# Problems

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1. Study from the singularity theory point of view the E.Cartan's theory of pfaffian systems. What is happening in the nonanalytical (smooth) case, a) generically b) outside a "neglectable" variety of codimension infinity?

**Example:** find the smooth version of the recent theory (by I.Ekeland) of the disaggregation of the analytical demand function in economics.

$\sharp$ For a model problem (convex Darboux theorem) the smooth version was provided by Zakalyukin.$\sharp$

2. Consider a small generic smooth perturbation of $\mathbb{RP}^2$ in $\mathbb{RP}^3$. A special point is a point of the parabolic curve of the perturbed surface, where the asymptotical direction is tangent to the parabolic curve (it corresponds to a swallowtail on the dual surface). a) Do there exist perturbations having less than 6 specials points? b) If the number of special points is 6, may the number of parabolic curves be smaller than 4? c) If the perturbed surface is hyperbolic at infinity, may the number of parabolic curves be smaller than 3?

$\sharp$ D.Panov (1997) has constructed perturbed surfaces having one parabolic curve and 12 special points. B.Segre (1942) has constructed perturbed surfaces having 4 parabolic curves and 6 special points. F.Aicardi (1997) has also formulated conjectures on the number of "hyperbolic" and of "elliptic" special points. $\sharp$

3. Let $[x : y : z : w]$ be homogeneous coordinates in $\mathbb{RP}^3$. Define a surface in $\mathbb{RP}^3$ by the formula $w = U(x, y, z)$, where $U$ is homogeneous of degree 1. Denote by $u$ the restriction of $U$ to the unit sphere in $\mathbb{R}^3$. The function $u$ is *hyperbolic* at a point of the sphere if the surface is hyperbolic at the corresponding point.

**Example.** *Generalized spherical functions* are functions verifying the equation $\Delta u + 2u = g$, where $g$ is a finitely supported distribution and $\Delta$ is the spherical Laplacian). These functions are hyperbolic except at the poles and at the "monkey saddles" where the second quadratic form of the surface vanishes identically.

Consider an odd hyperbolic function on $S^2$, whose only singularities are simple logarithmic poles. a) May the number of these poles be smaller than 6? b) May the number of the (small) parabolic curves on a generic surface, obtained from an odd hyperbolic function by its generic smoothening, be smaller than 4?

♯ An odd generalized spherical function has at least 6 poles. If the number of poles is equal to 6, they lie on a great circle. If they form a regular hexagon (with alternating positive and negative equal charges), the resulting function has 8 monkey saddles. It is perhaps an interesting (elliptic? Hypergeometric?) special function on the sphere. The generalized spherical functions, which are restrictions of the rational harmonic functions $p_{n+1}/q_n$ deserve a special study. See *Functional Analysis N*4*,* 1997.♯

4. Consider a small generic perturbation of the smooth surface in $\mathbb{RP}^3$, defined in the affine coordinates by the equation $z = \frac{1}{x^2+y^2}$. a) Prove that the number of parabolic curves of the perturbed surface at a neighbourhood of infinity is at least 3. b) For $z = \frac{x}{x^2+y^2}$ the number of parabolic curves of the perturbed surface at a neighbourhood of infinity is at least 2.

♯ Both surfaces have at infinity a degenerate parabolic line. For the perturbations, corresponding to sufficiently small values of the parameter in generic one-parameter families of perturbed surfaces, the above minorations follow from the Sturm-Hurwitz theorem. See *Functional Analysis* 1997 *N*4.♯

5. Describe the singularities of the parabolic curves of smooth surfaces at their flattening points (where the second quadratic form vanishes identically).

♯ The $E_6$ singularity (diffeomorphic to that of $x^3 + y^4 = 0$) is impossible. At a non-flattening parabolic point any finite multiplicity singularity may occur (up to a diffeomorphism at least).♯

6. Let $g(x, y)$ be a smooth germ of a function. Do there exist a smooth function $f$ such that the gaussian curvature of the surface $z = f(x, y)$ at the point with given $(x, y)$ be $g(x, y)$. Same problem for the Hessian and for the parabolic curve is also open.

$\sharp$ The answer is "yes" for the formal series, but is open for the smooth (analytic?) case, even for the critical points of finite multiplicity with critical value 0.$\sharp$

7. Let $z = f(x, y)$ be a smooth surface. May its asymptotic line be a closed line?

$\sharp$ Its projection to the plane $(x, y)$ can't be a star-like curve (D.Panov). In the periodic case ($x \in \mathbb{R}/\mathbb{Z}$) the projection curve can't be a graph of a function $y = g(x)$ (F.Aicardi), but might be arbitrary close to a union of tangent semicircles. Perhaps one might construct a similar close asymptotic line whose projection to $\mathbb{R}^2$ is arbitrary close to a non star-like union of circles of a fixed radius.$\sharp$

8. Describe the typical curves belonging to the boundary of the domain of those curves in $\mathbb{RP}^3$, who admit convex projections (from some outside point depending on the curve).

9. Complexify the notion of finite order invariants of the points of the complement to a given (algebraic) "discriminant" hypersurface.

$\sharp$ One way is to study usual finite order invariants of the loops which do not intersect the discriminant. Say, for the discriminant hypersurface consisting of one point in $\mathbb{C}$ this leads to the subring of polynomials in the ring of all functions on $\mathbb{Z}$. However examples like the "discriminant" $xy = 0$ in $\mathbb{C}^2$ show that the resulting structure should live rather in $H^*$ (and might be related to the weight filtration of the mixed Hodge structure). The natural examples of discriminants to which one should apply the required theory are the $A_n$ mirrors arrangements in $\mathbb{C}^n$, the complex swallowtails and their projective versions.$\sharp$

10. Consider a holomorphic mapping germ $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. Evaluate the maximal value of the multiplicity $\mu$ occurring in generic families with a fixed number $m$ of parameters (say, for $n = 2$).

$\sharp$ Khovansky and Gabrielov have recently proved some estimations, but the upper and lower bounds leading coefficients differ even for $n = 2$: one has no intuition, which kind of mappings do have the maximal multiplicity.$\sharp$

11. Consider the coadjoint representation of the ideal hydrodynamic group SDiff $M^n$ (of the diffeomorphisms of a compact manifold $M$ preserving

the volumes). Are all the local invariants (of the form $I = \int_M F(j_x^N(\omega))dx$, where $j_x^N$ is the $N$-jet at $x$) already known?

$\sharp$ This might be easier than the corresponding problem for the Euler equation, where there are more invariants.$\sharp$