Figure 14. Evaluation of the n-th jump on an easy relation

The 4-term relation appears naturally in the study of the generic triple points of immersions (where the tangents of the three branches are 3 linearly independent lines). Such points occur unavoidably in generic 3-parameter families of mappings of a curve in 3-space. The mappings with a triple point form a variety of codimension 3 in the space of mappings. Its transversal 3-space intersects the discriminant hypersurface along three surfaces, intersecting each other transversally (Figure 15). The first surface corresponds to the first return to a point visited by the immersion. The second and third surfaces correspond to the subsequent return to one of the two intersecting branches of the immersed curve (visited at the first and at the second instances).

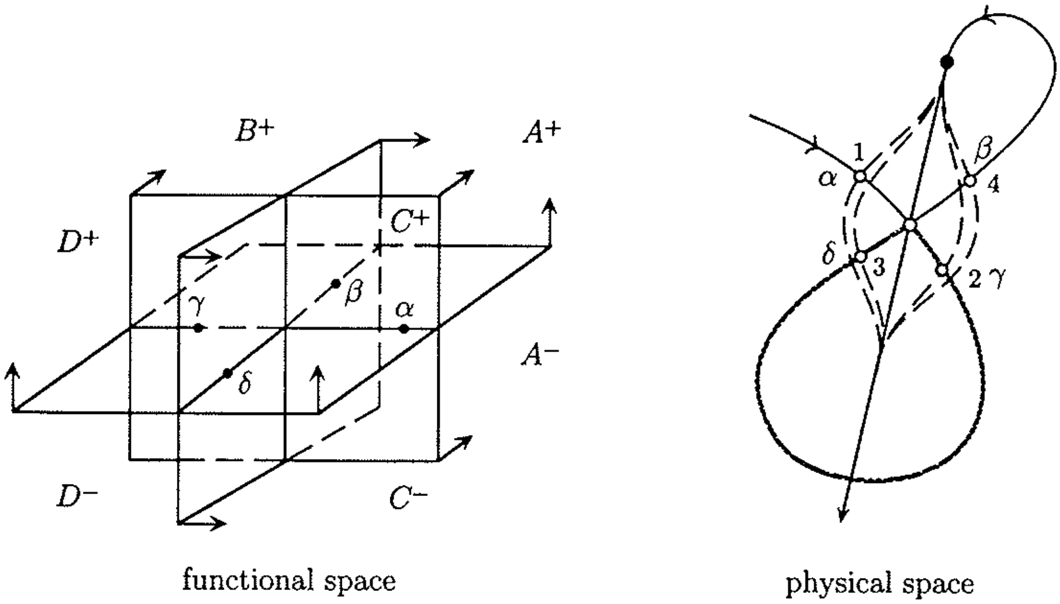

Figure 15. The origin of the 4-term relations: deformations of a triple point

Deform slightly the immersion near the third visit in such a way that the intersection with the initial part of the immersed curve at the triple point is replaced by the intersection with one of the 4 rays of the cross formed at the initial self-intersection. The four deformed immersions are shown in Figure 15 by the broken lines.

These four deformed immersed curves are represented in the functional space (and in the versal deformation 3-space, shown in Figure 15) by 4 points $(\alpha, \beta, \gamma, \delta)$ belonging to the codimension 2 strata of the discriminant hypersurface (namely, to its simple self-intersection strata). All these four points belong to one of the branches of the discriminant hypersurface (represented in Figure 15 by a horizontal plane).

Calculate the second jumps of an invariant at these points and denote them as the points themselves.

Lemma. $\alpha - \beta - \gamma + \delta = 0$

Proof. By definition

$$\begin{aligned} \alpha &= A^+ + C^- - A^- - C^+, \\ \beta &= A^+ + B^- - A^- - B^+, \\ \gamma &= B^+ + D^- - B^- - D^+, \\ \delta &= C^+ + D^- - C^- - D^+. \quad \square \end{aligned}$$

The relation between the values of the n-th jump of a Vassiliev invariant of order n on the 4 diagrams S_i (Figure 13) follows from the same arguments, applied to the four deformations of an immersion having one triple point, and $n-2$ double points. The value of the n-th jump of an invariant on the deformed immersion with n double points can be considered as the second jump of the $n-2$-th jump (as of a locally constant function on the space of mappings with $n-2$ double points.)

The parameters of the corresponding 4-term relation have the following meaning. The $n-2$ arcs form the diagram of an immersion in which the triple point disappears completely. The distinguished arc corresponds to the first return to the triple point (preserved under all the four deformations). The distinguished point describes the place of the last return among the moments of the other visits of the double points. $\square$

The theorem that we have proved implies that *any (rational) Vassiliev invariant of order n defines a homomorphism $A_n \to \mathbb{Q}$ and is defined by this homomorphism up to an addition of an invariant of a smaller order.*

Kontsevich has stated that any homomorphism $A_n \to \mathbb{Q}$ is the n-th jump of some (rational ?) Vassiliev invariant of order n. In other words, all the relations between the values of the n-jumps follow from relations 1 and 2 above:

$$(V_n/V_{n-1}) \otimes \mathbb{Q} \approx \text{Hom}\,(A_n,\ \mathbb{Q}).$$

Kontsevich's proof based on complete integration is sketched in Section 4 below. In the original approach of Vassiliev [1], the existence of his invariants was proved by purely combinatorial methods.

Remark 1. (cyclic invariance) The element of the group of diagrams, corresponding to an immersed closed curve with n double points, is well defined: *it does not depend on the place where we cut the circle to obtain a line* (which we have used in the construction of the diagram).

Indeed, consider any diagram and replace the leftmost point by a new point at the extreme right (connecting it by a new arc to the right end of the destroyed leftmost arc, see Figure 16).

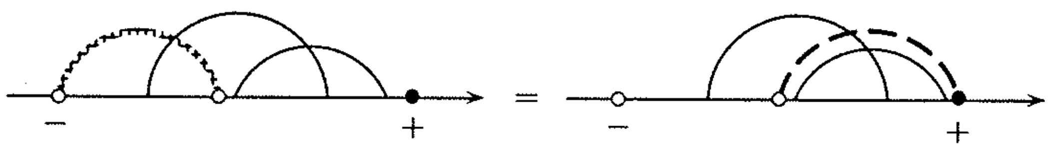

Figure 16. The cyclical invariance of a diagram's class in A_n

Lemma. *The new diagram defines the same element of the group of diagrams as the old one.*

Proof. Destroy the leftmost arc and sum the four-term relations, corresponding to all the choices of the distinguished arc among the remaining $n-1$ arcs, the distinguished point being at the right end of the left arc. □

Remark 2. The same reasoning proves that *any diagram containing an arc which does not intersect any other arc is equal to zero in the group of diagrams.* Indeed, one can transport the left end of this arc towards its right end, jumping over the intermediate arcs using the same operation as in the above proof (Figure 17).

Remark 3. One can combine the diagram groups into the *diagram ring* $A = \oplus A_n$, defining the product $A_n \otimes A_n \to A_{m+n}$ as the concatenation of corresponding diagrams (Figure 18).

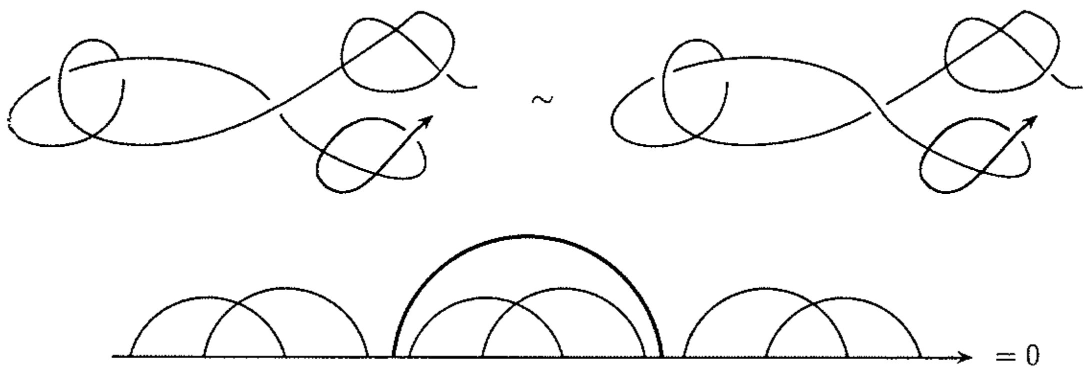

Figure 17. A corollary of relations 1 and 2

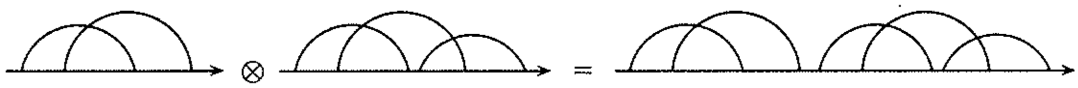

Figure 18. Multiplication of diagrams

This ring is commutative (by cyclical invariance (proved above in Remark 1).

In fact A has also a structure of a commutative and cocommutative Hopf algebra (the comultiplication is dual to the multiplication of the Vassiliev invariants). The elements of the ring A (or rather of its completion) may be viewed as models of knots—the Vassiliev invariants defining linear functions on it.

According to a general algebra theorem, the graded algebra A is isomorphic to the algebra of polynomials. It would be interesting to represent the multiplicative generators by linear combinations of special knots. The arithmetical properties of the coefficients of these combinations are also interesting.

4. Kontsevich integrals for Vassiliev invariants

Recently M. Kontsevich has presented some explicit formulas for the Vassiliev invariants of order n in a form of n-dimensional integrals, similar to the Gauss integral for the linking number.

Represent $\mathbb{R}^3$ as the product of the *horizontal plane* of a complex coordinate z and of the *vertical axis* of a real coordinate t. Represent a knot K as a *"very nice Morse embedding"* $S^1 \to \mathbb{R}^3$, for which all the critical points of the restriction of t to the knot curve are Morse nondegenerate and all the critical values are different.

The construction starts from the iterated integrals defining *Morse knot invariants*, which are constant along the components of the set of embeddings having only Morse critical points.

Choose n noncritical values $t_1 < \ldots < t_n$. Choose two different points (z_i, z_i') among the points of intersection of the knot with the horizontal plane $t = t_i$ (Figure 19).

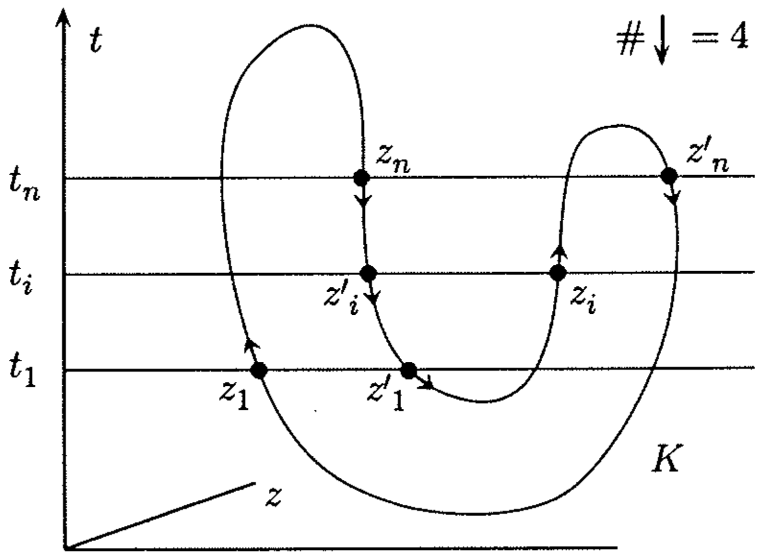

Figure 19. The construction of the Kontsevich integral

The knot branches define locally the smooth functions $z_i(t), z_i'(t)$. The n-times iterated *Kontsevich integral* is the integral with values in $A_n \otimes \mathbb{C}$,

$$\tilde{I}(K) = \int \ldots \int_{t_1 < \ldots < t_n} \sum_{\{z_i, z_i'\}} [w \bigwedge_{i=1}^{n} \frac{dz_i - dz_i'}{z_i - z_i'} (-1)^{\#\downarrow}],$$

where the *weight* $w \in A_n$ represents the Vassiliev diagram, formed by n arcs connecting (z_i, z_i') on the oriented circle K, and where $\#\downarrow$ is the number of *descending points* among $\{(z_i, z_i')\}$ (points where the orientation of K is opposite to that defined by dt). The summation is over all choices of the points z_i and z_i' for all i.

Remark. The integral is absolutely convergent. Indeed, the weight w vanishes at a neighborhood of a zero of the denominator (according to the easy relation 1 in A_n, see Section 3).

Thus *the integral depends on the Morse embedding K continuously.*

The crucial property of the Kontsevich integral is its *constancy along the deformations of the embedding K in the class of the Morse knots.* This property depends on the following, elementary but strange

Lemma. $\frac{dz_1 - dz_2}{z - z_2} \wedge \frac{dz_2 - dz_3}{z - z_2} +$ *cyclic permutations* $\equiv 0$.

Proof. Compute. □

Remark . This identity first appeared in [16] as the generator of the identities in the exterior algebra of the differential forms in the configuration space $\mathbb{C}^n -$diag, generated (over $\mathbb{C}$) by the standard forms $\omega_{i,j} = d\ell n(z_i - z_j)$. This subalgebra is isomorphic to the cohomology algebra of $\mathbb{C}^n - \cup(\text{diag})$.

The above identity is closely related to the Knizhnik-Zamolodchikov equation [17].

The Kontsevich construction depends on the choice of a closed complex $n-1$-form ω on $\mathbb{R}^n_1 \times \mathbb{R}^n_2 -$diag, verifying 3 conditions:

(1) the cohomology class $[\omega]$ is nonzero;
(2) ω is antisymmetric, that is $\sigma^*\omega = (-1)^n\omega$, where σ is the involution exchanging the factors;
(3) let $\omega_{i,j}$ be the form in $\mathbb{R}^n_1 \times \mathbb{R}^n_2 \times \mathbb{R}^n_3 - \cup(\text{diag})$, induced from ω on $\mathbb{R}^n_i \times \mathbb{R}^n_j -$diag under the natural projection (where $i, j \in \{1, 2, 3\}$); then

$$\omega_{1,2} \wedge \omega_{2,3} + \text{ cyclical permutations } \equiv 0.$$

For $n = 2$ such a form is given by the above lemma:

$$\omega = d\ell n(z - z').$$

For $n > 2$ no smooth form verifying the conditions 1–3 is known. The Kontsevich integrals correspond to a generalized solution in the class of currents. One represents $\mathbb{R}^n$ in the form $\mathbb{C} \times \mathbb{R}^{n-2}$ with coordinates $(z, t_1, \ldots, t_{n-2})$. The solution used by Kontsevich is the current

$$\omega = d\ell n(z - z') \wedge d\theta(t_1 - t_1') \wedge \ldots \wedge d\theta(t_{n-2} - t_{n-2}'),$$

where $(\theta)(t)$ is equal to 1 for positive t and to zero for negative t.

To prove the deformation invariance of a Kontsevich integral, one writes its variation as an integral of some differential form along K. *This form vanishes identically according to the preceding lemma.* (We leave the details to the interested reader. It is here that the four-term relations will be needed).

The deformation invariance implies the second crucial property of the Kontsevich integral; it can be considered as a "Vassiliev invariant of Morse knots with values in $A_n \otimes \mathbb{C}$". Kontsevich has stated that the n-th *jump of the integral* $\tilde{I}_n$, *evaluated at a Morse immersion* K with n double points, is equal to *the product of* $(2\pi i)^n$ *with the diagram of this immersion* (considered as an element of the diagram group A_n). The idea is to deform K near the singular points, as shown in Figure 20 for $n = 1$. It could follow from the Kontsevich iterated jump formula that *the* $n+1$-th *jump vanishes identically.* Thus $\tilde{I}_n$ may be considered as a generalized vector-valued Vassiliev invariant of Morse knots.

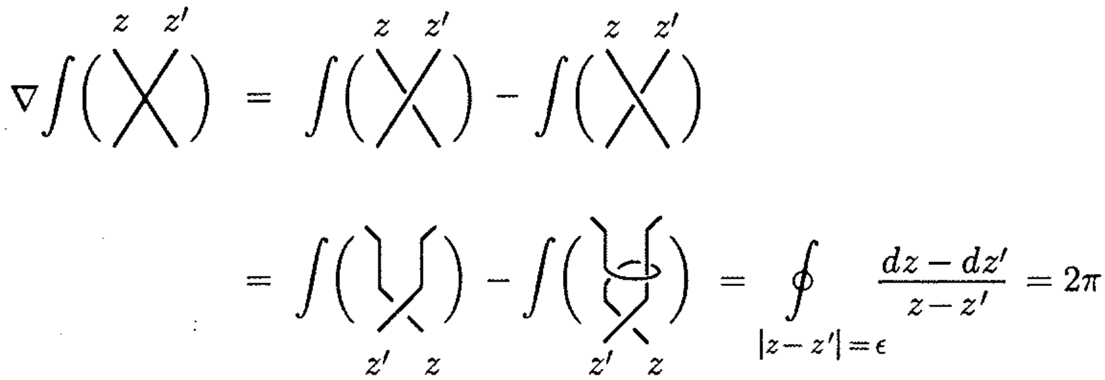

Figure 20. The residue as the jump of a Kontsevich integral

It would follow also, that *any element of* $A^* Hom(A_n \mathbb{C})$ *is equal to the* n*-th jump of some complex-valued Vassiliev invariant*:

$$V_n \otimes \mathbb{C} \approx (A_0 \oplus \ldots \oplus A_n)^* \otimes \mathbb{C}.$$

To write Kontsevich's formulas for the ordinary knots (providing invariants which do not depend on the choice of the Morse knot, representing a given knot class), introduce the *total integral* with values in the completion of the algebra $A \otimes \mathbb{C}$,

$$\tilde{I}(K) = \oplus \tilde{I}_n(K).$$

Consider an unknotted closed curve K_0 with two Morse maxima and two minima of t (to calculate the integrals, we may replace K_0 by the nonclosed plane curve $t = x^3 - x + i0$, since the integrals are deformation-invariant).

To make the Kontsevich integral invariant under the deformations which change the number m of maxima of the function t, Kontsevich had suggested to twist it in the following way:

$$I(K) := \tilde{I}(K)/\tilde{I}(K_0)^m.$$

The division here is understood in the sense of the completion of the algebra $A \otimes \mathbb{C}$: $\tilde{I}_0(K_0) = 1$ and $(1-a)^{-1} = 1 + a + a^2 + \ldots$, if the order of a in A is positive.

Example. The only number-valued Vassiliev invariant of order 2 (normed by the conditions that it vanishes on the unknot and takes value 1 on the trefoil knot) is equal to

$$\Phi(K) = \frac{1}{4\pi^2} \iint_{t_1 < t_2} \sum_{\{z,z'\} \frac{dz_1 - dz_1'}{z_1 - z_1'} \wedge \frac{dz_2 - dz_2'}{z_2 - z_2'} (-1)^{\#\downarrow} + \frac{m-1}{6}}$$

where K is a Morse embedding of a circle with $2m$ critical points of t on it and where the summation is over all the choices of the four points $(t_i, z_i), (t_i, z_i')(i = 1, 2)$, such that the points of the first pair $(i = 1)$ alternate with those of the second along the closed curve K.

The Kontsevich integrals would equip the ring $V \otimes \mathbb{C}$ with a $\mathbb{Z}^+$–grading (generated by that of A). However the arithmetical properties of this transcendental grading are not clear. Conjecturally the values of $I_n(K)$ belong to $(2\pi i\mathbb{Q})^n \otimes A_n$.

This arithmetic reflects the arithmetical nature of the constants involved in the formulas for the integer-valued invariants (like $4\pi^2$ and $1/6$ in the preceding formula). These constants depend on the values of the D. Zagier *ζ-functions of several variables* at the positive integer points,

$$\zeta(a_1, \ldots, a_n) := \sum k_1^{-a_1} \ldots k_n^{-a_n}$$

(the summation over the integer points in the Weyl chamber $0 < k_1 < k_2 < \ldots < k_n$).

The integer linear combinations of the numbers $\zeta(a)$ form a ring Z. Kontesevich has stated that his integrals values on any knot, $\tilde{I}_n(K)$ and $\tilde{I}(K)$, belong to $A_n \otimes Z$.

To understand how the ζ-function enters in the formulas, it suffices to consider the simplest case of the double Kontsevich integral

$$i_2(K_0) = 1 + \frac{1}{4} + \frac{1}{9} + \ldots = \frac{\pi^2}{6}$$

for the plane curve $K_0 : t = x^3 - x,\ z = x + i0$ (Figure 22).

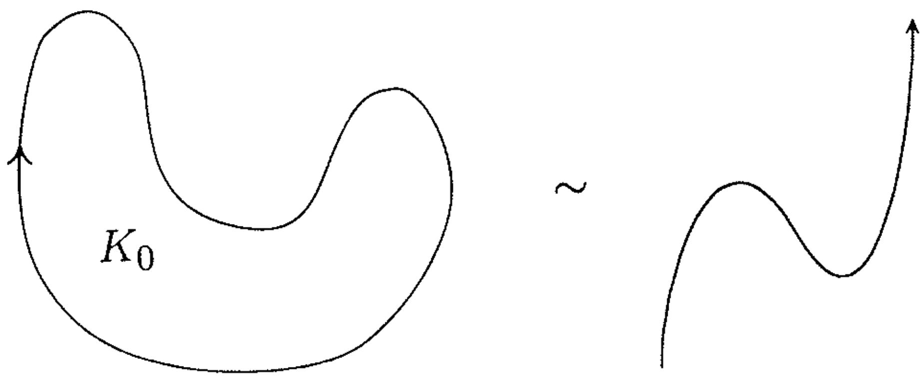

Figure 21. The standard curve K_0

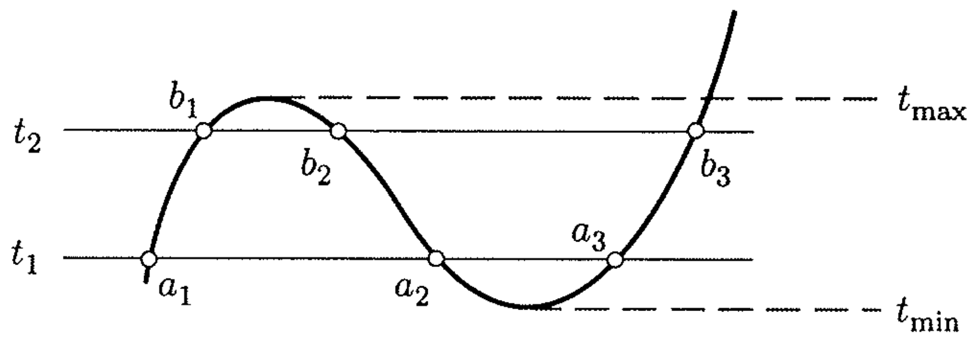

Figure 22. $\frac{\pi^2}{6}$ as a Kontsevich's integral

Below consider the points in any pair $\{z_i, z_i'\}$ as being unordered (otherwise one has to multiply the 2^n).

The choices containing a pair $\{z_1, z_1'\} = \{a_2, a_3\}$ or $\{z_2, z_2'\} = \{b_1, b_2\}$ are not admissible, since the corresponding quadruples cannot alternate. The remaining 4 possibilities of the choices provide 4 terms in the integrand, of the form

$$d\ell n a_{1,2} \wedge d\ell n b_{1,3}(-1) + \dots$$

where $a_{1,2} = a_1 - a_2$, and so on. Taking the signs into account, one can reduce the integrand to the form

$$d\ell n(a_{1,2}/a_{1,3}) \wedge d\ell n(b_{2,3}/b_{1,3}).$$

This expression explains the rather mysterious *invariance of the integral under the deformations of K_0 in the class of Morse embeddings.* Indeed, the integration domain is the triangle

$$t_{\min} < t_1 < t_2 < t_{\max}.$$

The ratios $u = a_{1,2}/a_{1,3}$, $v = b_{2,3}/b_{1,3}$ send the boundary of this triangle onto the boundary of the standard triangle $u+v \geq 1$, $u \leq 1, v \leq 1$ which is invariant under the deformations in the class of Morse embeddings. Thus the integral is reduced to the standard integral along the standard triangle,

$$\iint d\ell n u \wedge d\ell n v = \int_0^1 \ell n(1-u)\frac{du}{u}.$$

Taylorizing the logarithm, one obtains the value

$$i_2(K_0) = \sum_{k=1}^{\infty} \int_0^1 \frac{u^k}{k}\frac{du}{u} = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} = \zeta(2).$$

Similar wonderful cancellations are responsible for the independence of other Kontsevich integrals on the choice of the Morse representative in a knot class. The *standard integrals* occurring in these calculations always have the form

$$\iint_{0<t_1<\ldots<t_N<1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\cdots\frac{dt_{a_1}}{t_{a_1}}\frac{dt_{a_1+1}}{1-t_{a_1+1}}\cdots\frac{dt_N}{t_N}$$

(n groups similar to the first product of a_1 forms.) *This number is the value of* $\zeta(a_1,\ldots,a_n)$.

It is clear that the theories described above will be soon developed in many directions.

Vassiliev has started from the stabilization problem ([2]–[4]) of the cohomology rings of the complements to the discriminants and to the caustics in the complex versal deformation spaces of critical points of holomorphic functions of n complex variables ([5]–[8]). These stable rings are isomorphic respectively to the rings

$$H^*(\Omega^{2n}S^{2n+1}),\ H^*(\Omega^{2n}\Sigma^{2n}U(n)/O(n)).$$

generalizing the May-Segal ([18],[19]) result for the braid groups cohomology ($n=1$) and providing a homological version of the Gromov h-principle while its homotopical version fails there.

Then, applying his methods to the real functions of one variable with restricted singularities (see [9],[10]) Vassiliev has realized that these also work for the vector functions, for instance, for the knots.

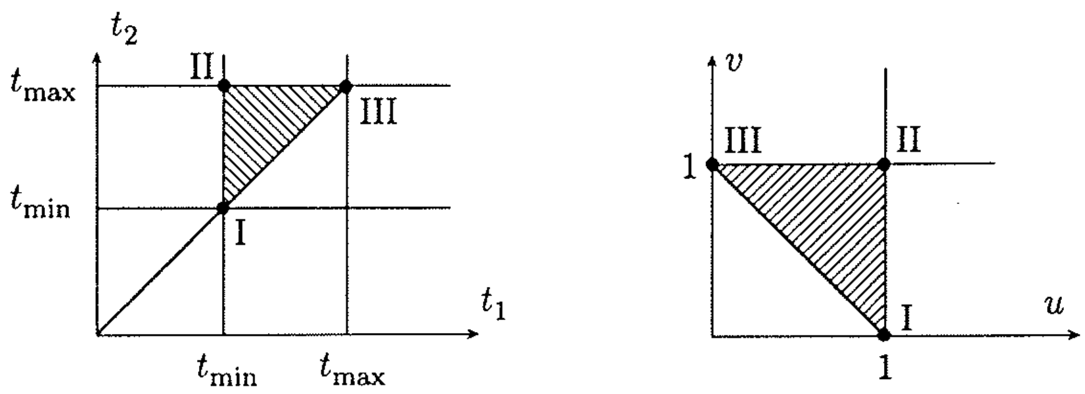

Figure 23. Integration domains for the Kontsevich integral equal to $\zeta^{(2)}$

Vassiliev has also discussed the applications of his theory to the higher dimensional embeddings. Bar-Natan, Lin and Kontsevich have defined Feynman diagram groups, starting from more general Feynman diagrams than those of Vassiliev, and using more relations (inspired by the Jacobi identity in Lie algebras).

The resulting diagram groups are isomorphic to those of Vassiliev. Birman, Lin, Bar-Nathan and Kontsevich have used these constructions to associate a Vassiliev invariant to any representation of a simple Lie algebra; Kontsevich has promised applications to the topology of 3- and 4-manifolds, to the cohomology of infinite-dimensional Lie algebras and to associative algebras.

The success of the singularity technique in knot theory should not obscure the fact that many fundamental problems of the topology of the functional spaces of mappings with restricted singularities are still open both in the real and in the complex domain, even for the functions of one variable (see [3], [4], [9], [10], [20], [21], [22]). Other probable domains of application include symplectic and contact geometry and the theories of immersed plane curves and of evolution of wave fronts.

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