Problems

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Foreword of the publisher

These are the notes of V.I.Arnold's talk at the seminar taken down *in vivo*. I take the responsibility for the possible inaccuracies in my interpretation. A more complete and authentic Russian edition is being prepared by PHASIS.

1. Pseudoring of a Lie group

Let G be a Lie group. Denote by $H(G)$ the set of homotopy classes of continuous mappings of (G, e) into itself equipped with two operations:

$$\begin{aligned} (a * b)(g) &= a(g)b(g), \\ (ab)(g) &= a(b(g)). \end{aligned}$$

Although we use additive notation for the first operation, it may be non-commutative, e.g. for the group $SU(2) \times SU(2)$. The two operations obey one one-sided distributive law $(a + b)c = ac + bc$.

Examples. In the following examples, $H(G)$ turns out to be a ring.

1. $H(U(1)) = \mathbb{Z}$.
2. $H(O(1)) = \mathbb{Z}_2$.
3. $H(SU(2)) = \mathbb{Z}$.

The problem is to study this algebraic object as a characteristic of the Lie group G. The simplest examples where the answer is unknown are $G = SO(3)$ and $G = \mathrm{Spin}(4)$.

2. Quaternionic matrices

Invent a definition of the determinant of a quaternionic matrix.

Let $A : \mathbb{H}^n \to \mathbb{H}^n$ be a quaternionic linear operator (say, left-handed, i.e. such that $A(qv) = qA(v)$ for $q \in \mathbb{H}$). Then its complexification is a complex linear operator ${}^{\mathbb{C}}A : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$. Let $H(A) = \det({}^{\mathbb{C}}A)$. It is easy to see that H is a real-valued polynomial, $H(A) \geq 0$ for any A and the equation $H(A) = 0$ defines a cone Σ of real codimension 4.

The problem is to study the ideal I of the algebraic variety Σ. In particular, if one finds that the polynomial function H can be represented as a sum of 4 squares:

$$H(A) = \alpha^2 + \beta^2 + \gamma^2 + \delta^2,$$

then a natural candidate for the quaternionic determinant would be

$$\det A = \alpha + i\beta + j\gamma + k\delta.$$

Remark. The polynomial $H(A)$ can be decomposed into a sum of 4 squares of rational functions, but cannot be written as a sum of squares of any number of real polynomials, if $n \geq 2$. — S.D.

3. Quaternionic vector bundles

Find the analogue of holomorphic linear vector bundles, their connection and curvature in the quaternionic case. Conjecture: the counterpart of Cauchy–Riemann equations should be non-linear.

4. Contact version of Liouville's theorem

Find a contact version of Liouville's theorem about completely integrable Hamiltonian systems.

The classical Liouville theorem about the motion along the invariant tori is based on the following lemma: if M^{2n} is a symplectic manifold and $p : M^{2n} \to B^n$ is a fibering with Lagrangian fibers, then these fibers bear a natural affine structure.

This fact has the contact analogue: if M^{2n+1} is a contact manifold and $p : M^{2n+1} \to B^n$ is a fibering with Legendrian fibers, then these fibers bear a natural projective structure.

The problem is to find the analogue of Liouville's theorem which is a consequence of this lemma, and its applications in differential equations. It is interesting already in the case $n = 1$ (ordinary differential equations).

5. Hofer fields

A *Hofer field* is a divergence free vector field $v \in \mathrm{Vect}(M^3)$ such that $i_v(d\alpha) = 0$, where α is a 1-form defining a contact structure on M^3 ($\alpha \wedge d\alpha \neq 0$).

Consider the motion of a charged particle on a 2-surface S. A magnetic field B perpendicular to the surface gives rise to the vector field v on $M^3 = T_1(S)$.

Problem: find the conditions on B which guarantee that v is a Hofer field.

6. Heisenberg uncertainty principle for lattices in $\mathbb{R}^n$

Let $\mathbb{R}^n$ be a Euclidean space and $\Gamma \subset \mathbb{R}^n$ be a closed subgroup such that $\mathbb{R}^n/\Gamma$ is compact (i.e. a torus $T^{k \leq n}$). Denote by $\Gamma^\vee$ the dual subgroup, i.e. the set of all $k \in (\mathbb{R}^n)^*$ such that $(k, x) \in \mathbb{Z}$ for all $x \in \Gamma$.

Theorem 1. If $\mathbb{R}^n \setminus \Gamma$ contains a ball of radius ρ, then there is a $k \in \Gamma^\vee$ such that $0 < |k| \leq c/\rho$, where c is a constant depending only on n.

Problem 1. Find the constant c.

Now suppose that $\Gamma \subset \mathbb{R}^n$ is a discrete lattice ($\Gamma \cong \mathbb{Z}^n$). Let $r(\Gamma)$ be the maximal radius of open balls centred at points of Γ and having no common points. Let $R(\Gamma)$ be the minimal radius of closed balls centred at points of Γ and covering all $\mathbb{R}^n$.

Theorem 2. There are constants c and C depending only on n such that

$$c < r(\Gamma)R(\Gamma) < C.$$

Problem 2. Find c and C. (E.Korkina claims that $c = 1/4$ and $C \leq \frac{\sqrt{3}}{4}\sqrt{\left(\frac{4}{3}\right)^n - 1}$.)

7. Simple singularities of curves in contact geometry

Classify the stable simple singularities of curves in a contact manifold with respect to contact diffeomorphisms.

A simple singularity in $\mathbb{R}^n$ is said to be stable, if it remains simple when considered in $\mathbb{R}^m$, $m > n$.

8. Causality in terms of linking

This problem was communicated by R.Penrose who referred to a paper by R.Low called "Causality in terms of linking...", Class. Quant. Gravity 7 (1990), 11 (1994).

Every point in timespace defines a submanifold in the manifold of light rays. Two events A, B may have causal relation, if and only if the corresponding submanifolds L_A^{n-1}, L_B^{n-1} are linked in M_{rays}^{2n-1}.

Problem: study this phenomenon with the machinery of knot theory.

9. Triangulation of knots

For any polygonal knot in $\mathbb{R}^3$ or S^3 there is a triangulation of the ambient manifold that contains the given triangulation of the knot. Estimate the number of simplices that are required, in terms of the number of segments in the polygonal knot.

10. Nekrasov's discriminant

(Proposed by N.Nekrasov).

Pairs of germs of Poisson commuting functions on a symplectic manifold, considered up to symplectic equivalence, make up an infinite dimensional symplectic manifold with a discriminant of codimension 2. Find the topology of its complement (fundamental group etc).

11. Non-Eulerian hydrodynamics

(Proposed by A.Varchenko.)

Construct an analogue of hydrodynamics using, instead of Euler equations, the following equation: $u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy} = 0$. This equation has the property that if $u(x,y)$ is a solution, then $f(u)$ is also a solution for any function f. Find other equations having the same property and their applications to topological invariants, variational principles etc.

Remark. Using the standard techniques of group analysis, it is easy to describe all equations that have this property. For example, in Monge's notation $p = u_x$, $q = u_y$, $r = u_{xx}$, $s = u_{xy}$, $t = u_{yy}$, the general form of a second order equation for a function $u(x,y)$ resolved with respect to s and such that any function of a solution is again a solution, is

$$s = \frac{1}{2}\left(\frac{pt}{q} + \frac{qr}{p}\right) + \phi\left(x, y, p, q, \frac{pt}{q} - \frac{qr}{p}\right),$$

where ψ is an arbitrary homogeneous function of degree 1 in the last three arguments. — S.D.

12. Non-Jordan normal forms

(Proposed by M.Kontsevich.)

Let $\mathbb{C}^{n^2}$ be the space of complex matrices. Find a family of disjoint affine planes in $\mathbb{C}^{n^2}$ such that their union contains exactly one element of every conjugate class of matrices.

Example for $n = 2$:

$$\begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}.$$

Note that the usual Jordan normal form does not have this property, because, for example, the plane of diagonal matrices contains *pairs* of distinct equivalent elements $\operatorname{diag}(a,b)$ and $\operatorname{diag}(b,a)$.

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