Some classificational problems in four-dimensional geometry: distributions, almost complex structures and Monge-Ampère equations

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Abstract

In this paper we consider three deeply connected classificational problems on four-dimensional manifolds. First we consider and describe locally regular distributions. Second we give a classification of almost complex structures of general position in terms of distributions. Finally we classify nondegenerate Monge-Ampère equations with two variables in terms of $\{e\}$-structures.

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Introduction

The notion of a G-structure ([St], [ALV]) generalizes the notion of a tensor field on a manifold. In particular such objects as vector fields, distributions, almost complex structures, metrics, orientations (and many others) are examples of the G-structure. Some of G-structures in the list are easier for classification then the others. In this paper we show some connections between different G-structures, which also may be seen as a classification (in special terms).

The structure of the paper is the following.

In chapter 1 we consider and describe regular distributions on four-manifold, which we need in chapter 2. The results are generally well-known and we give a short summary.

In section 2.1 of chapter 2 we classify (locally) almost complex structures of general position (the notion developed in [Kr]) on four-dimensional manifolds in terms of distributions, metrics and orientations. We also show that any regular two-dimensional distribution on four-dimensional manifold may be locally the image of the Nijenhuis tensor. This may be also seen as a partial realization of the distributional invariant of an almost complex structure. In section 2.2 we develop a language of projected almost complex structures, which can be regarded as generalizations of the so called ([B], [Sa]) cocomplex structures on odd-dimensional manifolds. We call the corresponding structures procomplex ones and introduce for dimension three the Nijenhuis operator, which can be generalized to the Nijenhuis operator-tensor in any odd dimension.

In chapter 3 we consider an elliptic Monge-Ampère equation on a two-dimensional manifold. The hyperbolic and mixed type case were considered in [L1] and [Ku]. Thanks to the paper [L2] an elliptic Monge-Ampère equation may be regarded as a pair of 2-forms on a four dimensional manifold with one form closed or as an almost complex structure on a symplectic four-manifold. Using invariants of almost complex structures we introduce the notion of nondegenerate Monge-Ampère elliptic equations and give their classification in terms of $\{e\}$-structures. This $\{e\}$-structure is a canonical structure, which is well-behaved w.r.t. the corresponding almost complex structure.

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Chapter 1

Distributions on four dimensional manifolds

Recall (cf. [ALV]) that a p-distribution (in Chevalley's sence or in another terminology a p-differential system, [St]) on a manifold M^m is a section $\Pi^p \in \Gamma\left(\mathrm{Gr}_p(M)\right)$ of the Grassmannian p-subspaces subbundle of the tangent bundle TM. A symmetry of the distribution Π^p on M is such a vector field v that the corresponding flow of diffeomorphisms (shifts by time) preserves the distribution: $\varphi_t^*(\Pi_x^p) = \Pi_{\varphi_t(x)}^p$. A symmetry $v \in \mathrm{Sym}\,\Pi^p$ is called *characteristic* ([L1]) if it is tangent to the distribution: $v \in \mathrm{Sym}\,\Pi^p \cap \Gamma(\Pi^p)$, where $\Gamma(\Pi^p)$ is the module of sections of Π^p.

Let us also recall the basic facts about 2-distributions on $\mathbb{R}^3$. Every 2-distribution is given by 1-form $\alpha \neq 0$. The distribution is locally *regular* if in some neighborhood it holds either $\alpha \wedge d\alpha \equiv 0$ or $\alpha \wedge d\alpha \neq 0$. In the first case the distribution is integrable i.e. the tangent bundle to some foliation according to the theorem of Frobenius. In the second case according to the Darboux's theorem ([St], [ALV]) the distribution is locally isomorphic to the standard contact distribution. Obviously in the first integrable case the distribution has (locally) two characteristic and one transversal (independent) symmetries. In the second case the contact distribution has only two transversal symmetries and none characteristic. Actually, the transversal symmetries in canonical coordinates, in which a contact form is $\alpha = p\,dq - du$, are exactly $\partial_u \pm \partial_q$ and if there were a characteristic one v then the form $i_v d\alpha$ restricted to the contact plane would vanish or $d\alpha$ would be degenerate on it.

Now let us turn to the distributions on $\mathbb{R}^4$. All 1-distributions are standard according to the basic theorem of ordinary differential equations theory. Due to the generalized Darboux's theorem ([ALV], [St]) every regular distribution of codimension 1 is isomorphic to the suspension over the standard contact one, i.e. to the product $\Pi^2 \times \mathbb{R}$ on $\mathbb{R}^3 \times \mathbb{R}$, where the distribution Π^2 is contact. This follows from the contact Darboux's theorem and the fact that every regular distribution of odd dimension and codimension 1 possesses a characteristic symmetry: $v \in \mathrm{Ker}\,(d\alpha|_{\Pi^{2n-1}}) \Leftrightarrow v \in \mathrm{Char}(\Pi)$.

So it remains the only case which seems nontrivial: 2-distributions Π^2 on $\mathbb{R}^4$. But these also allows canonical forms ([C]): in integrable case $\Pi^2 = \{dx^1 = dx^2 = 0\}$, in contact $\Pi^2 = \{dx^1 = 0, x^2dx^3 - dx^4 = 0\}$ and in general regular the distribution Π^2 is determined in special coordinates by the forms

$$\left\{ \begin{array}{rcl} dx^1 + x^2dx^4 & = & 0 \\ dx^2 + x^3dx^4 & = & 0. \end{array} \right.$$

Note that in the first and the second case the distribution Π^2 obviously possesses two transversal symmetries. The same is true for the last case also, the symmetries are $\partial_1 \pm \partial_4$. This fact we need in chapter 2 and we prove it below by other method using the Tanaka invariant which we briefly recall.

The Tanaka invariant (see [T],[Y]) of a regular distribution Π^p on a manifold M is a collection of graded Lie algebras $\mathfrak{g}(x)$ for each point $x \in M$, associated to the filtered algebras $\{D^p(x) \subset T_x\}_{p\geq 1}$, where the module of sections $\mathcal{D}^{(p)}$ of the distribution D^p is defined as $(p-1)$-th derivative: $\mathcal{D}^{(p)} = \partial^{(p-1)}\mathcal{D}^{(1)} = \partial^{(p-2)}\mathcal{D}^{(1)} + [\mathcal{D}^{(1)}, \partial^{(p-2)}\mathcal{D}^{(1)}]$, $[\cdot,\cdot]$ being the commutator of the vector fields, and $\partial^{(0)}\mathcal{D}^{(1)} = \mathcal{D}^{(1)}$ is the sections module of the initial distribution $D^1 = \Pi^p$; the Lie product is induced by the commutator of vector fields.

Proposition 1. *Every regular 2-distribution on $\mathbb{R}^4$ possesses locally two independent transversal fields of symmetry.*

Proof. Let us have a 2-distribution Π^2 on $\mathbb{R}^4$. Let us call the distribution Π^2 *a distribution of general position* if the second derivative of the distribution Π^2 does not coincide with the first, $\partial^{(2)}\Pi^2 \neq \partial^{(1)}\Pi^2 \neq \Pi^2$, i.e. $(\partial^{(2)}\Pi^2)_x = T_x\mathbb{R}^4$. In this case the Tanaka invariant — graded Lie algebra — has the underlying space of the form

$$\mathfrak{g}(x) = \mathfrak{g}_1(x) \oplus \mathfrak{g}_2(x) \oplus \mathfrak{g}_3(x) \simeq \Pi^2_x \oplus \Theta^1_x \oplus \Xi^1_x.$$

Hence because of the gradation, the Lie product is determined by a 2-form on Π^2_x with values in Θ^1_x and by 1-form on Π^2_x with values in $\mathrm{Hom}(\Theta^1_x, \Xi^1_x)$. Since we may trivialize the bundles Θ^1 and Ξ^1 we are given a two- and a one-form on the distribution Π^2. In the general case these forms are nonzero and we have the canonically determined distribution Π^1, which is the kernel of the 1-form above. Let us denote the first derivative by $\Pi^3 = \partial^{(1)}\Pi^2 \supset \Pi^2$ and $\Pi^4 = T_\bullet \mathbb{R}^4$. Note that the distribution Π^1 is singled out by the property that $[v, w] \in \Gamma(\Pi^3)$ for any $v \in \Gamma(\Pi^1)$ and $w \in \Gamma(\Pi^3)$.

If 2-distribution is regular then the underlying space of Tanaka invariant has stable dimensions of the gradations. Hence only three cases are possible:

1. $\varrho(x) = \varrho_1(x)$, i.e. the distribution Π^2 is integrable.
2. $\varrho(x) = \varrho_1(x) \oplus \varrho_2(x)$, i.e. $\Pi^2 = C^2 \times \{pt\}$ is the distribution on $\mathbb{R}^3 \times \mathbb{R}$ generated by a contact distribution C^2 on $\mathbb{R}^3$.
3. The case of general position described above.

In the first case we obviously have two transversal symmetries. In the second we have one (even two) transversal symmetry in $\mathbb{R}^3$ and the second obvious symmetry is the generator of the factor $\mathbb{R}$. Consider the last general case.

Let us straighten the canonical 1-distribution Π^1 described above. This means we have coordinates x^i, $1 \le i \le 4$, in $\mathbb{R}^4$ such that Π^1 is generated by $\partial_1 = \dfrac{\partial}{\partial x^1}$. Consider the subspace $\mathbb{R}^3 = \{x^1 = \mathrm{const}\}$ with the parameter x^1 on it. Let us denote the differentiation by this parameter with the dot: $\dfrac{d}{dx^1} f = \dot{f}$. Two-dimensional subspace Π^2 and $\mathbb{R}^3$ intersect transversely by 1-dimensional distribution. Let it be generated by a vector field

$$v = \sum_{i=2}^{4} a_i \partial_i \in \mathbb{R}^3 =< \partial_2, \partial_3, \partial_4 > .$$

Hence $\Pi^2 =< \partial_1, v >$ and we have:

$$[\partial_1, v] = \dot{v} = \sum_{i=2}^{4} \dot{a}_i \partial_i \in \mathbb{R}^3.$$

Let us denote $\Phi^2 =< v, \dot{v} >\subset \mathbb{R}^3$ the two-dimensional distribution. It does not depend on the parameter x^1 on $\mathbb{R}^3(x^2, x^3, x^4)$. Actually, according to the choice of ∂_1 we have $[\partial_1, \dot{v}] \in< \partial_1, v, \dot{v} >$. Hence for the vector field

$$[\partial_1, \dot{v}] = \ddot{v} = \sum_{i=2}^{4} \ddot{a}_i \partial_i \in \mathbb{R}^3$$

we have: $\ddot{v} = \alpha v + \beta \dot{v} \in \Gamma(\Phi^2)$ with some $\alpha, \beta \in C^\infty(\mathbb{R}^4)$. Note that the general position property means that the 2-distribution Φ^2 on $\mathbb{R}^3$ is contact. Hence it possesses two transversal symmetries. Let w be one of them. Since w is a symmetry then $[w, v] \in \Gamma(\Phi^2)$, i.e. $[w, v] = \lambda v + \mu \dot{v}$, $\lambda, \mu \in C^\infty(\mathbb{R}^4)$. Let us consider the vector field $\hat{w} = -\mu\partial_1 + w$. We have:

$$[\hat{w}, v] = -[\mu\partial_1, v] + [w, v] = (\partial_v \mu) \cdot \partial_1 - \mu\dot{v} + \lambda v + \mu\dot{v} = (\partial_v \mu) \cdot \partial_1 + \lambda v \in \Gamma(\Phi^2),$$

$$[\hat{w}, \partial_1] = [\partial_1, \mu\partial_1] - [\partial_1, w] = \dot{\mu}\partial_1 - \dot{w} = \dot{\mu}\partial_1 \in \Gamma(\Phi^2).$$

Thus we obtain a transversal symmetry and applying the same procedure again we obtain another. □

Remark 1. The statement of the theorem is not valid for general p-distributions on $\mathbb{R}^m$. Actually if one takes the distributions with Tanaka invariant (not only underlying space but also the Lie structure) changing locally from one point to another, $\mathfrak{Q}(x) \not\simeq \mathfrak{Q}(y)$, then the required is obvious. Another way: one takes a partial differential equation with no symmetries, then the corresponding distribution ([ALV]) possesses none either. □

Chapter 2

Four-dimensional almost complex geometry

2.1 Almost complex structures in $\mathbb{R}^4$

Recall that an almost complex structure j on a manifold M is such an endomorphism of the tangent bundle $j \in T^*M \otimes TM$ that $j^2 = -\mathbf{1}_M$. The Nijenhuis tensor $N_j \in \wedge^2 T^*M \otimes TM$ of this structure is defined by the formula:

$$N_j(\xi, \eta) = [j\xi, j\eta] - j[j\xi, \eta] - j[\xi, j\eta] - [\xi, \eta],$$

where ξ and η are some prolongation vector fields of the vectors given at a point. A theorem of Newlander-Nirenberg ([NN], [NW]) states that j is integrable (or complex) iff $N_j \equiv 0$. It was proved in [Kr] that actually the only invariant for formal classification of almost complex structures is the jet of the Nijenhuis tensor (see the cited paper for details).

Let us consider an almost complex structure j in $\mathbb{R}^4$ in a neighborhood of zero. Suppose that $(N_j)_0 \neq 0$. In this case the image $\operatorname{Im} N_j(\cdot, \cdot)$ is two-dimensional and complex generated by a vector $N_j(\xi, \eta) \neq 0$ for any two complex independent vectors ξ and η. Consider a two-dimensional distribution $\Pi^2_x = \operatorname{Im}(N_j)_x \subset T_x = T_x\mathbb{R}^4$. This distribution is an invariant of almost complex structure and hence invariants of this distribution lead to invariants of the almost complex structure.

Let us suppose that the first derivative of the distribution Π^2_* is nontrivial at the point: $(\partial^{(1)}\Pi^2)_0 \neq \Pi^2_0$. Then we have a three-dimensional distribution

$\Pi^3 = \partial^{(1)}\Pi^2$ on $\mathcal{O}(0)$. There exist vectors $\xi_i = \xi_i(x) \neq 0$, $i = 1,2,3$, in the space $\Pi^3_x \subset T_x$ such that $N_j(\xi_1, \xi_3) = \xi_1$, $N_j(\xi_2, \xi_3) = -\xi_2$. Actually, for any $\xi_3 \in \Pi^3 \setminus \Pi^2$ the mapping $\eta \mapsto N(\eta, \xi_3)$ is an orientation reversing isomorphism of Π^2, and hence there exist two one-dimensional invariant subspaces. Consider a vector ξ_1 on one of them. We have: $N(\xi_1, \xi_3) = f\xi_1$, $f \neq 0$. Let's change the vector $\xi_3 \mapsto \frac{1}{f}\xi_3$. Then $N(\xi_1, \xi_3) = \xi_1$ and $N(j\xi_1, \xi_3) = -j\xi_1$, i.e. we may take $\xi_2 = j\xi_1$.

Denote by $\Upsilon_1 =< \xi_1 >$ and $\Upsilon_2 =< \xi_2 >$ the one-dimensional subspaces generated by the vectors ξ_1 and ξ_2, $\Pi^2_x = (\Upsilon_1)_x \oplus (\Upsilon_2)_x$. Let us note that if the half-space $(\Pi^3_x)^+ \subset \Pi^3_x \setminus \Pi^2_x$, in which lies the vector ξ_3, is fixed then the vector is defined up to Π^2_x-shifts: $\xi_3 \mapsto \xi_3 + \alpha_1\xi_1 + \alpha_2\xi_2$. If one changes the half-space: $\xi_3 \mapsto \tilde{\xi}_3 = -\xi_3$, then the vectors ξ_1 and ξ_2 (and hence the distributions Υ_1 and Υ_2) interchanges: $\tilde{\xi}_1 = \pm\xi_2$, $\tilde{\xi}_2 = \pm\xi_1$. Selecting a half-space in $\Pi^3_x \setminus \Pi^2_x$, i.e. an orientation on $\Theta^1_x = \Pi^3_x/\Pi^2_x$, we fix which one of the distributions Υ_1 and Υ_2 is the first and which is the second; changing this orientation we change the numeration. In other words, there is a canonical orientation on the two-dimensional space $\Theta^1_x \times P(\Pi^2_x)$. Let us call this orientation the Θ-orientation. Note also that there's singled out a pair of two-dimensional affine subspaces in the space Π^3_x: $\{\pm\xi_3 + \alpha_1\xi_1 + \alpha_2\xi_2\}$, i.e. there's fixed a metric on Θ^1_x. Let us call it the Θ-metric.

Consider now the cofactor $\Xi^1_x = T^4_x/\Pi^3_x$ of the subspace $\Pi^3_x \subset T_x$. There exists a vector $\xi_4 \notin \Pi^3_x$ such that $N_j(\xi_1, \xi_4) = \xi_2$, $N_j(\xi_2, \xi_4) = -\xi_1$. In its half-space $T^4_x \setminus \Pi^3_x$ it is determined up to Π^2_x-shifts: $\xi_4 \mapsto \xi_4 + \alpha_1\xi_1 + \alpha_2\xi_2$, and so there exists a natural metric on Ξ^1_x. Let us call it the Ξ-metric. Under the change of half-space $\xi_4 \mapsto \tilde{\xi}_4 = -\xi_4$ in one of two subspaces Υ_1 or Υ_2 the orientation changes: $\xi_1 \mapsto -\xi_1$ $\xi_2 \mapsto -\xi_2$. Therefore there's determined the natural orientation of the space $\Upsilon^1_1 \times \Upsilon^1_2 \times \Xi^1$. Let us call it the Ξ-orientation.

Let us call the $\Upsilon\Theta\Xi$-invariant the pair of distribution Υ_i and Θ- and Ξ-metrics and orientations. It was proved in [Kr] the following statement:

Theorem 1. *Let the Nijenhuis tensors of almost complex structures j_1 in $\mathcal{O}_{\mathbf{R}^4}(x)$ and j_2 in $\mathcal{O}_{\mathbf{R}^4}(y)$ do not equal to zero and the first derivative of the distribution Π^2 is nontrivial, $\partial^{(1)}\Pi^2 \neq \Pi^2$. Pseudoholomorphic or antipseudoholomorphic mapping $u : (\mathcal{O}_{\mathbf{R}^4}(x), j_1) \to (\mathcal{O}_{\mathbf{R}^4}(y), j_2)$ exists if and only if there exists a mapping $\mathcal{O}_{\mathbf{R}^4}(x) \to \mathcal{O}_{\mathbf{R}^4}(y)$, which transforms one $\Upsilon\Theta\Xi$-invariant to the other.* $\square$

Remark 2. Because of the gradation, the Lie product is determined by a 2-form on Π^2_x with values in Θ^1_x and by 1-form on Π^2_x with values in $\mathrm{Hom}(\Theta^1_x, \Xi^1_x)$. Due to existence of the Θ- and Ξ-metrics and orientations and due to the decomposition $\Pi^2_x = (\Upsilon_1)_x \oplus (\Upsilon_2)_x$, the Lie algebra structure on $\mathfrak{g}(x)$ is given by elements $\omega^2_x \in (\Upsilon_1)^*_x \otimes (\Upsilon_2)^*_x = \mathrm{Hom}(\Upsilon_1, \Upsilon_2^*)_x$ and $\omega^1_x \in (\Upsilon_1)^*_x \oplus (\Upsilon_2)^*_x$. Thus the Tanaka invariant gives us two additional invariants: ω^1 and ω^2. $\square$

Proposition 2. *If v is a symmetry of 2-distribution $\Pi^2 = \mathrm{Im}\, N_j$ then for the vector field $w = jv$ it holds: $L_w j \in \mathrm{Aut}\, \Pi^2$.*

Proof. Actually, from the definition of the Nijenhuis tensor it follows that

$$L_w j(\xi) = L_w(j\xi) - jL_w(\xi) = [j\xi, jv] - j[\xi, jv] \in \Pi^2,\ \ \xi \in \Pi^2. \qquad \square$$

But we can take $w = jv$ to be also a field of symmetry and so we get:

Theorem 2. *Any regular 2-distribution on $\mathbb{R}^4$ is locally the image of Nijenhuis tensor $\Pi^2 = \mathrm{Im}\, N_j$ for some almost complex structure j.*

Proof. Let us take two sections $\xi_1, \xi_2 \in \Gamma(\Pi^2)$ and let $\eta_1, \eta_2 \in \mathrm{Sym}\, \Pi^2$ be two transversal symmetries. In every point x the four vectors $((\xi_1)_x$, $(\xi_2)_x$, $(\eta_1)_x$, $(\eta_2)_x)$ form a basis. Define an almost complex structure by the formula:

$$\begin{array}{ll} j\xi_1 = \xi_2 & j\xi_2 = -\xi_1 \\ j\eta_1 = \eta_2 & j\eta_2 = -\eta_1 \end{array}$$

Then $N_j(\xi_1, \eta_1) = [\xi_2, \eta_2] - j[\xi_2, \eta_1] - j[\xi_1, \eta_2] - [\xi_1, \eta_1]$ and since $L_{\eta_r}\xi_s \in \Pi^2$ we have $N_j(\xi_1, \eta_1) \in \Pi^2$. Due to j-invariance of Π^2 this implies the equality $\mathrm{Im}\, N_j = \Pi^2$. $\square$

2.2 Almost procomplex and cocomplex structures

Definition 1. Let us call *a procomplex structure* on three-dimensional manifold Q a 1-parametric pair $(j^{(t)}, w^{(t)})_{t\in\mathbf{R}}$, where the endomorphism $j^{(t)} \in T^*Q \otimes TQ$ has the spectrum $\mathrm{Sp}\, j^{(t)}_x = \{0, \pm i\}$ and the vector field $w^{(t)} \in \mathrm{Ker}\, j^{(t)} \setminus \{0\}$.

Remark 3. This notion is exactly the same as the notion of almost complex structure j on M^4. To obtain one from the other one takes some coordinates (t, x^1, x^2, x^3) in $M^4 = \mathbb{R}^1 \times Q^3$ and sets $Q^3 \simeq Q^3_t = \{t = \mathrm{const}\}$. One may suppose $j\partial_t \in T_\bullet Q^3$. If pr is the projection on Q^3_t along ∂_t one sets $j^{(t)} := \mathrm{pr} \circ j : T_\bullet Q^3_t \to T_\bullet Q^3_t$ and $w^{(t)} := j\partial_1 \in T_\bullet Q^3_t$. □

Let us consider $j^{(t)}$-invariant 2-distribution $\mathcal{C}^{(t)}_x = \mathrm{Im}\, j^{(t)}_x$. We have: $T_x Q^3 = \mathcal{C}^{(t)}_x \oplus \mathbb{R} w^{(t)}_x$. An analog of the Nijenhuis tensor is the operator $\mathrm{pr} \circ N_j(\partial_t, \cdot)$.

Definition 2. Let us call *the Nijenhuis operator* of almost procomplex structure $(j^{(t)}, w^{(t)})$ the operator $A^{(t)}_j \in T^*Q^3 \otimes TQ^3$ defined by formulae:

$$
\begin{aligned}
&A^{(t)}_j(w^{(t)}) = 0 \quad \text{and} \\
&A^{(t)}_j(v) = \left(L_{w^{(t)}} j^{(t)}\right) v - \left(j^{(t)} \circ \left.\frac{d}{d\tau}\right|_{\tau=t} j^{(\tau)}\right) v, \ v \in \mathcal{C}^{(t)}_x.
\end{aligned}
$$

Define a 1-form $\alpha^{(t)}$ on Q^3 by the formulae:

$$
\alpha^{(t)}(v) = 0, \ v \in \mathcal{C}^{(t)}_x; \quad \alpha^{(t)}(w^{(t)}_x) = 1.
$$

Proposition 3. *The following formula holds true:*

$$
A^{(t)}_j \circ j^{(t)} = -j^{(t)} \circ A^{(t)}_j + L_{w^{(t)}} \alpha^{(t)} \otimes w^{(t)}.
$$

Proof. Let $\tilde{v}$ be a prolongation of v in a neighborhood of $x \in Q^3$ and $v^{(\tau)}$ be a prolongation by t such that both $\tilde{v}$ and $v^{(\tau)}$ be sections of the distribution $\mathcal{C}^{(t)}$. We have:

$$
\begin{aligned}
&A^{(t)}_j(j^{(t)} v) = L_{w^{(t)}}(j^{(t)} j^{(t)} \tilde{v}) - j^{(t)} L_{w^{(t)}}(j^{(t)} \tilde{v}) \\
&-j^{(t)} \left.\frac{d}{d\tau}\right|_{\tau=t} (j^{(\tau)} j^{(\tau)} v^{(\tau)}) + j^{(t)} j^{(t)} \left.\frac{d}{d\tau}\right|_{\tau=t} (j^{(\tau)} v^{(\tau)}) = \\
&-j^{(t)} A^{(t)}_j v - L_{w^{(t)}} \tilde{v} - [j^{(t)}]^2 L_{w^{(t)}} \tilde{v} + j^{(t)} \left.\frac{d}{d\tau}\right|_{\tau=t} v^{(\tau)} - j^{(t)} \left.\frac{d}{d\tau}\right|_{\tau=t} v^{(\tau)} = \\
&-j^{(t)} A^{(t)}_j v - \mathrm{Pr}^{(t)} \left(L_{w^{(t)}} \tilde{v}\right),
\end{aligned}
$$

where $\mathrm{Pr}^{(t)}$ is the projection of T_xQ^3 on $\mathbb{R}w_x^{(t)}$ along $\mathcal{C}_x^{(t)}$. It's easy to see that this projection is given by the formula $\mathrm{Pr}^{(t)} = \alpha^{(t)} \otimes w^{(t)}$ and

$$\mathrm{Pr}^{(t)}\left(L_{w^{(t)}}\tilde{v}\right) = \alpha^{(t)}\left(L_{w^{(t)}}\tilde{v}\right)w^{(t)} =$$
$$\left(L_{w^{(t)}}[\alpha^{(t)}(\tilde{v})] - (L_{w^{(t)}}\alpha^{(t)})(\tilde{v})\right)w^{(t)} = -\left(L_{w^{(t)}}\alpha^{(t)}\right)(v)w^{(t)}.$$

To prove the proposition it remains to note that for $w^{(t)}$ we have the identity:

$$\left(L_{w^{(t)}}\alpha^{(t)}\right)(w^{(t)}) = L_{w^{(t)}}\left[\alpha^{(t)}(w^{(t)})\right] - \alpha^{(t)}\left(L_{w^{(t)}}w^{(t)}\right) = 0. \qquad \square$$

Let us recall the definition of *cocomplex structure*, which according to [B] is a field of endomorphisms $j \in T^*Q \otimes TQ$ such that j has a 1-dimensional kernel and $j^2 = -\mathbf{1}_Q$ on the cokernel. We will use a little different definition from [Sa] where it is required the additional nonzero section of the kernel. Note that this is the same as a time-independent procomplex structure. Thus in the cocomplex case the Nijenhuis operator is just $A_j = L_w j$ as one easily checks.

Note that here the integrability criterion is not vanishing of the Nijenhuis operator. Actually to see this one might consider the boundary of a strictly pseudoconvex domain in $\mathbb{C}^2$ and a field of symmetry for it. So in this case the integrability criterion is stronger: the structure is integrable iff the 2-distribution $\mathcal{C} = \mathrm{Im}\, j$ is integrable and the Nijenhuis operator $A_j = L_w j$ vanishes.

Let us now using the above formulae prove the special case of theorem 3 that any regular 2-distribution on $\mathbb{R}^3$, which is trivially extended to a 2-distribution $\Pi^2 = \Pi^2 \times \{pt\}$ on $\mathbb{R}^3 \times \mathbb{R}$, is the image of Nijenhuis tensor $\Pi^2 = \mathrm{Im}\, N_j$ for some almost complex structure j. We use the connection between almost complex structures and procomplex ones. Since we have actually a 2-distribution on $\mathbb{R}^3$ we consider cocomplex structures.

Let $\alpha^{(t)} = \alpha$ be a nonzero 1-form on $\mathbb{R}^3$ with the kernel $\Pi^2 = \mathrm{Ker}\,\alpha$. Let $w^{(t)} = w$ be a symmetry of Π^2 and $\alpha(w) \equiv 1$, from where it follows that $L_w\alpha = 0$. Let us define in arbitrary way a structure $j : \Pi^2_\bullet \to \Pi^2_\bullet$ satisfying the condition $j^2(\xi) = -\xi$, $\xi \in \Pi^2_\bullet$. Then from proposition 3 we have: $\mathrm{Im}\, A_j = \mathrm{Im}\, j = \Pi^2$. Thus the image of the Nijenhuis tensor projects onto Π^2 and since it is j-invariant it coincides with Π^2, quod erat demonstrandum.

Chapter 3

Elliptic Monge-Ampère equations on a two-dimensional surface

In [L2] was established a deep connection between differential Monge-Ampère equations on a manifold L and effective forms on the manifold T^*L. In more abstract way this allows to consider the following construction as the model of Monge-Ampère equations (see also [L1], [LRC]). We will deal only with the four-dimensional case $\dim M = 4$, which corresponds to Monge-Ampère equations on a two-dimensional surface, $\dim L = \frac{1}{2}\dim T^*L = 2$, but first constructions are valid for general case.

Let us consider a symplectic manifold (M^4, ω) and a two-form $\theta \in \Omega^2(M)$. The pair (ω, θ) determines a Monge-Ampère equation (usually in the case $M = T^*L$ and thus always locally). Let us consider an endomorphism $j \in T^*M \otimes TM$ defined by the formula $\theta(X, Y) = \omega(jX, Y)$, $X, Y \in T_\bullet M$. Note that due to the skew symmetry $\omega(jX, Y) = \omega(X, jY)$.

Proposition 4. *The following conditions are equivalent:*

1. $\theta \wedge \omega = 0$, $\mathrm{Pf}(\theta) = \dfrac{\theta \wedge \theta}{\omega \wedge \omega} > 0$,

2. *j is a conformal almost complex structure* $j^2 = -f^2 \cdot \mathbf{1}$, $f \neq 0$.

Proof. 2) $\Rightarrow$ 1). Let $Y \notin < X, jX >$ be such a vector that $\omega(X, Y) = 0$.

Then the four vectors (X, jX, Y, jY) are linear independent, hence form a basis and we have:

$$\theta \wedge \omega(X, jX, Y, jY) = \frac{1}{4}[\theta(X, jY)\omega(jX, Y) + \theta(jX, Y)\omega(X, jY)] =$$
$$\frac{1}{2}\omega(j^2X, Y)\omega(jX, Y) = -\frac{f^2}{2}\omega(X, Y)\omega(jX, Y) = 0,$$

$$\begin{aligned}
&\theta \wedge \theta(X, jX, Y, jY) = \\
&\quad \frac{1}{4}[\theta(X, jY)\theta(jX, Y) - \theta(X, Y)\theta(jX, jY)] = \\
&\quad \frac{1}{4}[\omega(jX, jY)\omega(j^2X, Y) - \omega(jX, Y)\omega(j^2X, jY)] = \\
&\quad \frac{-f^2}{4}[\omega(jX, jY)\omega(X, Y) - \omega(jX, Y)\omega(X, jY)] = \\
&\quad f^2\omega \wedge \omega(X, jX, Y, jY).
\end{aligned}$$

1) $\Rightarrow$ 2). It follows from the condition $\omega(jX, Y) = \omega(X, jY)$ that if the operator j has two real eigenvalues $\lambda_1 \neq \lambda_2$ with eigenvectors X_{λ_1} and X_{λ_2} then they are skew orthogonal: $\omega(X_{\lambda_1}, X_{\lambda_2}) = 0$. Due to nonvanishing of Pfaffian, $\mathrm{Pf}(\theta) \neq 0$, we have $\mathrm{rk}(\theta) = 4$ and zero is not an eigenvalue. So there's three cases possible:

1. Two pair of real nonzero eigenvalues: (a, a, b, b),
2. A pair of real and a pair of complex conjugate nonzero eigenvalues: $(a, a, \alpha \pm i\beta)$,
3. Two pairs of complex conjugate eigenvalues: $(\alpha \pm i\beta, \gamma \pm i\delta)$.

In the first case the condition $\theta \wedge \omega = 0$ implies $a = -b$ and thus $\mathrm{Pf}(\theta) = -a^2 < 0$. In the second we have a two-dimensional irreducible invariant subspace W^2 in $T_\bullet M^4$ and a two-dimensional invariant subspace V_a corresponding to the eigenvalue a. There exist vectors $0 \neq X \in W^2$ and $Y \in V_a$ such that $\omega(X, Y) = 0$ and also Y is an eigenvector $jY = aY$. Then $\omega(jX, Y) = \omega(X, jY) = a\omega(X, Y) = 0$ and since $\omega(X, jX) = \theta(X, X) = 0$ we have a three-dimensional isotropic subspace $< X, jX, Y >$ which is impossible.

So there remains only the last case 3). Let us note that the eigenvalues are pure imagine. Actually, consider a two-dimensional irreducible invariant subspace W^2, choose vectors $X \in W^2$ and $Y \notin W^2$ such that $\omega(X,Y)=0$. Since the isotropic subspace is maximum two-dimensional we have: $\omega(jX,Y)\neq 0$. Further:

$$0 = \theta\wedge\omega(X,jX,Y,jY) = \frac{1}{2}\omega(j^2X,Y)\omega(jX,Y).$$

Hence $\omega(j^2X,Y)=0$, $j^2X\in W^2$ and $j^2X=-f^2X$. Now $\alpha\pm i\beta=\gamma\pm i\delta$ due to nondegeneracy of ω and to the equality $\omega(j^2X,Y)=\omega(X,j^2Y)$, from where $j^2=-f^2\cdot\mathbf{1}$. □

Remark 4. It follows from the proof that almost product structure j (hyperbolic equation) corresponds to the case $\mathrm{Pf}(\theta)<0$ and the almost projection structure j (parabolic equation) to the case $\mathrm{Pf}(\theta)=0$, $\mathrm{rk}\,\theta=2$ (see [L1]). □

If one requires $\mathrm{Pf}(\theta)\equiv -1$, which is equivalent to the rescaling of θ, then j is an almost complex structure. We will suppose further this is the case.

Definition 3. Let us call a pair (ω,θ) on M^4 or equivalently a pair (ω,j) *a Monge-Ampère pair* (correspondingly elliptic, hyperbolic or parabolic).

We will consider only elliptic case. The invariants for hyperbolic and mixed type case may be found in [L1] and [Ku] correspondingly.

One may construct invariants for a Monge-Ampère pair (ω,j) combining invariants of its components ω and j. According to Darboux's theorem ([St]) the symplectic structure has no local invariants. Local (formal) invariants for almost complex structure j are the Nijenhuis tensor N_j (see §2.1) and higher Nijenhuis tensors $N_j^{(2)},\ldots$, see [Kr].

One may also construct invariants for a Monge-Ampère pair (ω,θ) in the following way. By a theorem of Lepage ([L1], [LM]) the symplectic form ω divide the 3-form $d\theta$: $d\theta=\omega\wedge\alpha$. By 1-form α one may construct forms $\theta\wedge\alpha$, $d\alpha$ and so on.

Let us connect these two approaches. Introduce the vector 2-form:

$$R_j^\alpha(X,Y) = N_j(X,Y) - X\alpha(jY) + Y\alpha(jX) - jX\alpha(Y) + jY\alpha(X).$$

Proposition 5. *The following formula holds true:*

$$jR_j^\alpha = 2\omega \otimes X_\alpha,$$

where X_α is the vector field dual to the 1-form α: $\omega(X_\alpha, Z) = \alpha(Z)$.

Proof. From the formula from [FN]

$$i_{[X,Y]} = [L_X, i_Y] = [i_X d + d i_X, i_Y] = i_X d i_Y + d i_X i_Y - i_Y i_X d - i_Y d i_X$$

we obtain

$$\begin{aligned}
&i_{N_j(X,Y)}\theta = i_{[jX,jY]}\theta - i_{j[X,jY]}\theta - i_{j[jX,Y]}\theta - i_{[X,Y]}\theta = \\
&i_{jX} d i_{jY}\theta + d i_{jX} i_{jY}\theta - i_{jY} i_{jX} d\theta - i_{jY} d i_{jX}\theta - \\
&i_X d i_Y\theta - d i_X i_Y\theta + i_Y i_X d\theta + i_Y d i_X\theta + \\
&i_{[X,jY]}\omega + i_{[jX,Y]}\omega = \\
&-i_{jX} d i_Y\omega + 2d[\theta(X,Y)] - i_{jY} i_{jX} d\theta + i_{jY} d i_X\omega - \\
&-i_X d i_Y\theta + i_Y i_X d\theta + i_Y d i_X\theta + \\
&i_X d i_{jY}\omega + d i_X i_{jY}\omega - i_{jY} i_X d\omega - i_{jY} d i_X\omega + \\
&i_{jX} d i_Y\omega + d i_{jX} i_Y\omega - i_Y i_{jX} d\omega - i_Y d i_{jX}\omega = \\
&-i_{jX} d i_Y\omega + 2d[\theta(X,Y)] - i_{jY} i_{jX} d\theta + i_{jY} d i_X\omega - i_X d i_Y\theta + \\
&i_Y i_X d\theta + i_Y d i_X\theta + i_X d i_Y\theta + 2d[\theta(Y,X)] - i_{jY} i_X d\omega - \\
&i_{jY} d i_X\omega + i_{jX} d i_Y\omega - i_Y i_{jX} d\omega - i_Y d i_X\theta = \\
&d\theta(X,Y,\cdot) - d\theta(jX,jY,\cdot).
\end{aligned}$$

Note that this identity might be also obtained from the expression for the differential of θ coming from the Cartan formula:

$$\begin{aligned}
d\theta(X,Y,Z) \;=\; & \partial_X\omega(jY,Z) - \partial_{jX}\omega(Y,Z) + \\
& \omega([jX,Y] - j[X,Y], Z) + \omega(Y, [jX,Z] - j[X,Z]).
\end{aligned}$$

Let us use the obtained expression for $i_{N_j(X,Y)}\theta$ and the formula $d\theta = \omega \wedge \alpha$. We have:

$$d\theta(X,Y,Z) = \omega(X,Y)\alpha(Z) + \omega(Z,X)\alpha(Y) + \omega(Y,Z)\alpha(X),$$

and similarly for $d\theta(jX, jY, Z)$. Hence:

$$\theta(N_j(X,Y),Z) = \omega(jN_j(X,Y),Z) = \omega(Z,X)\alpha(Y) + \omega(Y,Z)\alpha(X) - \omega(Z,jX)\alpha(jY) - \omega(jY,Z)\alpha(jX) + 2\omega(X,Y)\alpha(Z).$$

Carrying over all the terms but the last into the left part of the equality we obtain $\omega(jR_j^\alpha(X,Y),Z) = 2\omega(X,Y)\alpha(Z)$, quod erat demonstrandum. $\square$

Corollary. The integrability of almost complex structure j is equivalent to the closeness of the form θ, $d\theta = 0$.

Proof. In one side this statement is contained in the proof of theorem 1.5 from [LRC] and follows from proposition 5 and Newlander-Nirenberg theorem ([NN]). Actually, if $d\theta = 0$ then $\alpha = 0$, $X_\alpha = 0$, $R_j^\alpha = 0$ and $N_j = 0$. Consider now the inverse statement, let j be an integrable almost complex structure. Then $N_j = 0$. Suppose $\alpha \neq 0$. Then there exists a basis (X, jX, Y, jY) such that $X, jX, Y \in \operatorname{Ker}\alpha$, $\alpha(jY) = 1$. We have $-X = R_j^\alpha(X,Y) \in < jX_\alpha >$ and $-jX = R_j^\alpha(jX,Y) \in < jX_\alpha >$, which is a contradiction. Hence $\alpha = 0$ and $d\theta = 0$. $\square$

Let us denote by $\Gamma_\alpha^1 = < X_\alpha >$ the one-dimensional distribution generated by the vector field X_α. The proposition 5 implies that $jR_j^\alpha(X,Y) \in \Gamma_\alpha^1$ for all $X, Y \in T_\bullet M$. Since by definition $(\Gamma_\alpha^1)^{\perp\omega} = \operatorname{Ker}\alpha$ then $\Gamma_\alpha^1 \subset \operatorname{Ker}\alpha$.

Lemma 1. *Let almost complex structure j be nonintegrable at a point $x \in M$ (and hence in some neighborhood $\mathcal{O}_M(x) \subset M$), i.e. $N_j|_x \neq 0$. Then the image of the Nijenhuis tensor is (locally) a two-dimensional distribution $\Pi_j^2 = \operatorname{Im} N_j$. We have the following inclusions: $\Gamma_\alpha^1 \subset \Pi_j^2 \subset (\Gamma_\alpha^1)^{\perp\omega}$, and moreover $\Pi_j^2 = (\operatorname{Ker}\alpha) \cap j(\operatorname{Ker}\alpha)$.*

Proof. The first statement $\dim(\Pi_j)_{x'} = 2$, $x' \in \mathcal{O}_M(x)$, is obvious (§2.1). By the definition we have:

$$\alpha(jN_j(X,Y)) = \alpha(jN_j(X,Y)) - \alpha(jX)\alpha(jY) + \alpha(jY)\alpha(jX) + \alpha(X)\alpha(Y) - \alpha(Y)\alpha(X) = \alpha(jR_j^\alpha(X,Y)) = 0,$$

from where $\Pi_j^2 \subset \operatorname{Ker}\alpha = (\Gamma_\alpha^1)^{\perp\omega}$. Further, since the vector X_α is skew orthogonal to $\operatorname{Ker}\alpha$ every its complementary plane in $\operatorname{Ker}\alpha$ is symplectic, and hence every Lagrangian plane in $\operatorname{Ker}\alpha$ contains Γ^1_α. Due to $\omega(X, jX) = 0$ every two-dimensional j-invariant subspace is Lagrangian and hence $\Gamma^1_\alpha \subset \Pi^2_j$. The last statement is obvious. □

Let the distribution $\Pi^3_j = \partial^{(1)}\Pi^2_j$ be the first derivative of the distribution Π^2_j (see [T] and §2.1). The following lemma gives the regularity condition for the distributions Π^2_j and Π^3_j at a neighborhood of a point $x \in M$.

Lemma 2. *If $N_j|_x \neq 0$ then Π^2_j is a two-dimensional distribution on $\mathcal{O}_M(x)$. If $R^\alpha_j\Big|_x = 0$ then $N_j|_x = 0$. Π^3_j is a three-dimensional distribution different from $\operatorname{Ker}\alpha$ on M such that $\Pi^3_j \cap \operatorname{Ker}\alpha = \Pi^2_j$ iff $d\alpha|_{\Pi^2_j} \neq 0$.*

Proof. If $R^\alpha_j = 0$ at the point $x \in M$ then

$$N_j(X, Y) = X\alpha(jY) - Y\alpha(jX) + jX\alpha(Y) - jY\alpha(X), \text{ and}$$

$$-jN_j(X, Y) = N_j(jX, Y) = jN_j(X, Y),$$

where the first equality follows from the definition of the Nijenhuis tensor and the second from the obtained formula. Thus $N_j(X, Y) = 0$ for all $X, Y \in T_xM$.

If $X, Y \in \Pi^2_j$ then by Cartan formula $d\alpha(X, Y) = \alpha([Y, X])$ and the equivalence of equalities $d\alpha|_{\Pi^2_j} \neq 0$ and $\Pi^3_j \neq \operatorname{Ker}\alpha$ is obvious. □

Definition 4. Let us call a Monge-Ampère pair (ω, j) or (ω, θ) *nondegenerate* at a point $x \in M$ if the almost complex structure j is nonintegrable at this point, $N_j|_x \neq 0$, i.e. there's defined on $\mathcal{O}_M(x)$ the two-dimensional distribution $\Pi^2_j = \operatorname{Im} N_j$, and if $d\alpha|_{\Pi^2_j} \neq 0$, i.e. there's defined the three-dimensional distribution $\Pi^2_j \subset \Pi^3_j \neq \operatorname{Ker}\alpha$.

From now on we consider only nondegenerate (elliptic) pairs.

Let $X_0 \in \Gamma^1_\alpha \setminus \{0\}$. There exists a vector $Y_0 \notin \Pi^2_j$ such that $N_j(X_0, Y_0) = X_0$.

Lemma 3. $\alpha(X_0) = \alpha(jX_0) = \alpha(Y_0) = 0$, $\alpha(jY_0) = 1$.

Proof. Since X_0, jX_0, Y_0, jY_0 is a basis in T_xM it follows from the formula

$$\Gamma^1_\alpha =< X_0 >\ni jR^\alpha_j(X_0, Y_0) = \\ jX_0 - jX_0\alpha(jY_0) + jY_0\alpha(jX_0) + X_0\alpha(Y_0) - Y_0\alpha(X_0)$$

that $\alpha(X_0) = \alpha(jX_0) = 0$, $\alpha(jY_0) = 1$, $jR^\alpha_j(X_0, Y_0) = X_0\alpha(Y_0)$.

Also:

$$\Gamma^1_\alpha =< X_0 >\ni jR^\alpha_j(X_0, jY_0) = \\ X_0 + jX_0\alpha(Y_0) - Y_0\alpha(jX_0) + X_0\alpha(jY_0) - jY_0\alpha(X_0),$$

from where the equality $\alpha(Y_0) = 0$. □

Remark 5. The formulae from the proof give the values of jR^α_j on the basis (X_0, jX_0, Y_0, jY_0):

$$jR^\alpha_j(X_0, Y_0) = jR^\alpha_j(jX_0, jY_0) = 0,\ jR^\alpha_j(X_0, jY_0) = jR^\alpha_j(jX_0, Y_0) = 2X_0,$$

due to the formula $R^\alpha_j(jX, jY) = -R^\alpha_j(X, Y)$, and for other pairs of basis vectors the form is zero: $R^\alpha_j(X, X) = R^\alpha_j(X, jX) = 0$. □

The vector Y_0 is determined up to Π^2_j-shifts and the similar is true for jY_0. Thus the vector $P_1 \in \Gamma^1_\alpha$ satisfying the condition $\omega(P_1, jY_0) = 1$ is uniquely determined (remind that $jY_0 \notin \operatorname{Ker}\alpha = (\Gamma^1_\alpha)^{\perp\omega} = (P_1)^{\perp\omega}$). We set $P_2 = jP_1$.

Due to the nondegeneracy of the Monge-Ampère pair (ω, j) we have for the vector fields $P_1, P_2 \in \Gamma(TM)$: $Z_0 = [P_1, P_2] \in \Pi^3_j \setminus \Pi^2_j$. Let us denote by L^1 the intersection of the subspace $\operatorname{Ker}\alpha$ with the complex line $\mathbb{C}z_0 =< Z_0, jZ_0 >$. There exists a unique vector Q_2 on the transversal line L^1 to Π^2_j in $\operatorname{Ker}\alpha$ satisfying the equality $N_j(P_1, Q_2) = P_1$. We set $Q_1 = jQ_2$.

The arguments above prove the following statement:

Theorem 3. *A nondegenerate elliptic Monge-Ampère pair (ω, j) canonically determines an $\{e\}$-structure, i.e. the field of basis frames (P, Q). This structure is a complete invariant, i.e. two nondegenerate elliptic Monge-Ampère pairs are isomorphic iff the corresponding $\{e\}$-structures are. The classifying $\{e\}$-structure satisfies the following relations:*

$\omega(\uparrow,\leftarrow)$	P_1	P_2	Q_1	Q_2
P_1	0	0	1	0
P_2	0	0	0	1
Q_1	-1	0	0	0
Q_2	0	-1	0	0

$N_j(\uparrow,\leftarrow)$	P_1	P_2	Q_1	Q_2
P_1	0	0	$-P_2$	P_1
P_2	0	0	$-P_1$	$-P_2$
Q_1	P_2	P_1	0	0
Q_2	$-P_1$	P_2	0	0

X	P_1	P_2	Q_1	Q_2
jX	P_2	$-P_1$	$-Q_2$	Q_1

X	P_1	P_2	Q_1	Q_2
$\alpha(X)$	0	0	1	0

$\square$

Remark 6. An $\{e\}$-structure (f_i) is a field of basis frames on a manifold M. Let $[f_i, f_j] = c_{ij}^k f_k$. The set of functions c_{ij}^k (invariants) on the manifold M is called a structure function (structure constants in the case of Lie group). One may write a Monge-Ampère equation in terms of these invariants. The $\{e\}$-structure is a particular case of the G-structure when the group $G \subset \mathrm{Gl}(T_\bullet M)$ is trivial $G = \{e\}$ ([St], [ALV]). $\square$

Remark 7. Let us note that in general case the 1-distributions $< P_1 >$ and $< P_2 >$ are different from Υ_1 and Υ_2 (see §2.1). Since all these distributions lie in the 2-distribution Π_j^2 we may get new invariants combining them. Consider for example the slope of U_1 in the basis (P_1, P_2) of Π_j^2 *modulo* π. This invariant of the Monge-Ampère pair (ω, j) is connected with the $\{e\}$-structure in the following way. Define the complex line action on the tangent bundle by the formula: $(x+iy)W = xW + y(jW)$, $W \in T_\bullet M$. Let $w \in C^\infty(M, \mathbb{C})$ be such a complex-valued function that for the vector field $Z_0 = [P_1, P_2]$ it holds: $\alpha(wZ_0) = 0$, i.e. $wZ_0 \in L^1$. Let us require additionally that $|w| = 1$ and at the given point holds: $\mathrm{Re}(w) > 0$ or $\mathrm{Re}(w) = 0$, $\mathrm{Im}(w) > 0$. By these conditions the function w is determined uniquely (with the condition of continuity). The slope of U_1 in the basis (P_1, P_2) is given by the formula $\varphi = \frac{1}{2}\arg w$. Note that changing the parametrization function $w \mapsto -w$ is equivalent to the change of 1-distributions $\Upsilon_1 \mapsto \Upsilon_2$ or $\varphi \mapsto \varphi + \frac{\pi}{2}$. $\square$

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