corresponding to a function, is the graph of its differential. The general Lagrange submanifold does not need to be a section of the cotangent bundle, and the graph of the corresponding *"multivalued potential"* function does not need to be an immersed curve : it may have cusps.

It is intersting to understand, which would be the 4 cusps property version for the caustics of such Lagrange exact submanifolds, and what will happen to the *Sturm-Hurwitz theorem on the Fourier series zeros* (being the infinitesimal version of the caustics cusps theorem) *for such extended "multivalued periodic functions"*.

The past experience showed that one should extend the Morse theory of ordinary functions to the *Lagrange manifolds intersection thoery of the symplectic topology* (the "Arnold conjectures" of 1965, extending the "Poincaré last theorem" and being the starting point of the Floer homology and of many other things in symplectic and contact topology).

Recently Chekanov and Puschker has used this extension to prove my 1993 conjecture on *the necessity of the 4 cusps at any wave front eversion* (V. Arnold. Topological invariants of Plane Curves and Caustics, AMS, University Lecture Series, Vol. **8**, Providence, RI, 1994, 60pp). However it is unclear to me, whether their version of the Chekanov contact cohomology theory suffices to understand the caustics of the exact Lagrange submanifolds (that is of the Legendrian knots in the manifold of the 1-jets of periodic functions).

13. The theory of the caustics of periodic functions and of Lagrange submanifolds, discussed above, depends on the functions $x = \cos\varphi$ and $y = \sin\varphi$ on the circle $x^2 + y^2 = 1$. *Replacing the circle by a different curve*, say, by algebraic curve $C : f(x, y) = 0$, and g by the restriction to C of a function on the plane, say of a polynomial $P(x, y)$, we define the C-caustic (as the curve of non-Morse restrictions of the functions $P + Ax + By$ to C).

The problem is now *to extend the Sturm-Hurwitz Fourier series theorems* (as well as their extensions described in the problem 12 and in more details in the article on the astroidal geometry in "Russ. Math. Rev." 2001, Vol. **35** N^{o}6) *to the C-caustics*, associated to more general curves C, than the circle used in problem 12.

These caustics are algebraic curves (of the same genus as C). In the case of problem 12 they were rational curves, since the circle is, and thus the caustics of the trigonometric polynomials are unicursal curves and their Riemann surfaces are spheres.

By the way, to each periodic function in problem 12 and to each P in the present case one can associate a 1-*parameter family of "wavefront curves"*, consisting each of those points of the plane $\{(A,B)\}$ for which the restriction $P + Ax + By/_C)$ has a fixed critical value p (which is the parameter of the family).

These fronts are algebraic curves (of the same genus as C) rational in the case of the trigonometric polynomials of problem 12.

The problems on the cusps, alternative lenghts e.t.c. for the C-caustics and C-fronts for the algebraic curves C of higher gene are especially interesting in the complex case, since the real 4 cusps properties are known, for instance, for the smooth convex curves case (and with more cusps - for some generalizations of te convexity). But even in the rational (genus zero) case, and even for the degenerate elliptic curve, $y^2 = x^2 + x^3$, there are interesting problems relating the cusps of the caustic and the double points of the degenerate curve. There has been published some extensions of the Sturm-Hurwitz theorem on te Fourier series to the higher genera case (by S. Natanson). But I am not able to use them to obtain any usefull knowledge on the C-caustics and C-fronts for the algebraic curves C of higher genera.

14. Study the *torus T^2 triangulations, associated to the cubical algebraic numbers fields (by the theory of the higher-dimensional continued fractions).* One starts from a matrix $A \in SL(3,\mathbb{Z})$ having 3 positive eigenvalues. The 3 corresponding invariant planes subdivide $\mathbb{R}^3$ into 8 invariant octants. Each open octant contains the semigroup of its integer points. The boundary of the convex hull of this set (of integral points, $\{\mathbb{Z}^3 \cap$ octant $\}$) is called the sail. The sail is invariant under A. It is an (infinite) polyhedral surface, whose faces are bounded by convex compact polygons. it had been proved (by Dirichlet and H. Tsuchihashi, Tohoku Math. J., **35**, 1983, 607-639) that the sail is invariant under the action of a commutative subgroup $\mathbb{Z}^2$ of $SL(3,\mathbb{Z})$, formed by the matrices with the same eigenplanes.

The torus dealed with in problem 14 is the quotient-space

$$T^2 \ = \ (\text{sail of } \ A) \ \ /\mathbb{Z}^2 \ .$$

It is subdivided into the images of the faces of the sail under factorisation mapping. Each image contains some "integer points" (the images of the integer points of the face). Thus we have associated to A a geometrical object : a decomposition of T^2 into "convex polygones", containing "integral points".

The problem 14 is to understand, *which decompositions of T^2 (and which distinguished "integral-points" sets) are attainable this way from different matrices A* ?

The simplest example is provided by the "*cubical golden ratio*" matrice

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

(the ordinary golden ratio, $(\sqrt{5}+1)/2$, is the eigenvalue of the "quadratic golden ratio" matrice)

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} .$$

The torus triangulation, corresponding to the cubical golden rario case, can be described as the decomposition of the square into two triangles by a digonal (no interiour distinguished points). The simplest triangulations had been studied in the paper by E. Korkina, Proceedings of the Steklov Math. Inst., V. **209**, 1995, p. 143-166.

Studying one by one other $SL(3,\mathbb{R})$-matrices, starting with those which are not too large, one might obtain some hints, *what are all the possile triangulations and the distinguished points sets.* One might also study the statistics of these triangulations : the proportions of the triangles, of the quadrangles, of the pentagones and so on, the statistics of the numbers of the neighbouring faces of a vertice, that of the number of the sides of a polygone of the decomposition, that of the numbers of the distinguished points on the faces and on the edges,... .
It would be interesting to *compare these statistics with that of the sailes of the randomly choosen triangular cone* (whose vertice is the origin).

One might even compare the results with the similar statistics for the boundaries of the convex hulls of the sets of the integral points in the bodies bounded by the large smooth surfaces. Another interesting object to compare with would be the plane decompositions by the *Voronoi domains of the random distributions of the domains centers* (Voronoi domain of a discrete set of centers, in the euclidean plane consists of the points of the plane, for which the given center is the closest one among all the centers).

Studying these statistics one should compare the "domains areas" distributions, their "boundaries perimeters" distributions, the "edges lenghts" distributions, the "number of edges" distributions, and especially their simultaneous distributions, since they are far from being independent.

I would even suggest to study the dimensionless characteristics like the ratio "area / (perimeter squared)" and "number of vertices of a polygonal domain", whose correlation is also an interesting characteristic of the decomposition.

While averaging I would rather study the more stable *averages per unity of the area* than the average per a domain, (which overestimates the contributions of the small domains, due to their larger number).

15. While comparing problem 14 with the situation for the *ordinary continued fractions* (where there are no restrictions to the "triangulation of the torus and to this distinguished points", since every finite sequence of integers is a period of some quadratic irrational number), I must repeate the old interesting problem to compare the random irrational number continued fraction elements statistics with that of the elements of the periods of the random quadratic irrational numbers and of the eigenvalues of the random matrices of $SL(2,\mathbb{Z})$ with real eigenvalues.

I mean first to consider the integer points (p,q) defining the quadratic equations $x^2+px+q=0$ whouse roots are reall, such that $p^2+q^2\leq N$. Among the elements of the periods of the continued fractions of the roots of these equation we consider the proportion of the 1-s, of the 2-s and so on. The limit of the proportion of ks for N tending to infinity is by definition the ks statistics for the random quadratic irrational numbers.

The statistics for the random matrices eigenvalues periods is defined similarly, starting with those integral matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, for which $ad-bc=1$, the eigenvalues are real and $a^2+b^2+c^2+d^2\leq N$.

The guess is that these two statistics are the same, in both cases as for the statistics of the element of a random real number continued fraction.

The last statistics (frequency of k) $=\ln\left(1+\dfrac{1}{k(k+2)}\right)/\ln 2$ was discovered by Gauss and had been proved by R. Kusmin, 1928, using the Birkhof ergodic theorem (for every real number except those belonging to some exceptional numbers set of the Lebesgue measure zero).

However already in 1900 there appeared two papers whose titles mention the same problem :

T. Brodén. Wahrscheinlichkeibs Bestimmungen bei der Grewöhnlichen Kettenbruchentwicklung Reeller Zahlen. Acad. Föhr, Stockholm, v.**57**, 1900, pp. 239-266.

A. Wiman. Über eine-Wahrscheinlichkeits Auffage bei Kettenbruchentwicklungen, Acad. Föhr, Stockholm. v. **57**, 1900, pp. 589-841.

Unfortunatly, I have not chequed whether these papers contained the Kusmin 1928 theorem, and hence to read and to evaluate them from the modern view point is also a problem for our Seminar.

The Gauss-Wiman-Kusmin theorems and their relation to the *dynamical system* $x \mapsto \{1/x\} = 1/x - [1/x]$ *invariant measure* $\mu(A) = \int_A \rho(x)dx$, $\rho = \dfrac{1}{1+x}$, are discussed in the book : V. Arnold, Continued Fraction, MCNMO, Moscow, 2001, "Mathematical Education", **14**, 40pp.

The statistics for the sails of the random triangular pyramides in $\mathbb{R}^3$ are unknown, but M. Kontsevich and Yu. Sukhov have proved my conjectures on their *existence and universality* (pyramide independence) : their article is contained in the book "Pseudoperiodic Topology", AMS 1999 (V.I. Arnold, M.E. Kontsevich, A.V. Zorich Editors).
Unfortunatly, this mathematical existence proof is not answering the natural "physical" problems, like : *are there more or less integer point on the sail edges than on the random segments of similar lenghts between two integral points ?* What are the proportions between *the numbers of triangles and quadrangles* (or trianges and pentagons) among the faces of the sail of a random pyramide ? What is the distribution of the number of integral points on a face : more or less, than in a random plane domain of the same integral area ? Are the ratios "areas / (squarred perimetre of a face)" distributed for the faces of the sail the same way than for the random plane polygones or differently ?

All these questions are also problems in *experimental mathematics*, since it would be interesting to have the empirical data for, say, the "pseudorandom" cones, generated by three integral vectors (a, b, c) of total norm $|a|^2 + |b|^2 + |c|^2 < N$. The average statistics along the set of all such triples depend on N, and the empirical mean values for, say $N = 10$ and 100, might suggest whether there is a limit for N tending to infinity.

These empirical statistics might also be compared with those obtained from the cubical algebraic numbers fields (problem 14) : *is there a significative difference between the cubical field sails statistics and that of the random piramides sails ?* .