Singularities of fractions and behaviour of polynomials at infinity

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1 Classification of singularities of fractions

In this paper we mean by *fraction* the ratio (ordered pair) of two germs of (holomorphic or smooth) functions in n variables at a point (which usually will be refered as the origin). For the sake of simplicity mostly the complex case is considered below. The results for the real case are similar, only certain formulas contain non-equivalent real normal forms (which differ usually by $\pm$ signes at certain monomials).

The singularities of fractions will be classified up to the following equivalence relations.

Definition 1. Two fractions are called *coinciding*, if their numerators and denominators are proportional (by *non-zero* function).

Example 1. On $\mathbb{C}^2$ with coordinates x and y the fraction x^2/xy does not coincide neither with x/y, nor with x^3/x^2y, but it coincides with x^2e^x/xye^x.

Definition 2. Two fractions are called *R-equivalent*, if one of them becomes coinciding with the other when transfered by a germ of biholomorphism of the space of variables.

Example 2. The fraction $1/xy$ is R-equivalent to the fraction $1/(x^2+y^2)$.

Definition 3. Two fractions are called *R^+-equivalent* if one of them is R-equivalent to the sum of the other one with a germ of a holomorphic function.

Example 3. The fraction y^2/x is R^+-equivalent, (though not R-equivalent) to the fraction $(y^2+x)/x$.

Remark. The *sum* of the fraction P/Q and the holomorphic function A is by definition the fraction $(P+AQ)/Q$.

The main result of the paper is the begining of the classification of fraction germs up to R^+-equivalence. First singularities occur to be simple (to have no continious moduli). They are classified by simple Lie algebras $A_k, B_k, C_k, D_k, E_6, E_7, E_8, F_4$ (no G_2!) – the same list as in the classification of boundary singularities (comp. [1]).

Germs of generic fractional functions are R^+-simple at any point. Codimension in the space of all fractions on $\mathbb{C}^n$ of the subspace of fractions, which have somewhere a non R^+-simple germ, is equal to 4 (for $n \geq 2$).

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Thus, non R^+-simple fraction germs appear first (in an unremovable way) in families of fractions on n-dimensional manifold, depending on four parameters (provided that $n > 1$).

Definition. The class Y of fraction germs *is adjacent to* the class X, if the germs from the class X belong to any arbitrary small deformations of germ Y. Here by small deformation of fraction $p(x)/q(x)$ we mean the fraction of the family $P(x,\lambda)/Q(x,\lambda)$, $P(\cdot,0) = p, Q(\cdot,0) = q$ with small λ.

The table of adjacencies of R^+-simple fraction singularities is given in Fig. 1.

$$
\begin{array}{*{23}{c}}
& & & & & & & & & & & & & & (P_8) & & (X_9) & & (J_{10}) \\
& & & & & & & & & & & & & & \downarrow & & \downarrow & & \downarrow \\
& & A_0^* & \leftarrow & A_0 & & & & & & & & & & E_6 & \leftarrow & E_7 & \leftarrow & E_8 \\
& & & & & & & & & & & & & & \downarrow & & \downarrow & & \downarrow \\
& & & & & & & & & & & & D_4 & \leftarrow & D_5 & \leftarrow & D_6 & \leftarrow & D_7 & \leftarrow & D_8 & \leftarrow & \cdots \\
& & & & & & & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & & & & & & & A_1 & \leftarrow & A_2 & \leftarrow & A_3 & \leftarrow & A_4 & \leftarrow & A_5 & \leftarrow & A_6 & \leftarrow & A_7 & \leftarrow & \cdots \\
& & & & & & & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
& & & & & & A_1^* & \leftarrow & B_2 & \leftarrow & C_3 & \leftarrow & C_4 & \leftarrow & C_5 & \leftarrow & C_6 & \leftarrow & C_7 & \leftarrow & C_8 & \leftarrow & \cdots \\
& & & & & & \uparrow & & \uparrow & & \uparrow & (C_{2,2}) & & (P_8^\#) \\
& & & & & & A_2^* & \leftarrow & B_3 & \leftarrow & F_4 & \leftarrow & (X_9^\#) \\
& & & & & & \uparrow & & \uparrow & & \uparrow \\
& & & & D_4^* & \rightarrow & A_3^* & \leftarrow & B_4 & \leftarrow & (J_{10}^\#) \\
& & & & \uparrow & & \uparrow & & \uparrow \\
(P_8^*) & \rightarrow & E_6^* & \rightarrow & D_5^* & \rightarrow & A_4^* & \leftarrow & B_5 \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
(X_9^*) & \rightarrow & E_7^* & \rightarrow & D_6^* & \rightarrow & A_5^* & \leftarrow & B_6 \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
(J_{10}^*) & \rightarrow & E_8^* & \rightarrow & D_7^* & \rightarrow & A_6^* & \leftarrow & B_7 \\
& & & & \uparrow & & \uparrow & & \uparrow \\
& & & & D_8^* & \rightarrow & A_7^* & \leftarrow & B_8 \\
& & & & \uparrow & & \uparrow & & \uparrow \\
& & & & \vdots & & \vdots & & \vdots
\end{array}
$$

$$
\begin{array}{rlrl}
A_k = \frac{y^{k+1}+z^2}{x}, & & A_k^* = \frac{1}{x^{k+1}+y^2+z^2}; \\
D_k = \frac{y^2z+z^{k-1}}{x} & & D_k^* = \frac{1}{x^2y+y^{k-1}+z^2}; \\
E_6 = \frac{y^3+z^4}{x} & & E_6^* = \frac{1}{x^3+y^4+z^2}; \\
E_7 = \frac{y^3+yz^3}{x} & & E_7^* = \frac{1}{x^3+xy^3+z^2}; \\
E_8 = \frac{y^3+z^5}{x} & & E_8^* = \frac{1}{x^3+y^5+z^2}; \\
B_k = \frac{x}{x^k+y^2+z^2} & & C_k = \frac{x}{xy+y^k+z^2}; \\
\end{array}
$$
$$
F_4 = \frac{x}{x^2+y^3+z^2}.
$$

Fig. 1. Classification of R^+-simple fraction singularities (formulas are written for the case of three variables).

The symmetry of this table is closely related to the Lagrange duality, found by I.G.Scherback [2] in the theory of boundary singularities. Hovewer I have no the apriory proof of this symmetry and I was astonished to discover it .

Remark. Besides the R^+-equivalence of fractions (gauge corresponding to the additive subgroup of the projective transformations of function value axis), one can define as well the $R^\times$-equivalence (which gauge coressponds to the subgroup of affine transformations $z \mapsto az+b$) and R^P- eqiuvalence (corresponding to the whole projective group $z \mapsto (az+b)/(cz+d)$).

Such an extension of underlying group might increase the number of simple singularity classes, but this does not happen if the dimension n of variable space is greater than 2 . For $n=2$ there appear additional $R^\times$-simple, but not R^+-simple, singularities, whose adjacencies are shown in Fig. 2 below. They are to be compared with the lists of Giusti [3] and Goryunov [4]. Unfortunately I was unable to deduce the table 2 from their lists.

Presumably, our results clear up the Giusti list which seems to have no good structure. It consists of parts of different nature (R^+ and $R^\times$-simple), fluxing together due to the equivalence relation (biholomorphic equivalence of complete intersections). Our definitions correspond rather to the classification of flags of submanifolds (see [8]), which has a better theory of basises of vanishing cycles and Dynkin diagrams, than the theory of complete intersections.

Theorem 1. *The R^+-simple fraction germs in $\mathbb{C}^n$ are listed (up to the R^+-equivalence) in the following table:*

$$\begin{array}{ll}
A_k = (y^{k+1}+Q_2)/x, & A_k^* = 1/(x^{k+1}+Q_1);\\
D_k = (y^2z+z^{k-1}+Q_3)/x, & D_k^* = 1/(x^2y+y^{k-1}+Q_2);\\
E_6 = (y^3+z^4+Q_3)/x, & E_6^* = 1/(x^3+y^4+Q_2);\\
E_7 = (y^3+yz^3+Q_3)/x, & E_7^* = 1/(x^3+xy^3+Q_2);\\
E_8 = (y^3+z^5+Q_3)/x, & E_8^* = 1/(x^3+y^5+Q_2);\\
B_k = x/(x^k+Q_1), & C_k = x/(xy+y^k+Q_2);\\
F_4 = x/(x^2+y^3+Q_2). &
\end{array}$$

Here $k \geq 0$ for A_k, A_k^*; $k \geq 2$ for B_k; $k \geq 3$ for C_k; and $k \geq 4$ for D_k. The letter Q denotes the quadratic form $Q_r = z_{r+1}^2 + \cdots + z_n^2$, thus the coordinates on $\mathbb{C}^n$ are denoted by:

$$\begin{array}{ll}
(x; z_2, \ldots, z_n)\,(n \geq 1) & \textit{for } A_k^* \textit{ and } B_k;\\
(x, y; z_3, \ldots, z_n)\,(n \geq 2) & \textit{for } A_k, D_k^*, E_k^*, C_k, F_4;\\
(x, y, z; z_4, \ldots, z_n)\,(n \geq 3) & \textit{for } D_k, E_k.
\end{array}$$

Remark. The singularity A_0^* is R-equivalent to $1/x$. For a generic fractional function the singular points of this class form a hypersurface (of dimension $n-1$ for functions in n variables).

The singularity A_0 is R-equvalent to y/x. For a generic fractional function the singular points of this class form a submanifold of codimension 2 (of dimension $n-2$ for functions in n variables).

At the remaining points a generic fractional function is holomorphic and therefore is R^+-equivalent to zero. The R-classification of these (holomorphic) germs is well known (see [5]).

The singularities from theorem 1 appear unremovably only in the families of functions. The simple singularity denoted by $\cdot_k$ in the list, can be found in generic family with $p \geq k$

parameters when the parameters belong to a certain submanifold of dimension $p-k$. For these special values of parameters the singularity occurs at isolated points of the source space of the function.

Example. Consider the singularity A_1. It occurs if the restriction of the numerator to the smooth hypersurface of zeros of denominator has Morse critical point. Such points are inherent already in generic fractions. But they belong to the class A_1 only if the critical value is equal to zero.

This condition defines certain submanifold of codimension $k=1$ in p-dimensonal parameter space. In this sense the codimension of the class A_1 is equal to 1.

In Fig. 1. the classes of equal codimension are connected by lines. In brackets there are given the non R^+-simple "fencing" classes. For function in two variables the class $C_{2,2}$ has the minimal codimension (equal to 4) among them. This class is formed by fractions whose nominator and denominators have a common critical point with zero common critical value.

For greater number of variables $n>2$ the codimension of this class grows (it equals $n+2$), but there appear other classes ($F^{\#}$) of non R^+-simple (even non $R^\times$-simple) singularities with codimension 5 (they are described in theorem 2 below).

Corollary. *Generic families of meromorphic functions in $n\geq 2$ variables depending on $p\leq 3$ parameters contain only such functions whose germ at any point of non-holomorphy is R^+-simple and R^+-equivalent to one of the following simple germs*

$$0; A_0^*; A_0; A_1^*, A_1; A_2^*, B_2, A_2; A_3^*, B_3, C_3, A_3$$

of theorem 1.

Remark. Proving theorem 1 (see §§ 3–4 below) we actually will show more:

Theorem 2. *Any germ of fraction on $\mathbb{C}^n$ either is R^+-simple (and therefore is R^+-equavelent to one of the germs from theorem 1), or belongs to a class of infinite codimension (A_∞, B_∞, C_∞, D_∞, A_∞^*, D_∞^*), or is adjacent to one of the following "fencing" classes of fractions:*

$$\begin{array}{ll}
(P_8)=(y^3+z^3+u^3+ayzu+Q_4)/x, & (P_8^*)=1/(x^3+y^3+z^3+axyz+Q_3); \\
(X_9)=(y^4+ay^2z^2+z^4+Q_3)/x, & (X_9^*)=1/(x^4+ax^2y^2+y^4+Q_2); \\
(J_{10})=(y^3+ayz^4+z^6+Q_3)/x, & (J_{10}^*)=1/(x^3+axy^4+y^6+Q_2); \\
(P_8^{\#})=x/(xy+y^3+ay^2z+z^3+Q_3), & \\
(X_9^{\#})=x/(x^2+axy^2+y^4+Q_2), & \\
(J_{10}^{\#})=x/(x^3+ax^3y+y^3+Q_2), & \\
(C_{2,2})=(a_2y_2^2+\cdots+a_ny_n^2)/Q_0. &
\end{array}$$

Here a, a_i are parameters, $Q_r=y_{r+1}^2+\cdots+y_n^2$. Any germ from classes $(P_8^{\#})$ and $(X_9^{\#})$ is adjacent to the classes B_4, F_4, C_4, while the germs from the class $(J_{10}^{\#})$ are adjacent only to the singularities B_4 and F_4.

Classes of fractions which are R^+-equivalent to those listed above have the following codimension and R^+-modality:

Class	P_8	X_9	J_{10}	P_8^*	X_9^*	J_{10}^*	$P_8^{\#}$	$X_9^{\#}$	$J_{10}^{\#}$	$C_{2,2}$
codim	7	8	9	7	8	9	5	5	5	$n+2$
m	1	1	1	1	1	1	1	1	1	$n-1$

Example. Generic four-parametric families of fractions on $\mathbb{C}^2$ contain only germs which are R^+-equivalent either to $ay^2/(x^2+y^2)$ (for certain isolated values of parameters) or to the R^+-simple fractions of the types $D_4, A_{k\le 4}, C_{k\le 4}, F_4, B_{k\le 4}, A^*_{k\le 4}, D^*_4$.

Remark. The notations of the " fencing " boundary singularities with sign # are chosen in such way that the double covering of the bounary singularity of denominator (that is the substitution $x = X^2$ in it) defines a function from the corresponding class of the fencing unimodal singularities of holomorphic functions P_8, X_9, J_{10} in variables X, y, z, y_i.

2 Additional series of $R^\times$-simple fractions

Definition. Two fractions are called $R^\times$-*equivalent*, if one of them is R^+-equivalent to the product of the other with non-zero (at the given point) holomorphic function.

In other words, the fraction P/Q is $R^\times$-equivalent at 0 to any fraction of the form $(AP+BQ)/Q$, provided that A is non-zero at the origin.

Example 1. Let P and Q be generic quadratic forms in two variables x and y. Then the fraction P/Q is $R^\times$-equivalent to the fraction $(x^2+y^2)/(xy)$.

Example 2. Let P and Q be generic quadratic forms in $n > 2$ variables. Then the fraction P/Q is not $R^\times$-simple.

This fact follows immediately from the eigenvalue theory of pencils of quadratic forms. One may suppose $Q = x_1^2 + \cdots + x_n^2$, $P = \lambda_1 x_1^2 + \cdots + \lambda_n x_n^2$. Eigenvalues of the form $aP + bQ$ with respect to Q are equal to $a\lambda_i + b$. For $a = A(0), b = B(0)$ it implies, that the classes of $R^\times$-equivalent fractions P/Q correspond to submanifolds of codimension 2 in the n-space of fractions having the special form described above. If $n > 2$ then all our fractions are not simple (even with respect to liberal $R^\times$-equivalence): any neighbourhood of every fraction contains a continuum of pairwise non-equivalent germs.

The difference between $n \le 2$ and $n \ge 3$ cases discussed above, constitutes the main reason why the new $R^\times$-simple series appear for $n = 2$ (and does not for $n > 2$).

The adjacencies of these singularities are shown in Fig. 2.

$$
\begin{array}{ccccccccccc}
(A_1) & \leftarrow & (A_2) & \leftarrow & (A_3) & \leftarrow & (A_4) & \leftarrow & (A_5) & \leftarrow & \cdots \\
\uparrow & \diagup & \uparrow & \diagup & \uparrow & \diagup & \uparrow & \diagup & \uparrow & \diagup & \\
(B_2) & \leftarrow & (C_3) & \leftarrow & (C_4) & \leftarrow & (C_5) & \leftarrow & (C_6) & \leftarrow & \cdots \\
\uparrow & \diagup & \uparrow & \diagup & \uparrow & \diagup & \uparrow & \diagup & \uparrow & \diagup & \\
(B_3) & & C_{2,2} & \leftarrow & C_{3,2} & \leftarrow & C_{4,2} & \leftarrow & C_{5,2} & \leftarrow & \cdots \\
\uparrow & \diagup & \uparrow & \diagup & \uparrow & \diagup & \uparrow & \diagup & \uparrow & \diagup & \\
(F_4) & \leftarrow & F_5 & & C_{3,3} & \leftarrow & C_{4,3} & \leftarrow & C_{5,3} & \leftarrow & \cdots \\
 & & \uparrow & \diagup & \uparrow & \diagup & \uparrow & \diagup & \uparrow & \diagup & \\
 & & F_6 & \leftarrow & F_7 & & C_{4,4} & \leftarrow & C_{5,4} & \leftarrow & \cdots \\
 & & & & \uparrow & \diagup & \uparrow & \diagup & \uparrow & & \\
 & & & & F_8 & \leftarrow & F_9 & & \vdots & & \\
 & & & & & & \uparrow & & & & \\
 & & & & & & \vdots & & & &
\end{array}
$$

$$F_{p+q+1} \to C_{p,q} \to (A_{p+q-1})$$

Fig. 2. Adjacencies of $R^{\times}$-simple , but not R^{+}-simple germs,
to R^{+}-simple ones and among themselves (here $n = 2$).

Theorem 3. *The $R^{\times}$-simple fraction germs in $\mathbb{C}^n$ are exhausted (up to $R^{\times}$-equivalence) by R^{+}-simple germs and besides that for $n = 2$ by the germs from the following list:*

$$C_{p,q} = \frac{x^p + y^q}{xy}, \quad F_{2k+1} = \frac{y^k}{x^2 + y^3}, \quad F_{2k+2} = \frac{xy^{k-1}}{x^2 + y^3}, \quad (p \geq q \geq 2, k \geq 2).$$

In the space of fractions with zero values of numerator and denominator at the origin the codimension of the corresponding class is equal respectively to $p + q, 2k + 1, 2k + 2$.

Remark. One may identify $C_{p,1} = C_{p+1}, C_{1,1} = B_2, F_3 = B_3$ (all these germs as well as F_4 are already R^{+}-simple). In Fig. 2 the R^{+}-simple germs are given in brackets.

Corollary. *Generic four-parameteric families of meromorphic functions in two variables have only $R^{\times}$-simple germs at any point (of non-holomorphy).*

Generic five-parametric families may contain non $R^{\times}$-simple germs, which can not be removed by small perturbations.

3 Classificator of singularities of fractions

The following 16 theorems permit to find the location of any fraction P/Q (where P and Q are either holomorphic at the origin function on $\mathbb{C}^n, n > 2$, or power series) in our classification .

Below the sign "$\approx$" denotes R-equivalence, the sign "$\sim$" means R^{+}-equivalence. The sign "$\models\!\Rightarrow$**n**" is the abbreviation of "see theorem **n**". Sign "$\Rightarrow$" means "implies". And the sign "$j^k f$" denotes the k-jet (Taylor polynomial of degree k) of function f at the origin. By x, y, z, u we denote coordinates on $\mathbb{C}^n$.

We will use the notations of singularity classes $(A_k, \ldots, J_{10}^{\#})$, as introduced in § 1.

1. *If $j^0 Q \neq 0$, then $P/Q \sim 0$.*

2. *If $j^0 Q = 0$, then*

- *either $j^1 Q \neq 0$, $\models\!\Rightarrow$**3**;*
- *or $j^1 Q = 0$, $\models\!\Rightarrow$**9**.*

3. *If $j^0 Q = 0, j^1 Q \neq 0$, then $P/Q \sim \mathcal{F}(y_2, \ldots, y_n)/x$, $\models\!\Rightarrow$**4**.*

4. *If $P/Q \sim \mathcal{F}(y)/x$, then*

- *either $j^0 \mathcal{F} \neq 0$, $\models\!\Rightarrow$**5**;*
- *or $j^0 \mathcal{F} = 0$, $\models\!\Rightarrow$**6**.*

5. *If $P/Q \sim \mathcal{F}(y)/x$, $j^0 \mathcal{F} \neq 0$, then $P/Q \approx 1/x \in A_0^*$.*

6. *If $P/Q \sim \mathcal{F}(y)/x$, $j^0 \mathcal{F} = 0$, then*

– *either* $j^1\mathcal{F} \neq 0$, $\models\!\Rightarrow$**7**;

– *or* $j^1\mathcal{F} = 0$, $\models\!\Rightarrow$**8**.

7. *If* $P/Q \sim \mathcal{F}(y)/x$, $j^0\mathcal{F} = 0$, $j^1\mathcal{F} \neq 0$, *then* $P/Q \approx y/x \in A_0$.

8. *If* $P/Q \sim \mathcal{F}(y)/x$, $j^1\mathcal{F} = 0$, *then*

– *either* $P/Q \in A_k\,(k \geq 1)$, $D_k\,(k \geq 4)$, E_6, E_7, E_8;

– *or* P/Q *is adjoint to* $(P_8), (X_9)$ *or* (J_{10}) *and is not* R^+*-simple fraction;*

– *or* P/Q *belongs to the class of infinite codimension* A_∞ *or to* D_∞ *(which in holomorphic case contains the reducible fractions and fractions with non-isolated singularity of numerator with zero critical value).*

Remark. The location of fraction P/Q in $A-D-E$- classification may be found applying the classificator from [5] to $\mathcal{F}$.

The classes of non R^+*-simple fractions adjacent to* P_8, X_9 *and* J_{10} *have codimensions respectively* $7, 8, 9$ *(provided that* $n \geq 4, 3, 3$*).*

That is the germs from these classes appear unremovably first in generic families of fraction on $\mathbb{C}^n$, depending on $7, 8$ and 9 parameters respectively.

The classes D and E are not void if $n \geq 3$.

Proceed with the classificator.

9. *If* $j^1Q = 0$, *then:*

– *either* $j^0P \neq 0$, $\models\!\Rightarrow$**10**;

– *or* $j^1P \sim x$, $\models\!\Rightarrow$**11**;

– *or* $j^1P = 0$, $\models\!\Rightarrow$**16**.

10. *If* $j^1Q = 0, j^0P \neq 0$, *then* $P/Q = 1/\mathcal{F}, j^1\mathcal{F} = 0$.

The classification of such fractions P/Q (up to R-equivalence) reduces to R-classification of holomorphic function germs $\mathcal{F}$ having a critical point with zero critical value at the origin. We get:

– *either* $\mathcal{F} \approx a_k = x^{k+1} + y_2^2 + \cdots + y_n^2\,(k \geq 1)$;

– *or* $\mathcal{F} \approx d_k = x^2y + y^{k-1} + y_3^2 + \cdots + y_n^2\,(k \geq 4)$;

– $$\begin{array}{lll} \text{or } \mathcal{F} & \approx & e_6 = x^3 + y^4 + y_3^2 + \cdots + y_n^2, \text{ or} \\ & \approx & e_7 = x^3 + xy^3 + y_3^2 + \cdots + y_n^2, \text{ or} \\ & \approx & e_8 = x^3 + y^5 + y_3^2 + \cdots + y_n^2; \end{array}$$

– *or* $\mathcal{F}$ *is adjacent to one of the classes* $(P_8), (X_9), (J_{10})$ *of non* R*-simple singularities of functions;*

– *or* $\mathcal{F}$ *belongs to one of the classes* A_∞ *or* D_∞ *of infinite codimension (in holomorphic case they consist of functions having non-isolated critical point with zero critical value).*

Respectively, we get:

- *either* $P/Q \approx 1/a_k \in A_k^*, k \geq 1$;
- *or* $P/Q \approx 1/d_k \in D_k^*, k \geq 4$;
- *or* $P/Q \approx 1/e_k \in E_k^*, k = 6,7,8$;
- *or the fraction* P/Q *is adjacent to one of the classes* $(P_8^*), (X_9^*), (J_{10}^*)$ *of non* R^+*-simple fractions;*
- *or* P/Q *belongs to one of the classes* A_∞^* *and* D_∞^* *of codimension* ∞ *(For holomorphic* P *and* Q*, these classes consist of fractions having non-isolated critical point of denominator with zero critical value).*

Remark. The codimensions of classes $(P_8^*), (X_9^*), (J_{10}^*)$ are equal to $7, 8$ and 9 respectively, provided that $n \geq 3, 2, 2$.

11. *If* $j^1 Q = 0, j^1 P = x$*, than* $P/Q \approx x/\mathcal{F}, j^1\mathcal{F} = 0$.

The classification of these fractions up to R-equivalence follows from the theory of boundary singularities [1], applied to the function $\mathcal{F}$. This theory implies that using diffeomorphisms preserving the " boundary" $x = 0$ (that is having the form $(x, y) \mapsto (xh(x, y), g(x, y)))$ one can reduce function $\mathcal{F}$ to one of the following forms:

- *either* $\mathcal{F} \approx b_k = x^k + y_2^2 + \cdots + y_n^2\,(k \geq 2)$;
- *or* $\mathcal{F} \approx c_k = xy + y^k + y_3^2 + \cdots + y_n^2\,(k \geq 3)$;
- *or* $\mathcal{F} \approx f_4 = x^2 + y^3 + y_3^2 \cdots + y_n^2$;
- *or* $\mathcal{F}$ *is adjacent to one of the classes* $(P_8^\#), (X_9^\#), (J_{10}^\#)$*, which fence in simple boundary singularities (see. [1]);*
- *or* $\mathcal{F}$ *belongs to one of the classes* B_∞ *and* C_∞ *of infinite codimension.*

Thus we get respectively:

- *either* $P/Q \approx Ex/b_k, k \geq 2, j^0 E \neq 0, \models\!\!\Rightarrow$**12**;
- *or* $P/Q \approx Ex/c_k, k \geq 3, j^0 E \neq 0, \models\!\!\Rightarrow$**13**;
- *or* $P/Q \approx Ex/f_4, j^0 E \neq 0, \models\!\!\Rightarrow$**14**;
- *or* P/Q *is adjacent to one of the classes* $P_8^\#, X_9^\#, J_{10}^\#$ *of fractions* $Ex/\mathcal{F}$*, where* $\mathcal{F}$*-is a "fencing " boundary singularity,* $j^0 E \neq 0, \models\!\!\Rightarrow$**15**;
- *or* P/Q *belongs to one of the classes* B_∞ *and* C_∞ *of infinite codimenion (including, for example , the reducible fractions).*

12. *If* $j^0 E \neq 0, k \geq 2$*, then* $Ex/b_k \approx x/b_k \in B_k$.

13. *If* $j^0 E \neq 0, k \geq 3$*, then* $Ex/c_k \approx x/c_k \in C_k$.

14. *If* $j^0 E \neq 0$*, then* $Ex/f_4 \approx x/f_4 \in F_4$.

15. *The fractions $Ex/\mathcal{F}$, where $\mathcal{F}$ is a "fencing " boundary singularity (and hence all fractions adjacent to them) are not R^+-simple. For all regular values of parameter a they are $R^\times$-unimodal and R^+-unimodal.*

Remark. The codimension of the class of such fractions is equal to 5 (when $n \geq 2$).

16. *If $j^1Q = j^1P = 0, n \geq 2$, the the fraction P/Q is not R^+-simple.*

Remark. The codimension of the class of such fractions is equal to $n + 2 \geq 4$.

The theorems **1**–**16** imply that all fractions of theorem 1 (from § 1) are R^+-simple and vice versa. Really, as we have seen any other fraction is adjacent to some class of non R^+-simple (having moduli with respect to the R^+-equivalence relation) fractions. The simplicity of the fractions from the list of the theorem 1 is implied by the semi-continuity arguments of the multiplicity k, of ranks and other similar invariants. These agruments show that deforming a singularity of our list one gets only finite number of other singularities. All of them also belong to the list.

Example. Deforming the fraction with $j^kP \neq 0$ one can not get the fraction with $j^kP = 0$.

In particular, this implies that the singularities A^*, D^*, E^* are not adjacent to A, B, C, D, E, F, and hence to P, X, J. Similarly, the singularities A, D, E are not adjacent to A^*, D^*, E^*, B, C, F, and hence to P^*, X^*, J^*. The singularities B are not adjacent to A_2, and the singularities C to A_2^*. In a similar way one can check that all singularities of our list are adjacent only to the previous classes of this very list.

Remark. However not all adjacencies are shown in Fig. 1: for example, E_k is adjacent to A_{k-1}.

4 Proofs of the clasificator theorems

Theorems **1**, **2**, **4**,**5**, **6**, **7** and **9** are evident. Theorem **3** is implied by Hadamard Lemma. Theorems **8**, **10**, **11**, **15** are essentially the clasification of ordinary and boundary simple and fencing singularities of holomorphic functions (see [1],[5]). The only really new statement (which we shall now prove) is that from theorems **8**, **10** and **15** concerning R^+- (and even about $R^\times$-) unimodality of fencing singularities of fractions. These singularities have the form $1/\mathcal{F}, \mathcal{F}(y)/x$ and $x/\mathcal{F}(x, y)$ in theorems **10**, **8** and **11** respectively.

The proofs of unimodality of these fractions are bases on quasi-homogeneity of fencing singularities of holomorphic functions $\mathcal{F}$. It permits to reduce the $R^\times$-equivalence of fractions to the R-equivalence of holomorphic functions $\mathcal{F}$.

Lemma 1. *Let $\mathcal{F}$ be quasi-homogeneous holomorhic function of non-zero weight. Then any fraction being $R^\times$-equivalent to the fraction $1/\mathcal{F}$, is in fact R- equivalent to it.*

Proof. A quasi-homogeneous function $\mathcal{F}$ of non-zero weight is R-equivalent to the product of itself by any non-zero function (see [5] or lemma 3 below). Hence, when $H(0) \neq 0$, we obtain

$$(H/\mathcal{F}) + G = (H + \mathcal{F}G)/\mathcal{F} \approx 1/\mathcal{F}.$$

Thus, the class of $R^\times$-equivalence of the fraction $1/\mathcal{F}$ is determined by the class of biholomorphic equivalence of the denominator (provided that the latter is quasihomogeneous). The lemma is proved.

Corollary 1. *The singularity clases of fractions $(P_8^*), (X_9^*), (J_{10}^*)$ from theorem* **10** *are at least unimodal with respect to $R^\times$-equivalence (and hence with respect to R^+-equivalence of fractions). For almost all values of parameter a they are exactly R^+-unimodal (and hence $R^\times$-unimodal).*

Really, the singularities P_8, X_9 and J_{10} of corresponding holomorhic functions $\mathcal{F}$ are quasihomogeneous (with positive weights) and R-unimodal.

This completes the proof of theorem **10**.

Lemma 2. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be such a quasihomogeneous holomorphic functions in $y \in \mathbb{C}^m$ with non zero weights that the jerms of (irreducible) fractions $\mathcal{F}_i/x$ at the point $0 \in \mathbb{C}^{m+1}$ are $R^\times$-equivalent. Then $\mathcal{F}_1 \approx \mathcal{F}_2$ (that is one of the holomorphic function germs $\mathcal{F}_1$ and $\mathcal{F}_2$ at the origin in $\mathbb{C}^m$ might be obtained from the other via a biholomorphic diffeomorphism).*

Proof. Holomorphic type of zero hypersurface of numerator ($\mathcal{F}_i = 0$) in the smooth hupersurface of zeros of denominator ($x = 0$) is the same for $R^\times$-equivalent fractions. Meanwhile, the holomorphic type of zero hupersurface of quasihomogeneous function with non zero weghts determines this function up to R-equivalence. This proves lemma 2.

Corollary 2. *Fraction singularities of classes $(P_8), (X_9), (J_{10})$ from theorem* **8** *are at least unimodal with respect to $R^\times$-equivalence relation (and hence to R^+-equivalence). For almost all values of parameter a these fractions are exactly R^+-unimodal (and $R^\times$-unimodal).*

This statement follows from Lemma 2 since singularities P_8, X_9, J_{10} of corresponding holomorphic functions $\mathcal{F}$ are quasihomogeneous (with positive weights) and R-unimodal.

This completes the proof of theorem **8**.

Theorem **11** is an immediate consequence of classification of boundary singularities given in [5]. Really, the diffeomorphism, which reduces the boundary singularity of holomorphic function $\mathcal{F}(x, y)$ on the manifold with the boundary $x = 0$ to its normal form f, sends function x to the function $xE(x, y)$, where $E(0) \neq 0$. So the fraction $x/\mathcal{F}(x, y)$ takes the form $Ex/f(x, y)$. This proves theorem **11**.

The proofs of theorems **12–15** are bases on the following evident statement on quasihomogeneous functions.

Lemma 3. *Let P and Q be quasihomogeneous functions with different quasihomogeneous degrees, let E be a holomorphic function with non zero value at the origin. Then the fraction germ EP/Q at the origin is biholomorphic equivalent to the fraction germ P/Q: $EP/Q \approx P/Q$.*

Proof. Denote by z_i the quasihomogeneous coordinates and by α_i their weights. Thus we have

$$P(t^{\alpha_1} z_1, \ldots, t^{\alpha_n} z_n) \equiv t^\beta P(z_1, \ldots, z_n),$$
$$Q(t^{\alpha_1} z_1, \ldots, t^{\alpha_n} z_n) \equiv t^\gamma Q(z_1, \ldots, z_n).$$

Put $t(z) \equiv (\ln E(z))/(\beta - \gamma)$. Then the diffeomorphisms $(Z_i \equiv t^{\alpha_i}(z) z_i)$ is the required equivalence, as

$$(P/Q)(Z(z)) \equiv E(z)(P/Q)(z).$$

Lemma 3 is proved.

Normal forms b_k, c_k, f_4 are quisihomogeneous polynomials (whose degrees are greater than that of x, provided that $k \geq 2$). Therefore lemma 3 implies theorems **12–14** directly (the R-equivalence constructed in the proof of lemma 3 preserving the boundary $x = 0$).

Corollary 3. *Fencing fraction singularities of classes $(P_8^\#), (X_9^\#)$, $(J_{10}^\#)$ are at most R^+-unimodal (and hence at most $R^\times$-unimodal) provided that the values of parameter a are regular.*

In fact, by lemma 3 these fractions are R-equivalent themselves to normal forms of the type $x/f(x,y)$, where the quasihomogeneous polynomial f contains only one parameter a. All germs of their small deformations (values of a being regular) are either R^+-simple or belongs to the same class. The corollary is proved.

To complete the proof of theorem **15** claiming the unimodality of these fractions it rests to show that the module a of the normal form of fencing fraction x/f remains invariant not only with respect to R-equivalences, which preserves the boundary $x = 0$, but also to R^+-equivalences (in fact, even to $R^\times$-equivalences).

Lemma 4. *Let $f(x,y)$ be a quasihomogeneous function holomorphic at the origin, whose weight is greater than that of variable x. Then any fraction which is $R^\times$-equivalent to the fraction x/f is ,in fact, R-equivalent to it.*

Proof. Consider a fraction $Ex/f + G$, which is $R^\times$-equivalent to x/f. Here $E(0) \neq 0$. By lemma 3 the first term is R-equivalent to x/f, hence the sum is R-equivalent to $x/f + G' = (x + G'f)/f$. Consider $x + G'f = X$ as a new variable. Then

$$f(x,y) \equiv f(X,y) + [f(X - G'f(x,y), y) - f(X,y)].$$

Now the expresion in square brackets might be written in the form $K(X,y)f(x,y)$, where function K is holomorphic at the origin. The value of K at the origin equals zero, since $\partial f/\partial X$ vanishes at the origin (as the weight of f is greater than that of x). Therefore $f(x,y) \equiv H(X,y)f(X,y)$, where the function H being equal to $(1-K)^{-1}$ is holomorphic at the origin. Thus we get the identity

$$(x + G'f)/f(x,y) \equiv X/(H(X,y)f(X,y)).$$

This proves R-equivalence of the sum $Ex/f + G$ and the fraction $x/(Hf)$. The latter is R-equivalent to the fraction x/f by lemma 3, hence finally

$$Ex/f + G \approx x/f.$$

Lemma 4 is proved.

Lemma 5. *Let f_1, f_2 be quasihomogeneous holomorphic functions whose weights differ from that of quasihomogeneous variable x. Than the R-equivalence if irreducible fractions x/f_1 and x/f_2 implies boundary R-equivalence of denominators (i.e. the equivalence of function germs f_1 and f_2 at the origin via a diffeomorphism preserving the hypersurface $x = 0$).*

Proof. Initial R-equivalence maps x to Ex, and f_1 to Ef_2, where the holomorphic function E is non zero at the origin. Thus the diffeomorphism preserving the hyperplane $x = 0$ and converting f_1 into Ef_2 is already found. By lemma 3 $Ef_2 \approx f_2$ via the diffeomorphism preserving hyperplane $x = 0$. Lemma 5 is proved.

Corollary 4. *Let be quasihomogeneous functions in x and y. Let weight of f_1 and f_2 be greater that the weight of variable x. Then $R^{\times}$-equivalence of irreducible fractions x/f_1 and x/f_2 implies boundary R-equivalence of the denominators (i.e. there exists a diffeomorphism preserving hyperplane $x = 0$ and converting f_1 into f_2).*

Proof. The fractions are R-equivalent by lemma 4. By lemma 5 there exists the boundary R-equivalence of the denominators.

Corollary 5. *Fractions of the classes $(P_8^{\#}), (X_9^{\#}), (J_{10}^{\#})$ from theorem 2 are at least unimodal with respect to $R^{\times}$-equivalence relation (the more so to R^{+}-equivalence).*

Proof. The $R^{\times}$-equivalence of fractions implies boundary R-equivalence of the denominators. Since parameter a is module of boundary R-equivalence of denominators (according to the classification of fencing boundary singularities, see [1]), then it ought to be module for $R^{\times}$- (as well as for R^{+}-) classification of fractions.

This corollary and corollary 3 imply that *the fencing fraction singularities, mentioned in these statements are exactly unimodal for non singular values of parameter a with respect to both of $R^{\times}$- and R^{+}- equivalences.*

This completes the proof of theorem **15**.

Theorem **16** is proved in § 2 (example 2).

Theorems **1–16** together imply both theorem 1 § 2 and theorem 2 (the statement of the latter about adjacency and codimension is a consequence of boundary singularity theory and corollary 4).

5 Classification of $R^{\times}$-simple fractions germs

All R^{+}-simple fraction germs are so more $R^{\times}$-simple. The above classificator and corollaries 1, 2, 5 imply the following statement. Every non R^{+}-simple fraction germ:

- either belongs to one of the classes of infinite codimension;
- or is adjacent to one of the fencing classes, which consists of $R^{\times}$-unimodal germs;
- or belongs to the class of non R^{+}-simple fraction germs P/Q from theorem **16** (their numerator and denominator have common critical point with zero critical values).

Singularities from the latter class are not $R^{\times}$-simple provided that the fractions are defined on $\mathbb{C}^n$, $n > 2$ (example 2, § 2). Hence, the list of $R^{\times}$-simple fraction singularities consists of two parts: a) R^{+}-simple singularities; b) special fraction singularities in two variables, which are $R^{\times}$-simple but not R^{+}-simple.

These special classes are denoted in § 2 by $C_{p,q}$ and $\mathcal{F}_m$ (to be more strict, there are also infinitely degenerated limit cases $C_{\infty,q}, C_{\infty,\infty}$ and $\mathcal{F}_{\infty}$, forming the classes of infinite codimension). The location of each fraction in $R^{\times}$-equivalence hierarchy is determined by the following classificator (here $\underset{\times}{\sim}$ means $R^{\times}$-equivalence).

Classificator of $R^{\times}$-simple fraction singularities $P(x, y)/Q(x, y)$.

1. *If* $j^1P = j^1Q = 0$, *then*

- *either* $j^2Q \approx xy$, $\models\!\Rightarrow$**2**;
- *or* $j^2Q \approx x^2$, $\models\!\Rightarrow$**5**;
- *or* $j^2Q = 0$, $\models\!\Rightarrow$**13**.

2. *If* $j^2Q = xy$, *then* $P/Q \approx R/(xy)$, $\models\!\Rightarrow$**3**.

3. *If* $Q = xy, j^pR(x,0) = ax^p, j^qR(0,y) = by^q$, *then*

- *either* $ab \neq 0$, $\models\!\Rightarrow$**4**;
- *or* R/Q *belongs to a class of infinite codimension.*

4. *If* $Q = xy, j^pR(x,0) = ax^p, j^qR(0,y) = by^q, ab \neq 0$, *then* $R/Q \underset{\times}{\sim} (x^p + y^q)/(xy) \Rightarrow C_{p,q}$.

5. *If* $j^2Q = x^2, j^1P = 0$, *then*

- *either* $j^3Q \approx x^2 + y^3$, $\models\!\Rightarrow$**6**;
- *or* $j^3Q \approx x^2$, $\models\!\Rightarrow$**11**.

6. *If* $j^3Q = x^2 + y^3$, *then* $P/Q \approx R/(x^2 + y^3)$, $\models\!\Rightarrow$**7**.

7. *If* $Q = x^2 + y^3$, *then* $R/Q \sim (\alpha(y) + x\beta(y))/Q$, $\models\!\Rightarrow$**8**.

8. *If* $Q = x^2 + y^3, R = \alpha(y) + x\beta(y), j^1R = 0$, *then*

- *either for certain* $k \geq 2$ $j^k\alpha = ay^k, a \neq 0, j^{k-2}\beta = 0$, $\models\!\Rightarrow$**9**;
- *or for certain* $k \geq 2$ $j^k\alpha = 0, j^{k-1}\beta = by^{k-1}, b \neq 0$, $\models\!\Rightarrow$**10**;
- *or* R/Q *belongs to a class of infinite codimension.*

9. *If* $Q = x^2 + y^3, R = \alpha + x\beta, j^k\alpha = ay^k, k \geq 2, a \neq 0, j^{k-2}\beta = 0$, *then* $R/Q \underset{\times}{\sim} y^k/(x^2 + y^3), \Rightarrow \mathcal{F}_{2k+1}$.

10. *If* $Q = x^2 + y^3, R = \alpha + x\beta, j^k\alpha = 0, j^{k-1}\beta = by^{k-1}, k \geq 2, b \neq 0$, *then* $R/Q \sim xy^{k-1}/(x^2 + y^3), \Rightarrow \mathcal{F}_{2k+2}$.

11. *If* $j^3Q = x^2, j^1P = 0$, *then the fraction* P/Q *is ajacent to the fraction class* $x/(x^2 + axy^2 + y^4)$, $\models\!\Rightarrow$**12**.

12. *Fractions from the class* $X_9^{\#} = x/(x^2 + axy^2 + y^4)$ *are* $R^\times$*-unimodal for non singular values of parameter* a.

13. *If* $j^1P = 0, j^2Q = 0$, *then the fraction* P/Q *is adjacent to the fraction class* $x/(x^3 + ax^2y + y^3)$, $\models\!\Rightarrow$**14**.

14. *The fractions from the class* $J_{10}^{\#} = x/(x^3 + ax^2y + y^3)$ *are* $R^\times$*-unimodal for non singular values of parameter* a .

Proofs of the theorems on the classification of $R^{\times}$-simple fractions.

Theorems **1**, **3**, **8**, **11**, **13** are evident.

Theorems **2** and **5**, **6** are actually the R-classification of holomorphic functions of classes A_1 (Morse Lemma) and A_2.

Theorems **12** and **14** are the particular cases of the theorem **15** § 3.

The remaining statements (theorems **4**, **7**, **9**, **10**) are proven below .

Proof of theorem 4. By the hypothesis, $R = ax^p + by^q + (xy)S(x,y)$ (here we may assume a to depend on x and b – on y only), $ab \neq 0$ at the origin. Therefore, there exists a R^+-equivalence

$$R/(xy) \sim (ax^p + by^q)/(xy) = g(x,y).$$

Now transform the fraction $f = (x^p + y^q)/(xy)$ by neans of substitution $x = AX, y = BY$. We'll find such $A(X,Y)$ and $B(X,Y)$, that $f(x,y) = g(X,Y)$.

If $N = pq - p - q \neq 0$, then $A = a^{(q-1)/N} b^{1/N}, B = a^{1/N} b^{p-1/N}$ is the desired R-equivalence normalising the germ $g \approx f$. Despite this fact, all our singularities are even non R^+-simple (due to unique exeptional case $p = q = 2$, when $N = 0$). In this latter case we choose $X = \sqrt{a}x, Y = \sqrt{b}y$. The required $R^{\times}$-equivalence takes the form $g(x,y) = c(X^2 + Y^2)/(XY), c = 1/\sqrt{ab}$ (i.e. $g \underset{\times}{\sim} f$).

Proof of theorem 7. Decompose function R into its even and odd parts with respect to x

$$R(x,y) = A(x^2,y) + xB(x^2,y).$$

When $z = x^2 + y^3$ Hadamard lemma implies

$$\begin{aligned} R(x,y) &= A(z - y^3, y) + xB(z - y^3, y) = \\ &= A(-y^3, y) + xB(-y^3, y) + zC(x,y) = \\ &= \alpha(y) + x\beta(y) + QC, \end{aligned}$$

this proves theorem **7**.

We preface the proofs of theorems **9** and **10** with the following general remark.

Consider holomorphic (at the origin) vector fields tangent to level hypersurfaces of function Q. Such vector fields form module M over the ring of functions holomorphic at the origin. Phase flows of these fields preserve function Q.

Example. In the case of the function $Q = x^2 + y^3$ in two variables we are interested now in module M is generated by Hamiltonian field $H = 3y^2\partial/\partial x - 2x\partial/\partial y$ and consists of the fields hH, where function h are holomorphic at the origin.

Consider the ideal (Q, MP), generated by Q and by all functions from MP, where P is some chosen function. Assume this ideal to have finite codimension, so the factor by it is generated (as a linear space) by a finite set of functions $e_1, \ldots, e_N$. In other words, suppose that for any function (holomorphic at the origin) there exists a decomposition

$$f = \sum c_i e_i + gQ + hmP,$$

where $c_i \in \mathbb{C}$ are constants, functions g and h are holomorphic at the origin and $m \in M$.

Example 1. $Q = x^2 + y^3, P = ay^k, a \neq 0, k \geq 2$. In this case $HP = -2akxy^{k-1}$. Therefore ideal $(Q, MP) = (x^2 + y^3, xy^{k-1})$ has finite codimension. As the generators e_i one can choose the monomials $y^s, s < k + 2$ and $xy^s, s < k - 1$.

Examples 2. $Q = x^2 + y^3, P = axy^{k-1}, a \neq 0, k \geq 2$. In this case $HP = ay^{k-2}[3y^3 - 2(k-1)x^2]$. Hence ideal $(Q, MP) = (x^2 + y^3, y^{k+1}, x^2y^{k-2})$ has finite codimension. As functions e_i one can take monomials y^s and $xy^s, s < k + 1$.

Suppose now that function Q is quasihomogeneous (with positive weights of the arguments) and functions P and e_i are quasihomogeneous as well. Note, that in examples 1 and 2 all this assumptions are fulfilled.

Distinguish those e_i , whose quasihomogeneous degree is greater than that of quasihomogeneous function P. Denote them by f_j and call them *upper generators.*

Example 3. In example 1 there is only one upper generator: $f = y^{k+1}$. In example 2 it is alone ($f = xy^k$) as well.

Let now S be a quasihomogeneous function (holomorphic at the origin), whose degree is greater than that of quasihomogeneous function P.

Lemma. *Fraction $(P + S)/Q$ is R^+-equivalent to a fraction of the form $(P + T)/Q$, where T is a scalar linear combination of upper generators f_j.*

Proof. We adapt the algorithm of quasihomogeneous function normalisation (see [5]) to our situation, when the variable changes preserve all the values of Q (they are phase flow mappings of fields from M). As a result we reduce $P + S$ to the form $P + T + gQ$, preserving Q:

$$(P + S)/Q \approx (P + T + gQ)/Q \sim (P + T)/Q.$$

Lemma is proved.

Proof of theorem 9. Apply lemma to $Q = x^2 + y^3, P = ay^k, S = R - P$. Quasihomogeneous order of zero of perturbation S is greater than the degree of P by the assumption of the theorem. Acoording to example 3, we obtain the following R^+-equivalence

$$R/Q \sim Ey^k/Q,\ E = a + by,\ E(0) = a \neq 0.$$

By definition of $R^\times$-equivalence, $R/Q \underset{\times}{\sim} y^k/Q$. This proves the theorem.

Remark. If $k \neq 3$ we get even R^+-equivalence $R/Q \sim y^k/Q$ (by lemma 3 § 4).

Proof of theorem 10. Apply lemma to $Q = x^2 + y^3, P = axy^{k-1}, S = R - P$. quasihomogeneous order of zero of perturbation S is greater than the degree of P by theorem assumption. According to example 3, we get the R^+-equivalence

$$R/Q \sim Exy^{k-1}/Q,\ E = a + by,\ E(0) = a \neq 0.$$

Quasihomogeneous degrees of qusihomogenious functions xy^{k-1} and $x^2 + y^3$ are different (for arbitrary k). By lemma 3 § 4 we get the R^+-equivalence $R/Q \sim xy^{k-1}/Q$. This proves the theorem.

6 Singularities of polynomials at infinity.

Consider the family of level hypersurfaces of a polynomial in m complex variables near an infinitely distant point of m-dimensional projective space. The formulas given below describes this family of hypersurfaces (up to a biholomorphic equivalence) for generic polynomials and for generic familes of polynomials, depending on a few parameters, provided that the degrees of polynomials are large enough. Simplest singularies happen to be simple (non simple singularities arise in generic families only when the number of parameters are at least 5). All simple singularities (having no moduli for holomorphic equivalence relation) are described by means of ordinary (for boundary singularity theory) list of Weyl groups $A_k(k \geq 1), B_k(k \geq 2), C_k(k \geq 3), D_k(k \geq 4), E_k(k = 6, 7, 8), F_4$. The role of boundary is played of course by infinite hyperplane.

Consider polynomial $f(x, y_2, \ldots, y_m)$ in $\mathbb{C}^m, m \geq 2$. Near infinite distant point at the x axis introduce new affine coordinates $Y = y/x, Z = 1/x$ centered at this point ($[x : y : 1] = [1 : Y : Z]$). The explicit substitution of $1/Z$ instead of x and of Y/Z instead of y proves the following lemma.

Lemma. *In this new variables polynomials f is represented by fraction $f(x, y) = H(Y, Z)/Z^n$, where H is a polynomial of degree n (its coefficients are obtained from that of polynomial f by a permutation), and $Z = 0$ is the equation of infinite hyperplane.*

We will study the fraction germes of this special form H/Z^n with the numerator H being an arbitrary holomorphic function. Such a fraction will be called *R-simple*, if its neigbourhood *in the space of these fractions* is exhausted by a finite number of R-equivalence classes.

Theorem. *Simple (having no moduli with respect to biholomorphic R-equivalence) fraction singularities of the form H/Z^n in $\mathbb{C}^m$ are exhausted (up to R-equivalence) by the following list:*

$$
\begin{aligned}
&A_0 = Y_m/Z^n;\ A_k = (Y^{k+1} + Z + Q_2)/Z^n, k \geq 1;\\
&B_k = (Y^2 + Z^k + Q_2)/Z^n, k \geq 2;\\
&C_k = (Y^k + YZ + Q_2)/Z^n, k \geq 3;\\
&D_k = (Y^2U + U^{k-1} + Z + Q_3)/Z^n, k \geq 4;\\
&E_6 = (Y^3 + U^4 + Z + Q_3)/Z^n;\\
&E_7 = (Y^3 + YU^3 + Z + Q_3)/Z^n;\\
&E_8 = (Y^3 + U^5 + Z + Q_3)/Z^n;\\
&F_4 = (Y^3 + Z^2 + Q_2)/Z^n.
\end{aligned}
$$

These singularities are R-simple proveded that n is sufficiently large (namely, when $n > 1$ for A_k, C_k, D_k, E_k, when $n > k$ for B_k and when $n > 2$ for F_4).

Codimension of simple singularity class indexed by k is equal to k.

All non R-simple singularities form the subset of codimension five (provided that n is sufficiently large).

Remark. A class of fraction germs H/Z^n has codimension c, if the germs from this class arise (at isolated points) in generic fraction families of the form H/Z^n, depending on $p \geq c$ parameters (when parameters belongs to a certain $p - c$-dimensional submanifold in p-dimensional parameter space).

Example. Class A_1 has codimension 1, as the corresponding function H is to have a critical point with *zero critical value* on manifold $Z = 0$.

The adjacency table of R-simple and fencing singularities of the form H/Z^n (with large enough n) is given in Fig. 3. Fencing singularities are determined by formulas

$$
\begin{aligned}
&(P_8) = (Z + Y^3 + U^3 + V^3 + aYUV + Q_4)/Z^n,\\
&(X_9) = (Z + Y^4 + aY^2U^2 + U^4 + Q_3)/Z^n,\\
&(J_{10}) = (Z + Y^3 + aYU^4 + U^6 + Q_3)/Z^n;\\
&(P_8^{\#}) = (ZY + Y^3 + aY^2U + U^3 + Q_3)/Z^n,\\
&(X_9^{\#}) = (Z^2 + aZY^2 + Y^4 + Q_2)/Z^n,\\
&(J_{10}^{\#}) = (Z^3 + aZ^2Y + Y^3 + Q_2)/Z^n.
\end{aligned}
$$

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & \diagdown & & & & & & & \\
 & & & & & & & & (P_8) & \leftarrow & (X_9) & \leftarrow & (J_{10}) & & \\
 & & & & & & & \diagdown & \downarrow & \diagdown & \downarrow & \diagdown & \downarrow & \diagdown & \\
 & & & & & & & & E_6 & \leftarrow & E_7 & \leftarrow & E_8 & & \\
 & & & & & \diagdown & & \diagdown & \downarrow & \diagdown & \downarrow & \diagdown & \downarrow & \diagdown & \\
 & & & & & & D_4 & \leftarrow & D_5 & \leftarrow & D_6 & \leftarrow & D_7 & \leftarrow & \cdots \\
 & \diagdown & & \diagdown & & \diagdown & \downarrow & \diagdown & \downarrow & \diagdown & \downarrow & \diagdown & & \diagdown & \\
A_0 & \leftarrow & A_1 & \leftarrow & A_2 & \leftarrow & A_3 & \leftarrow & A_4 & \leftarrow & A_5 & \leftarrow & A_6 & \leftarrow & \cdots \\
 & \diagup & \uparrow & \diagup & \uparrow & \diagup & \uparrow & \diagup & \uparrow & \diagup & \uparrow & \diagup & \uparrow & \diagup & \\
 & & B_2 & \leftarrow & C_3 & \leftarrow & C_4 & \leftarrow & C_5 & \leftarrow & C_6 & \leftarrow & C_7 & \leftarrow & \cdots \\
 & \diagup & \uparrow & \diagup & \uparrow & \diagup & \uparrow & \diagup & & & & & & & \\
 & & B_3 & \leftarrow & F_4 & \leftarrow & (P_8^{\#}) & & & & & & & & \\
 & \diagup & \uparrow & \diagup & \uparrow & & | & & & & & & & & \\
 & & B_4 & \leftarrow & (J_{10}^{\#}) & - & (X_9^{\#}) & & & & & & & & \\
 & \diagup & \uparrow & \diagup & & & & & & & & & & & \\
 & & B_5 & & & & & & & & & & & & \\
 & \diagup & \uparrow & & & & & & & & & & & & \\
 & & \cdots & & & & & & & & & & & &
\end{array}
$$

$$
A_k = \frac{Y^{k+1} + Z + U^2}{Z^n};\ B_k = \frac{Y^2 + Z^k + U^2}{Z^n};\ C_k = \frac{Y^k + YZ + U^2}{Z^n};
$$

$$
D_k = \frac{Y^2U + U^{k-1} + Z}{Z^n};\ E_6 = \frac{Y^3 + U^4 + Z}{Z^n};\ E_7 = \frac{Y^3 + YU^3 + Z}{Z^n};
$$

$$
E_8 = \frac{Y^3 + U^5 + Z}{Z^n};\ F_4 = \frac{Y^3 + Z^2 + U^2}{Z^n}.
$$

Fig. 3. Adjacencies of R-simple fraction singularities of the form H/Z^n.

Here by Q_r we denote the quadratic form $Y_{r+1}^2 + \cdots + Y_m^2$. Thus, the coordinates in $\mathbb{C}^m$ are denoted by $(Z, Y, Y_3, \ldots, Y_m)$ for $A_k, B_k, C_k, F_4, (X_9^{\#}), (J_{10}^{\#})$, by $(Z, Y, U, Y_4, \ldots, Y_m)$ for $D_k, E_k, (X_9), (J_{10})$, $(P_8^{\#})$ and by $(Y, Z, U, V, Y_5, \ldots, Y_m)$ for (P_8).

Fencing singularities are even not $R^{\times}$-simple (they have moduli with respect to $R^{\times}$-eqivalence of fractions of the form H/Z^n). Besides these fencing not $R^{\times}$-simple singularities and those adjacent to them there exist also classes of non R-simple singularities H/Z^n $A_\infty, B_\infty, C_\infty, D_\infty$ of infinite codimension. Singularities with non suffuciently large degrees n

of denominators ($n \leq k$ for B_k, $n = 1$ for A_k, C_k, D_k, E_k, $n = 1$ and 2 for F_4) are not R-simple as well.

In the fig. 3 the singularities of equal codimensions are connected by lines. The formulas in the figure correspond to the case $m = 3$.

Example. General four-parameter families of fractions H/Z^n contain only simple singularities: D_4, A_4, C_4, F_4, B_4 – at isolated points, A_3, C_3, B_3 – on certain curves, A_2,B_2 –on some surfaces, A_1 – on three dimensional submanifolds and A_0 at the remaining points of four-dimensional parameter space. Here we suppose, that $n > 4$, otherwise class B_4 in not R-simple.

Proof of the theorem. Boundary singularity classification implies that either function H might be reduced to the corresponding normal form of the numerator from the theorem by a diffeomorphism, which preserve hyperplane $Z = 0$, or this function is adjacent to one of the six fencing unimodal singularity classes, or belongs to one of the following classes of infinite codimension $A_\infty, \ldots, D_\infty$.

In the first case the fraction might be reduced (by biholomorphic transformation) to the form

$$H/Z^n \approx EH_0/Z^n,$$

where $E(0) \neq 0$. Normal forms H_0 for all these classes are quasihomogeneous. If n is sufficiently large (precise limits are described in the theorem), then the quasihomogeneous degree of the numerator is less than that of the denominator. According to lemma 3 § 4 coefficient E might be exterminated by an appropriate R-equivalence. So in this case

$$H/Z^n \approx H_0/Z^n.$$

Any fencing singularity is non $R^\times$-simple as the singularity of fraction H/Z^n. This might be proved by lemma 2 § 4 arguments (as the fencing singularities are quasihomogeneous). The equality of codimension of classes $A_\infty, \ldots, D_\infty$ to infinity is evident. The theorem is proved.

Corollary. *Generic family of fractions H/Z^n, containing simple singularity H_0/Z^n for zero parameter value , can be reduced for the nearby parameter values to the form $\mathcal{F}/Z^n$, where $\mathcal{F}$ is a versal deformation of boundary singularity of H (using a holomorphic change of variables holomorphically depending on parameters and a holomorphic change of parameters).*

Example. One-parameter generic family of fractions H/Z^n in $\mathbb{C}^2$ might be reduced to the form

(1) $A_1 = (Y^2 + Z + \lambda)/Z^n$,

where λ is a parameter.

Two-parameter generic family of fractions have local normal forms

(2) $A_2 = (Y^3 + \lambda_1 Y + \lambda_2 + Z)/Z^n$,

(3) $B_2 = (Y^2 + Z^2 + \lambda_1 Z + \lambda_2)/Z^n$.

Finally three-parameter generic families have normal forms:

(4) $A_3 = (Y^4 + \lambda_1 Y^2 + \lambda_2 Y + \lambda_3 + Z)/Z^n$,

(5) $B_3 = (Y^2 + Z^3 + \lambda_1 Z^2 + \lambda_2 Z + \lambda_3)/Z^n$,

(6) $C_3 = (Y^3 + \lambda_1 Y^2 + \lambda_2 Y + \lambda_3 + YZ)/Z^n$.

Appliing these formulae to generic families of polynomials in two variables at infinity, we obtain

Corollary. *If the polynomial degree n is sufficiently large ($n > p$, where p is the number of family parameters), then (provided that $p < 5$) a family of polynomials , which depends generically on p parameters, has local normal form $f = \mathcal{F}/Z^n$, where $\mathcal{F}$ is a versal deormation of the corresponding simple boundary singularity (A_k, B_k, C_k, F_4).*

Example. Level lines of generic polynomial ($p = 0$) intersect transversally the infinity line (A_0 case). Formula $A_0 = Y/Z^n$ implies that the level lines ($Y = cZ^n$) of generic polynomial f of degree $n > 1$ are tangent to each other at generic point at infinity-line (they have there a common point of multiplicity n).

In generic one parameter family of polynomials of degree $n \geq 2$ the level lines might have tangency with the infinity line for isolated parameter values (formula (1) for $\lambda = 0$ determines the level line equation of polynomial f: $Z = -Y^2 + cZ^n$). The latter implies that all level lines $f = c$ are quadratically tangent to infinity line. Meanwhile these lines become more tangent to each other (than for regular point). At the infinity they intersect at a common point with multiplicity $2n$.

Formula (1) shows how the bifurcation (i.e. the transition from transversal intersection with the infinity to quadratic tangency ($Z = -Y^2 - \lambda + cZ^n$) looks like. Naturally, the conjunction of two intersection points (distance between them being of the order $\sqrt{\lambda}$ when $\lambda \neq 0$) causes for $\lambda = 0$ the augmentation of their intersection multiplicity : Instead of two intersection points of multiplicity n there arise one of multiplicity $2n$.

Similarly, for $n \geq 3$ generic bifurcations in two parameter families of polynomials of degree n are described by formulae (2), (3). When $n \geq 4$ generic three parameter bifurcations are determined by formulae (4), (5), (6).

Our formulae imply various topological and analitic results on singularities and bifurcations at infinity of level surfaces of families of polynomials in arbitrary number of variables: theory of vanishing cycles, corresponding monodromy group theory (generalised braids group representations), asymptotic behaviour of various integrals, related to polynomials, corresponding period mappings, Poisson and Frobenius structures etc.

Even simplest examples (of small dimension and codimension) in this field provides non trivial results. I would like to emphasize the possibility to apply these ideas to the study of non subregular orbits of coadjoint representation of simple complex Lie algebra and of complicated hierarhy of conditions at infinity, appearing when "Milnor fibrations" of these non isolated singularities are investigated (cr. [6], [7]).

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