The Vassiliev Theory of Discriminants and Knots

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In these times, the angel of topology and the devil
of abstract algebra fight for the soul
of each individual mathematical domain

H. Weyl*

The study of the discriminant variety in a functional space of smooth mappings is a traditional and fundamental part of the theory of singularities. The discriminant variety is the set of those points of the functional space which represent the mappings having nongeneric singularities. The topological, homotopical and even homological invariants of the complement to the discriminant variety (that is, of the space of generic mappings) are important for many applications. However, progress in these difficult global problems of singularity theory was rather slow until Vassiliev [1] over the last few years has demonstrated the new perspectives opened up by the singularity theory approach in knot theory.

A *knot* is a connected component in the space of smooth embeddings of a circle into 3-space. Hence, we start with the functional space $\mathcal{F}$ of all smooth mappings of S^1 into $\mathbb{R}^3$ and we define the *discriminant variety* Σ as the set of mappings, which are not embeddings (that is, those that have either self-intersections or singularities (see Figure 1).

The discriminant variety 'is a *hypersurface* in the space $\mathcal{F}$ of all mappings since the self-intersections occur in generic one-parameter families of mappings of a curve in 3-space. This hypersurface subdivides the complement into connected domains which are the knots.

We wish to study the topological properties of the knot space $\mathcal{F} - \Sigma$. For instance, the elements of its 0-dimensional cohomology group are locally constant functions, that is knot invariants. Such functions can be multiplied, forming a ring:

$$H^\circ(\mathcal{F} - \Sigma) = \text{knot invariants ring.}$$

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* *Invariants,* Duke Math. J., **5** (1939). This description seems to be an allusion to a painting by Uccello (at the Urbino castle) "l'hostie profannée," representing an event that happened in Paris in 1290. The event is also represented in a series of pictures in the church Saint-Jean-Saint François in Paris.

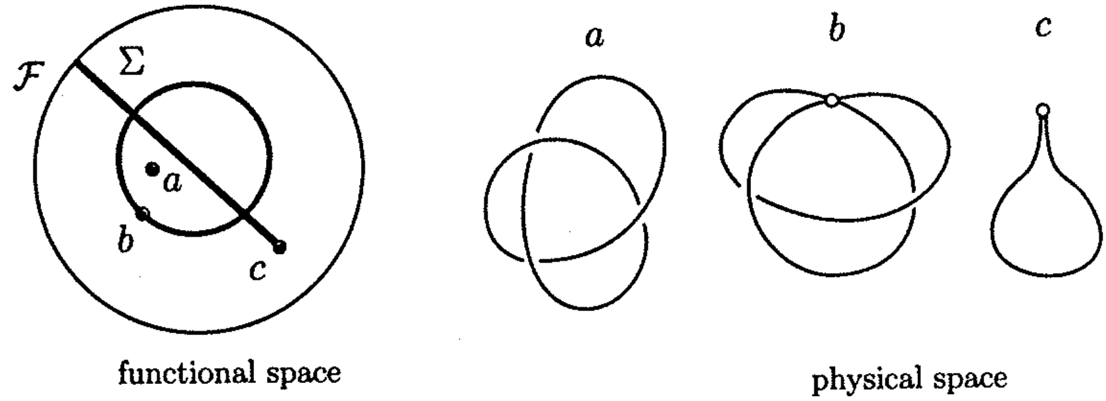


Figure 1. Generic mappings and the discriminant variety in the space $\mathcal{F}$ of mappings $S^1 \to \mathbb{R}^3$

The space $\mathcal{F}$ is linear and hence contractible. The study of the cohomology of the space of knots is therefore reducible to that of the discriminant (modulo ∞), by the Alexander duality. The difficulties of the infinite dimensionality of $\mathcal{F}$ can be overcome by the standard finite dimensional approximation technique of singularity theory (see e.g., [2]–[10]). For instance, one can replace $\mathcal{F}$ by the space $\mathcal{F}_N$ of trigonometric polynomials of degree at most N. Then each homology group $H^i(\mathcal{F}_N - \Sigma)$ stabilizes for $N \to \infty$:

$$H^i(\mathcal{F}_N - \Sigma) \approx H^i(\mathcal{F} - \Sigma) \text{ for } N \gg i.$$

Thus the Alexander duality is essentially used only in finite dimensional cases.

The advantage of the discriminant variety (over its complementary knot space which is our main object of study) is that this variety is naturally stratified according to the hierarchy of the singularities (while the knot space is smooth). Thus, to study homology, we need to cut the knot spaces into pieces, while for the discriminant variety, the pieces are provided by the strata of the stratification. This stratification induces an additional structure in the homology of the discriminant which survives also in the cohomology of the knot space, for instance, in the ring of its zero-dimensional cohomology. This talk is an introduction to the study of the Vassiliev structure in the ring of knot invariants.

The works of J. Birman, X.S. Lin, D. Bar-Natan, and M. Kontsevich ([11]–[15]) have shown that this Vassiliev structure is a fundamental general combinatorial mathematical object, related to the Jacobi identity, Yang-Baxter and Knizhnik-Zamolodchikov equations, the hierarchy of Feynman integrals of perturbative theory in the Chern-Simons action, the D. Zagier ζ-functions of several variables, and to the cohomology of the Lie algebra of Hamiltonian vector fields on infinite dimensional spaces.

1. Vassiliev invariants

These invariants form an increasing sequence of finite dimensional subspaces in the ring of knot invariants, similar to the sequence of spaces of polynomials of increasing degree in the ring of power series. Together these finite dimensional subspaces form the subring V of the Vassiliev invariants:

$$H^0(\mathcal{F}\backslash\Sigma) \supset V \supset \ldots \supset V_n \supset \ldots \supset V_1 \supset V_0.$$

The subspace V_n (or the subgroup, if we consider cohomology with integer coefficients) is called *the space (group) of Vassiliev's invariants of order n.* The product of invariants of orders m and n will be an invariant of order $m+n$.

The polynomials of degree at most n are defined by the condition $d^{n+1}p = 0$. The Vassiliev invariants of order n are defined by a similar condition

$$\nabla^{n+1} i = 0, \quad i \in H^0(\mathcal{F} - \Sigma),$$

with the *jump operator* ∇ replacing the derivative (and which we shall see is also similar to the residue) is defined by the following construction.

Lemma. *The discriminant hypersurface in $\mathcal{F}$ has a natural coorientation* (Figure 2).

Indeed, fix the orientations on the circle and in 3-space. A generic (nonsingular) point of the discriminant hypersurface is represented in the physical space by an immersed curve γ with one point of transversal self-intersection. A small displacement of the point from the discriminant hypersurface in a direction, transversal to it, transforms the curve γ into an embedded curve γ^+ or γ^-. The self-intersection point is represented on each of these embedded curves by two points 1 and 2. The velocity vectors of the embedding at points 1 and 2, together with vector 12, form a frame in 3-space. Its orientation is positive in one case (γ^+) and negative in the other (the result does not depend on the choice of points 1 and 2, for instance, not on their ordering). □

Definition. The *jump* of an invariant i at a point of the discriminant hypersurface is the difference of the values of the invariant, evaluated at

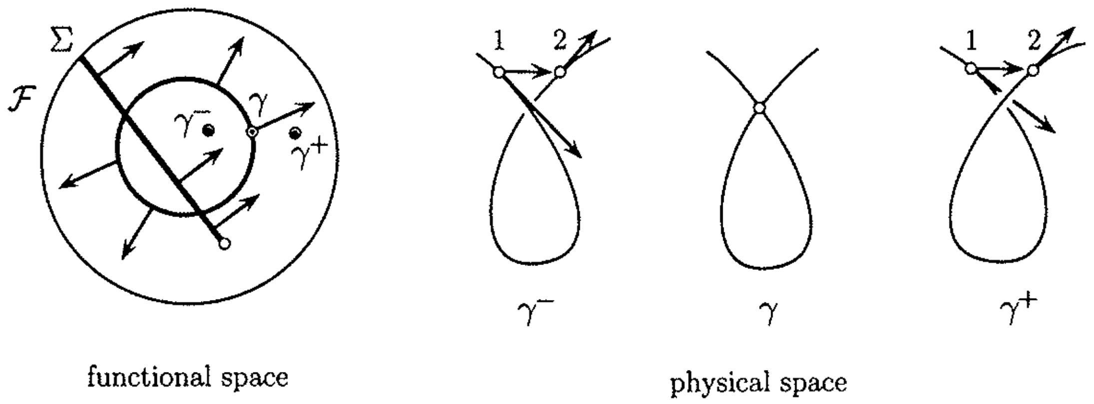


Figure 2. Coorientation of the discriminant variety

both sides of the hypersurface:

$$(\nabla i)\gamma = i(\gamma^+) - i(\gamma^-), \quad i \in H^0(\mathcal{F} - \Sigma).$$

Thus, ∇i is a locally constant function on the set of nonsingular points of the discriminant.

Iterating this construction, one defines the n-th jump, $\nabla^n i$, which is a locally constant function on the set of immersions whose images have n double points.

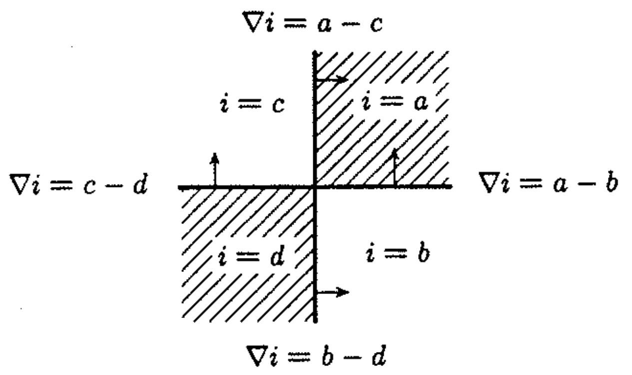
Example. The second jump of an invariant is defined at the self-intersection points of the discriminant hypersurface as the jump of the first jump of the invariant at the first branch of the discriminant hypersurface (Figure 3). Its value does *not* depend on the choice of the branch of the discriminant hypersurface which was called above the first one. Similarly, the higher jumps are well defined.

Definition. A *Vassiliev invariant of order* n is a knot invariant whose $n+1$-th jump vanishes identically.

Theorem. *The Vassiliev invariants form a subring of the ring of all knot invariants. Indeed, the following version of the Leibniz formula holds:*

$$\nabla(ij) = i^+j^+ - i^-j^- = i^+j^+ - i^+j^- + i^+j^- - i^-j^- = (i^+)\nabla j + (j^-)\nabla i.$$

Hence the product of Vassiliev invariants of orders m and n is a Vassiliev invariant of order at most $m+n$.



$$(\nabla^2 i) = (a - b) - (c - d)$$
$$= (a - c) - (b - d)$$

Figure 3. The independence of the second jump of an invariant on the ordering of the branches of the discriminant variety

Before we start to calculate the ring of Vassiliev invariants, let us discuss the motivations behind its definition.

The standard technique of topological work with discriminant varieties is the following *resolvent* construction. Replace each self-intersection point by two copies of it (one at each branch) and add a segment so that these are joined points. Then replace all the triple points by triads of points and glue a closed 2-simplex to each such triad. Glue 3-simplices to the resolved quadruple points, and so on (Figure 4).

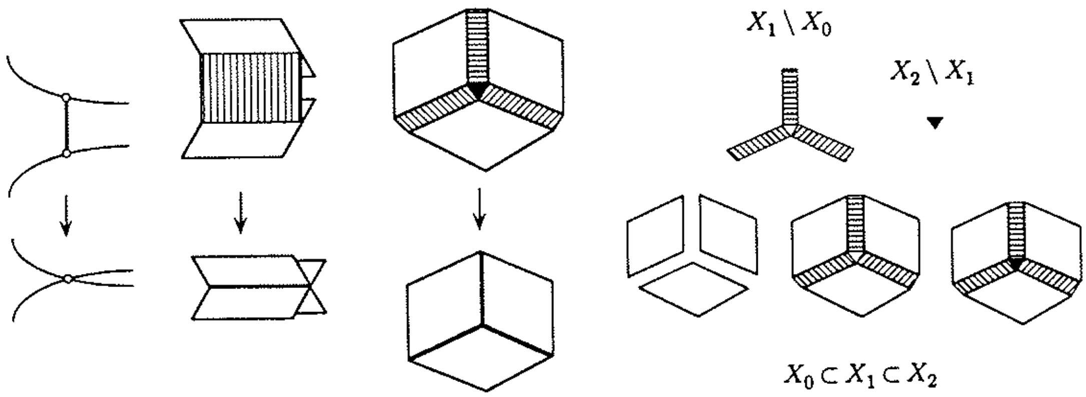


Figure 4. The resolution of self-intersections

The resulting topological space is homotopy equivalent to the initial one. It has an increasing filtration $X_0 \subset X_1 \subset X_2 \subset \ldots$, where $X - X_i$ replaces the self-intersections of multiplicity greater than i. The difference $X_i - X_{i-1}$ is the closure of the space of the fibration into open i-simplices over the set of self-intersection points of multiplicity i. The space X_0 is the closure of the set of those points of the initial (discriminant) variety, which are not self-intersection points.

Now one considers the spectral sequence associated to this filtration. Vassiliev's iterated jumps occur naturally in the study of the first differential of this spectral sequence (see [1]). If this sequence converges to the cohomology of the knots space, then the Vassiliev invariants distinguish all knots. This way of thinking, so natural from the singularity theory point of view, was rather unusual for the knot theorists. Vassiliev theory had not been noticed by the knot theory community until I explained it to Joan Birman, and posed the problem of whether Vassiliev invariants distinguish more knots than do the one variable Jones polynomials (a question which she and X. S. Lin ssubsequently settled affirmatively).

Kontsevich stated in his Bonn lectures in March 1992 that the Vassiliev spectral sequence degenerates at the first term (at least when tensored with $\mathbb{C}$).

Remark 1. Vassiliev has conjectured that his invariants distinguish any two knots. This conjecture has been neither proved nor disproved. In any case, the Vassiliev invariants distinguish at least as many knots as all other known invariants. For instance, if one substitutes e^t for the variable in the Jones polynomial and develops the resulting function in a Taylor series, then the coefficient of the term containing t^n will be a Vassiliev invariant of order n (Birman and Lin [11]). Hence all knots, distinguished by the Jones polynomials, are distinguished also by Vassiliev invariants. Similar results hold for all other known polynomial invariants.

Remark 2. The Vassiliev ring has not yet been computed explicitly. However Bar-Natan and Kontsevich announced that the corresponding graded ring (tensored with $\mathbb{C}$) is isomorphic to the graded ring of polynomials in an infinite set of indeterminates whose degrees are such that the number $\#(n)$ of indeterminates of any fixed degree n is finite:

n	1	2	3	4	5	6	7	8	9
$\# n$	0	1	1	2	3	5	8	12	?

One thus finds the dimensions of the spaces of Vassiliev invariants of small order n to be:

n	0	1	2	3	4	5	6	7	8	9
$\dim V_n$	1	1	2	3	6	10	19	33	60	?

For $n < 5$, these dimensions and spaces had been calculated by Vassiliev [1] and for higher n they have been calculated by Bar-Natan (using many hours of Cray computations).

2. Calculation of the Vassiliev invariants

The dual of the free finitely generated abelian group V_n/V_{n-1} admits an explicit combinatorial description: it is generated by the Feynman diagrams of a special form (Vassiliev diagrams), and their relations are described below. These relations, while rather complicated, are as fundamental as the relations in braid groups, the Jacobi identity, the Yang-Baxter and Knizhnik-Zamolodchikov equations mentioned above (which are closely related to the combinatorics of the relations between the Vassiliev diagrams).

To understand the nature of these relations, we shall start to calculate the Vassiliev invariants of small order n. It is technically convenient to represent the knots by embeddings $\mathbb{R} \to \mathbb{R}^3$ with boundary conditions at infinity (the corresponding functional space of mappings $\mathcal{F}$ is an affine space).

2.1 Invariants of order 0

The defining relation $\nabla i = 0$ means that the invariant i is constant globally. Hence *the space of zero order Vassiliev invariants is the space of constants.*

$$V_0 = \mathbb{Z}$$

(similar to the space of polynomials of degree zero)

2.2 Invariants of order 1

The defining equation $\nabla^2 i = 0$ means that *the first jump of the invariant i is constant on all the immersions with just one point of transversal self-intersection* (Figure 5).

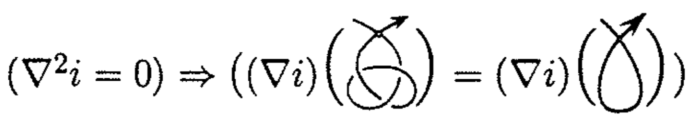
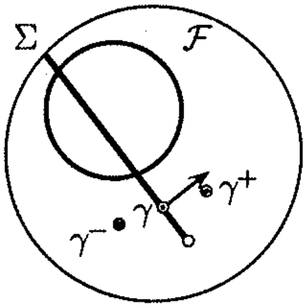


Figure 5. The constancy of the jump of an invariant of order 1

Indeed, each pair of immersions of this class is joined by a finite chain of surgeries ("perestroikas") during which one branch of the curve moves through the other (introducing at that moment one new double point of transversal self-intersection of the immersed curve). The jump of the first jump at any such surgery vanishes, since $\nabla^2 i = 0$. Hence the value of the jump is the same as for the standard plane curve γ (Figure 6)



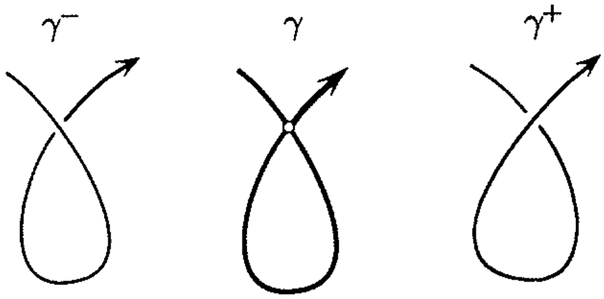


Figure 6. The calculation of the jump of an invariant of order 1

For the standard plane curve γ with one self-intersection point the discriminant is surrounded by the same component of the complement from both sides (the curves γ^- and γ^+ are regularly isotopic). Hence $(\nabla i)\gamma = 0$, and thus *any first order invariant is a zero order invariant*:

$$V_1 = V_0 = \mathbb{Z}.$$

Remark. In terms of the functional space $\mathcal{F}$, the preceding result expresses the following information on the discriminant hypersurface:

(1) the strata, corresponding to more complicated singularities of the discriminant hypersurface, as well as the transversal self-intersection of two branches, *do not divide* the discriminant hypersurface.

(2) the stratum (of codimension 2) in $\mathcal{F}$, formed by the simplest (cusped) singular curves in $\mathbb{R}^3$, is *the boundary* of the discriminant hypersurface.

The mini-versal deformation of a semi cubical cusp is two-parametrical, and the discriminant hypersurface intersects the plane of the parameters along a ray, ending at the point representing the cusped curve. (Figure 6 is thus rather realistic).

It is clear that the points of the plane at both sides of a ray belong to the same component of the complement to that ray. That explains the existence of a regular isotopy between the embeddings γ^+ and γ^-.

The calculations of the higher order invariants are similar to what we have done; only the simplest information on the stratification of the discriminant hypersurface, corresponding to the hierarchy of singularities, is used. This information is provided by the versal deformations of some few very simple singularities. The next step, where the relevant singularity is the triple point, is crucial for the whole theory.

2.3 Invariants of order 2

The defining equation $\nabla^3 i = 0$ means that the second jump of the invariant i does not change under surgery of an immersion whose image has two points of transversal self-intersection, which introduces for a moment a third self-intersection point

Unlike the immersions whose image has one self-intersection point, *the immersions with two such points cannot in general be connected by a finite chain of surgeries each of which introduces momentarily one more self-intersection point.*

Indeed consider the preimages of the double points on the oriented line by examining their mappings in 3-space. There are 4 preimages and they form two pairs (the two points of a pair have the same images in 3-space).

It is convenient to describe a decomposition of the set $\{1, 2, \ldots, 2n\}$ into n pairs by a system of arcs in the upper halfplane (connecting the i-th point with the j-th one iff (i, j) is a pair). I shall call any such system of n arcs a *Vassiliev diagram* of order n (Figure 7).

Figure 7. The Vassiliev diagrams of order 2

Of course, many people have previously studied these diagrams, which, for instance, describe the classes of complete flags in a linear symplectic space of dimension $2n$. The components of the knot space are the orbits of the coadjoint representation of $S\,\mathrm{Diff}\,\mathbb{R}^3$, which may be more than just a coincidence.

There exist exactly 3 Vassiliev diagrams of order 2 (Figure 7).

The Vassiliev diagram of an immersion with n double points does not change under a surgery, which introduces momentarily one more double point of the immersed curve. Hence *there exist at least three immersions of a line with 2 double points which cannot be reduced to one another by a chain of such surgeries* (Figure 8).

Any immersion with two self-intersection points on the immersed curve can be reduced to one of these three standard curves by a finite chain of standard surgeries, which introduces a third self-intersection point.

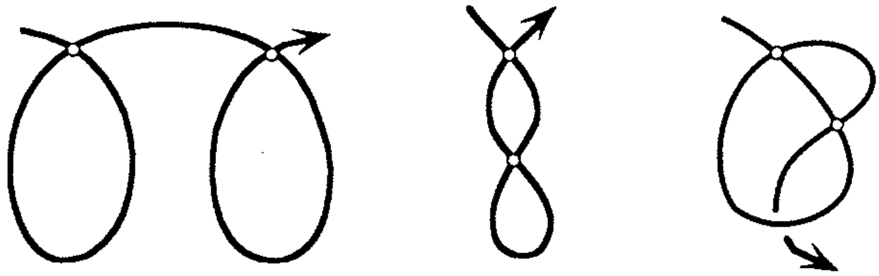


Figure 8. The three standard immersed curves with 2 double points

Therefore, any Vassiliev invariant of order 2 is determined (up to an additive constant) by the values of its second jump on the three standard curves of Figure 8.

The values of the second jump of any invariant on the first two standard curves of Figure 8 vanish. This follows from the fact that both resolutions of one of the self-intersections produce equivalent (smoothly isotopical) immersed curves with one transversal self-intersection. (Figure 9).

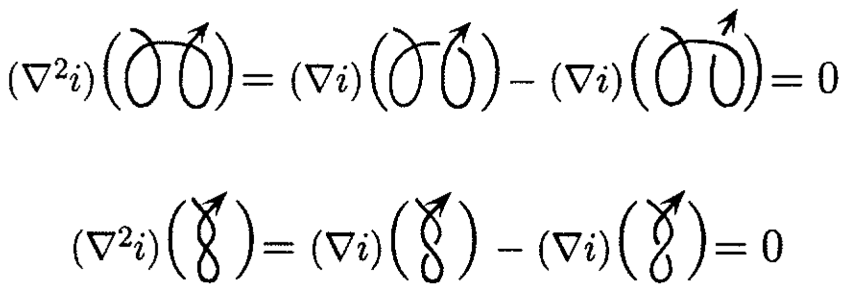


Figure 9. Evaluation of the second jump

Thus, the second jump of an invariant of the second order is unambiguously defined by its value on the third curve. If this value does not vanish (i.e., if the invariant is genuinely of second and not of the first order) then we can multiply it by a constant in such a way that the value on the third curve of Figure 8 will be equal to 1.

A second order Vassiliev invariant with these properties exists and is unique (up to an additive constant). Thus, $V_2 \approx \mathbb{Z}^2$. We can eliminate the constant, choosing the value of the invariant on an unknot to be zero. The calculation of this invariant for the trefoil knot is presented in Figure 10 (where, for simplicity, the signs are neglected):

$$1 = (\nabla^2 i)(K) = (\nabla i)(K^+) - (\nabla i)(K^-), \quad (\nabla i)(K^-) = 0;$$
$$1 = (\nabla i)(K^+) = i(K^{++}) - i(K^{+-}), i(K^{+-}) = 0;$$
$$1 = i(K^{++}) = \text{ the value of the invariant evaluated at a trefoil knot.}$$

The existence of this invariant is proved in [1].

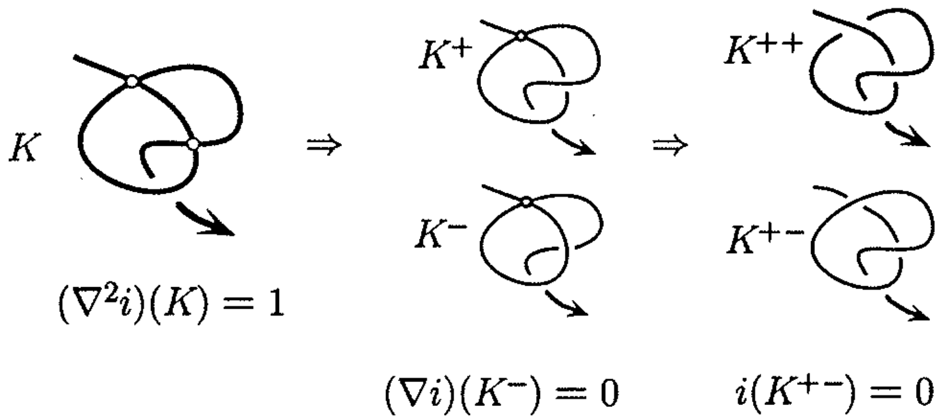


Figure 10. Calculation of the Vassiliev invariant of order 2

This invariant is nontrivial. But it can be reduced to the known ones (it is equal to the x^2 coefficient in the Conway version of the Alexander polynomial). In this case Vassiliev's approach gives an algorithm for calculation of an old invariant. In more complicated cases it generates invariants automatically, by standard combinatorial calculations similar to the preceding ones.

3. The group of diagrams

The calculation of the Vassiliev invariants of order n is similar to the calculations of those of order 2. The defining equation $\nabla^{n+1} i = 0$ means that the n-th jump of the invariant i is a locally constant function on the space of immersions, whose images have n self-intersection points, which does not change under the surgeries, which introduces momentarily one more double point. It follows that *the* n-th *jump*, $\nabla^n i$, *depends only on Vassiliev diagram of the immersion with* n *double points.* □

The number of Vassiliev diagrams formed by n arcs is equal to

$$(2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1).$$

An invariant of order n is defined by the values of its n-th jump on those diagrams up to the addition of an invariant of a smaller order. Hence we obtain the inequality

$$\dim V_n/V_{n-1} \leq (2n-1)!!$$

showing that *the space of Vassiliev invariants of any given order is finite-dimensional.*

To describe explicitly the space V_n/V_{n-1}, it is convenient to start from the free abelian group $\mathbb{Z}^{(2n-1)!!}$, whose generators are the diagrams of order n. The n-th jump of an invariant of order n is a linear function on the additive group, generated by the diagrams. However, as we have seen above for $n = 2$, some of these linear functions are not equal to the n-th jump of any n-th order Vassiliev invariant. For example, for $n = 2$ the values of this function on the first two diagrams of Figure 7 must vanish.

In the general case of arbitrary n, the admissible linear functions are those which vanish on some special diagrams or linear combinations of diagrams. We shall describe below these diagrams and combinations. It is convenient to introduce the following.

Definition. The group of diagrams of order n is the abelian group A_n whose generators are the Vassiliev diagrams consisting of n arcs and whose relations subgroup (in the free abelian group generated by the diagrams) is generated by the two types of relations, as described below:

$$A_n = \frac{\mathbb{Z}^{2n-1!!}}{\text{(relations 1 and 2)}}.$$

Relation 1. (The *easy relations*). Each diagram, containing an arc joining two neighboring points belongs to the relations subgroup (Figure 11).

Relation 2. (The *4-term relations).* The combination of four diagrams

$$S_1 - S_2 + S_3 - S_4$$

belongs to the relations subgroup. Here S_i are the diagrams consisting of n arcs, which are described in (Figure 13) below.

The 4-term relation is a fundamental combinatorial relation whose role in Vassiliev invariants theory is similar to that of the Jacobi identity in Lie algebra theory.

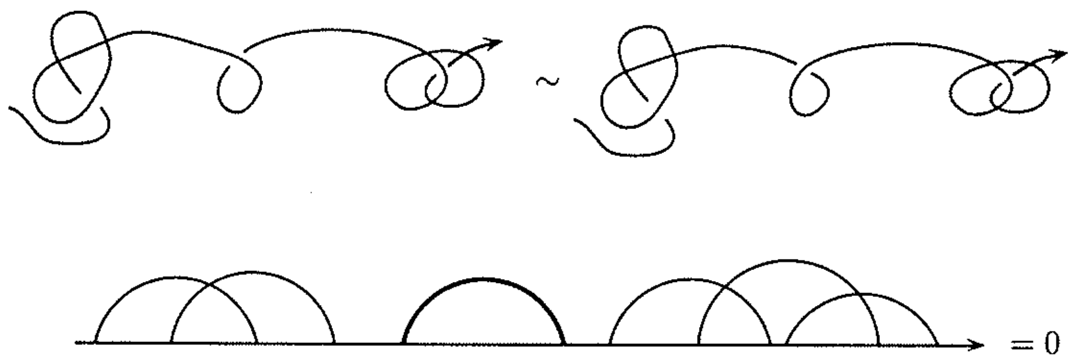


Figure 11. An easy relation and its motivation

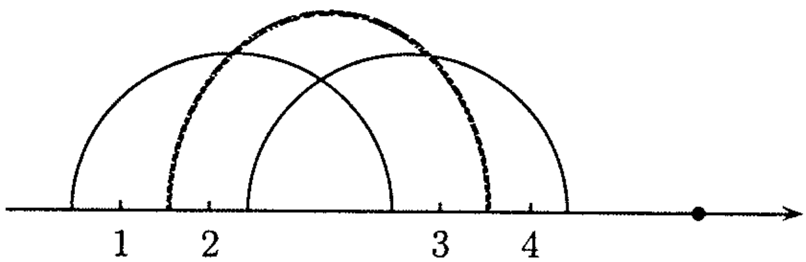


Figure 12. The construction of a 4-term relation

To write the Jacobi identity as a system of relations between the structure constants we have to fix the value of four indices (i, j, k, l) and then add the corresponding products of the structure constants with those indices. Thus the Jacobi identity is in fact a family of numerical equations, parameterized by the choice of indices.

The parameter of the 4-term relations of the group A_n consists of the following data:

(1) a Vassiliev diagram of order $n-2$ (shown in Figure 12 by the ordinary lines);
(2) one more distinguished arc in the upper halfplane (shown in Figure 12 by a wavy line);
(3) one more distinguished point on the border line (0 in Figure 12).

Thus the total number of points at the border line is $2n-1$. These points divide the line into parts. Let us consider the 4 parts adjacent to the endpoints of the distinguished arc (some of these parts may coincide). We denote them by the numbers $(1, 2, 3, 4)$ in the order defined by the orientation of the border line.

The diagram S_i is the union of the $n-1$ arcs defined by the above data and of one more arc joining the distinguished point to a point of the part i. A 4-term relation, corresponding to the data in Figure 12, is represented in Figure 13.

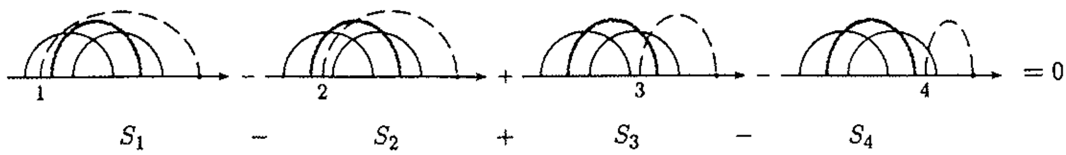


Figure 13. A 4-term relation

Remark. The 4-term relations, which were implicit in Vassiliev's initial work [1], have been written in the form described above by Birman and Lin [11].

The number of independent relations among the relations 1 and 2 is at present known only for small n. According to the computations of Vassiliev ($n < 5$) and Bar-Natan, the ranks of the free abelian groups A_n are given by the following table:

n	0	1	2	3	4	5	6	7	8	9
$\dim A_n$	0	0	1	1	3	4	9	14	27	?

Any function on the set of diagrams with n arcs defines a linear function on the free abelian group generated by the diagrams.

Theorem. *The value of the n-th jump of any Vassiliev invariant of order n on each relation of the diagram group A_n vanishes.*

Proof. Fix an easy relation, that is, a diagram containing a short arc. Consider an immersed curve with n double points whose diagram has a short arc. Introducing one more double point at the moment of the surgeries, we can transform this immersion into an immersion for which the short arc is represented by a standard short simple loop in a ball of 3-space containing no other parts of the curve (Figure 14).

The value of the n-th jump of any invariant on such a curve is equal to the difference of the values of the preceding jump on two regularly isotopical immersed curves with $n-1$ double points; hence it vanishes. □