Two constructions of weight systems for invariants of knots in non-trivial 3-manifolds

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Abstract

A new family of weight systems of finite type knot invariants of any positive degree in orientable 3-manifolds with non-trivial first homology group is constructed. The principal part of the Casson invariant of knots in such manifolds is split into the sum of infinitely many independent weight systems. Examples of knots separated by corresponding invariants and not separated by any other known finite type invariants are presented.

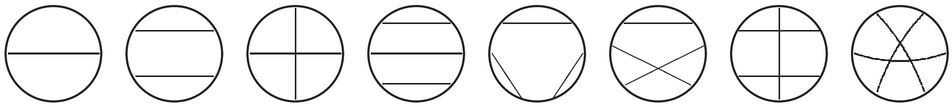
Keywords: knot invariant, weight system, chord diagram

1 Introduction

The starting point in the construction of finite type invariants of knots in a 3-dimensional manifold M^3 is the construction of corresponding *weight systems*, i.e. functions on the set of homotopy types of singular knots in M^3, satisfying certain natural conditions, see [6], [8]. This paper describes two new constructions of weight systems in 3-manifolds with $\pi_1(M^3) \neq \{1\}$.

Any weight system in $\mathbf{R}^3$ generates a weight system in an arbitrary orientable M^3. If $\pi_1(M^3)$ is non-trivial, then the obtained weight system obviously splits into the sum of independent weight systems corresponding to different homotopy types of loops in M^3 defined by these singular knots. This splitting does not help in separating knots. However any of these summands splits further into a sum of many weight systems that can generate independent knot invariants. In §2 such a splitting is described for the simplest (of degree 2) invariant of knots in $\mathbf{R}^3$; if rank $H_1(M^3) \geq 2$ then we obtain infinitely many independent weight systems in this way, in particular infinitely many independent degree 2 invariants of knots in $M^2 \times \mathbf{R}^1$.

In §3 we describe an infinite family of weight systems of arbitrary degrees for knots in 3-manifolds with non-trivial first homology group. For any k, these systems of degree k are parameterized by unordered collections of $k+1$ non-zero elements of the group $H_1(M^3)$. These weight systems are characterized by taking zero values on all singular knots whose chord diagrams have crossing chords. For M^3 of the form $M^2 \times \mathbf{R}^1$ the simplest (of degree 1) such weight



1_1 2_1 2_2 3_1 3_2 3_3 3_4 3_5

Figure 1: Examples of chord diagrams

systems coincide with the principal parts of Fiedler's invariants [4], and our degree 2 weight systems improve the principal parts of invariants $I_3^K(a,b)$ from Theorem 2.10 of [5].

In §4 we show how the invariants with these new weight systems separate knots that cannot be separated by previously known finite type invariants.

1.1 Definitions (see [6], [8])

A *chord diagram of degree* k (or simply a k*-chord diagram*) is an arbitrary collection of $2k$ distinct points in S^1 matched in pairs. For examples of such diagrams, see Fig. 1, where the matched points of the circle are connected by thin chords. A smooth map $f : S^1 \to M^3$ *respects* some chord diagram if it joins the points of any of its pairs. Two k-chord diagrams are *equivalent* if they can be transformed into one another by orientation-preserving diffeomorphisms of S^1. Given an equivalence class A of k-chord diagrams, two maps $f_1, f_2 : S^1 \to M^3$ belong to one and the same A*-route* of degree k, if they both respect some k-chord diagrams $\bar{A}_1, \bar{A}_2$ of class A, and can be reduced to one another by the composition of 1) a homotopy in the class of maps $S^1 \to M^3$ respecting $\bar{A}_1$, and 2) an orientation-preserving diffeomorphism of S^1 transforming $\bar{A}_1$ to $\bar{A}_2$. Thus, the A-routes in M^3 are the equivalence classes of singular maps $S^1 \to M^3$ under this equivalence relation.

A *degree* k *weight system* in M^3 is a numerical function on the set of all A-routes of degree k in M^3, satisfying some two sets of restrictions.

One, the simplest restriction, called 1T-relation, claims that this function should take zero value on any A-route of degree k such that:

1) any chord diagram of class A contains a pair of points x_i, y_i not separated in S^1 by points of other pairs of this diagram (i.e. one of segments $[x_i, y_i]$ or $[y_i, x_i]$ in S^1 does not contain points x_j or y_j, $i \neq j$, as e.g. in pictures 1_1, 2_1, 3_1, 3_2, 3_3 of Fig. 1), and

2) the loop $f : [x_i, y_i] \to M^3$ or $f : [y_i, x_i] \to M^3$, defined by the image of this segment under a map f from our A-route, is contractible in M^3.

The second series of restrictions (1), called 4T-relations, is more complicated; it can be derived from the study of singular maps with $k-2$ double points and one triple point. Let us consider any such generic map, i.e. a map $f : S^1 \to M^3$ with exactly $k-2$ generic self-intersections and one triple self-intersection point such that three derivatives of f at this triple point are linearly independent in

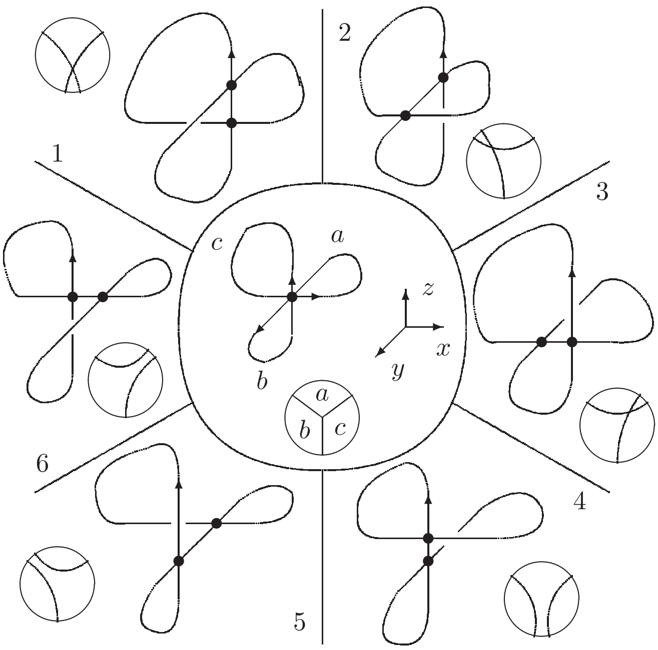


Figure 2: Resolutions of a triple point

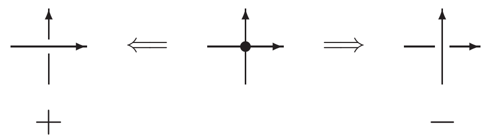


Figure 3: Possible resolutions of a transverse self-intersection

the tangent space of M^3. Then the triple point can be partially resolved in six different ways, moving it into two double self-intersection points, see Fig. 2, so that f splits in six different ways into singular knots with exactly k self-intersections.

Let I be a degree k weight system, and $I(m)$, $m = 1, \ldots, 6$, be its value on the singular knot obtained from f by a local move indicated in Fig. 2 in the sector labelled by m. Then 4T-relation claims that

$$I(1) - I(4) = I(2) - I(5) = I(3) - I(6). \tag{1}$$

The importance of 1T- and 4T-relations is determined by the fact that the *residues* of finite type knot invariants should satisfy these conditions; let us recall this notion. A self-intersection point $f(x) = f(y)$, $x \neq y$, of a smooth map $f : S^1 \to M^3$ is called *transverse* if the derivatives of f at x and y are not collinear in $T_{f(x)}M^3$. Any transverse self-intersection of a map $f : S^1 \to M^3$ can be resolved in two essentially different ways by small local moves of f, see Fig. 3. These two local resolutions cannot be connected by a short local path in the space of embeddings $S^1 \to M^3$ since they are separated in a neighborhood of f in this space by a piece of the *discriminant variety* consisting of maps with self-intersections as in the middle picture in Fig. 3. This variety is a singular hypersurface in $C^\infty(S^1, M^3)$; its regular points are exactly the maps with unique transverse self-intersection. If M^3 is oriented, then there is an invariant way to call one of these resolutions as *positive*, and the other as *negative*; for the canonical orientation in $\mathbf{R}^3$ this discrimination is indicated by indices $+$ and $-$ in Fig. 3. Indeed, if we fix an affine chart in M^3 close to the self-intersection point $f(x) = f(y)$ and a parameterization in S^1, then the determinant of the triplet of vectors $\{f'(x),\ f'(y),\ f(y) - f(x)\}$ is a well-defined function in a neighborhood of the point f in the space $C^\infty(S^1, M^3)$. The derivative of this function defines a transversal orientation of the discriminant variety at the point f, and hence the desired difference between two possible resolutions of f.

Given a numerical invariant I of knots in M^3 (i.e. of smooth embeddings $S^1 \to M^3$) and an arbitrary map $f : S^1 \to M^3$ with k transverse self-intersection points $f(x_i) = f(y_i)$, $df(x_i) \not\parallel df(y_i)$, $i = 1, \ldots, k$, which does not have any other self-intersections or singular points, we can resolve all these singularities in 2^k different ways, replacing any self-intersection point as it is shown in the left- or right-hand part of Fig. 3. The *residue* of the invariant I at the singular knot f is defined as the alternated sum of values of I at all these 2^k non-singular knots obtained from f; the value of invariant I at such a desingularization should be taken with the coefficient 1 or -1 depending on the parity of the number of negative local resolutions defining the desingularization.

By definition, a knot invariant is *of degree* $\leq k$ if its residue at any singular knot with more than k transverse self-intersections is equal to 0.

It is easy to see that the residue of any degree k invariant of knots in orientable M^3 is a weight system, i.e. it satisfies 1T- and 4T-relations.

In general, these necessary conditions are not sufficient. For example, there exists a degree 1 weight system in $S^2 \times S^1$ that does not correspond to any knot invariant, see [8]. For M^3 of the form $M^2 \times \mathbf{R}^1$ the situation is much better.

Proposition 1 (see [2], [1]) *Suppose that $M^3 = M^2 \times \mathbf{R}^1$, M^2 an orientable surface (maybe with boundary), and I_k is a non-zero $\mathbf{R}$-valued weight system of degree k in M^3. Then there exists an $\mathbf{R}$-valued degree k invariant of knots in M^3, whose residue coincides with this function I_k.* $\square$

2 Degree 2 invariants with the chord diagram $\oplus$

Any weight system I in $\mathbf{R}^3$ defines also a weight system (of the same degree) in any orientable 3-manifold M. Indeed, for any chord diagram A there exists only one A-route in $\mathbf{R}^3$. Thus, any degree k weight system in $\mathbf{R}^3$ is just a function on the set of equivalence classes of k-chord diagrams. We can define the desired weight system in M^3 as the function whose value on an A-route is equal to the value of I on the chord diagram A. The obtained function obviously satisfies the 1T- and 4T-relations, since the original weight system does.

If $\pi_1(M^2) \neq 0$ then weight systems of this origin can be split into sums of more specific independent systems. Consider for example the unique degree 2 weight system in $\mathbf{R}^3$ taking value 1 on the crossing 2-chord diagram $\oplus$ and value 0 on the non-crossing diagram. We denote by $I_{\oplus}(M^3)$ the corresponding degree 2 weight system in M^3. This system splits into many independent summands, corresponding to the splitting of the set of $\oplus$-routes in M^3 into equivalence classes generated by the following equivalence condition: two $\oplus$-routes are equivalent if one can approach one and the same singular knot with a generic triple point (see Fig. 2) along both of these $\oplus$-routes. Let us describe these equivalence classes algebraically.

In Fig. 2 the singular knots with crossed 2-chord diagram are shown in sectors 1, 2 and 3. Contracting one of four segments of such a knot, bounded by two self-intersection points, we obtain a singular knot with a triple point. Let a, b and c be three elements of the group $\pi_1(M^3)$ defined by three loops of this singular knot, taken in the cyclic order defined by the orientation of the knot.

Definition 1 Given a group π, the related *trefoil structure* $\clubsuit(\pi)$ is the set of equivalence classes of cyclically ordered triplets of non-unit elements of π, with equivalence relation generated by the following basic equivalences:

0) definition of the circular ordering: $(a, b, c) \sim (b, c, a) \sim (c, a, b)$;

1) simultaneous conjugation by an element of π: (a, b, c) is equivalent to $(g^{-1}ag, g^{-1}bg, g^{-1}cg)$ for any $g \in \pi$;

2) (a, b, c) is equivalent to the following six triplets: $(a, ba, a^{-1}c)$, (a, ba^{-1}, ac), (ac^{-1}, cb, c), $(ac, c^{-1}b, c)$, $(b^{-1}a, b, cb)$, and (ba, b, cb^{-1}).

The last six expressions have the following sense. Given a singular knot with a triple point as in Fig. 2, we choose 1) one of its three partial desingularizations having the crossed chord diagram as in pictures 1, 2 or 3 of this figure, and 2) an endpoint of the "short" segment in the corresponding singular knot, joining its two self-intersection points; this in total gives six possibilities. Then we expand this segment by moving the chosen self-intersection point until it meets another self-intersection point. The homotopy classes in $\pi_1(M^3)$ of three loops of the resulting curve with a triple point are expressed through the similar classes of the initial (central in Fig. 2) singular knot in one of six ways indicated in item 2) of the previous definition.

Definition 2 A singular knot in M^3 with exactly two transverse self-intersections *respects* the element τ of the trefoil structure $\clubsuit(\pi_1(M^3))$, if

1) the chord diagram, represented by this singular knot, is crossed, and

2) contracting an arbitrary segment of this knot, joining its two singular points, we obtain a singular knot with a triple point, whose three loops (cyclically ordered by the orientation of the knot) define a triplet of elements of $\pi_1(M^3)$ (with the basepoint at the triple point of f) belonging to τ.

Proposition 2 *A. The last definition is correct, i.e. the class of the obtained triplet in the trefoil structure does not depend on the choice of one of four segments to be contracted.*

B. All singular knots from one and the same $\oplus$-route in M^3 define one and the same element of $\clubsuit(\pi_1(M^3))$.

This proposition follows immediately from the definition. $\square$

Proposition 3 *Let M^3 be an orientable 3-manifold, and τ an element of the trefoil structure $\clubsuit(\pi_1(M^3))$. Then the function on degree 2 routes in M^3 which takes value 1 on all singular knots in M^3 respecting τ, and value 0 on all other singular knots with two transverse self-intersections, is a weight system.*

Indeed, only the $\oplus$-routes, representing one and the same element of the trefoil structure, can meet in one and the same 4T-relation. Therefore for any element τ the described function satisfies 1T- and 4T-relations. $\square$

There is an Abelian version of this notion which generally defines a smaller number of independent weight systems, but is simpler. Let us describe it.

If the group H is Abelian, then the related trefoil structure $\clubsuit(H)$ is the set of equivalence classes of cyclically ordered triplets of non-zero elements of H, with equivalence relation generated by elementary equivalences as follow:

$$(a,b,c) \sim (a,b+a,c-a) \sim (a,b-a,c+a) \sim (a-c,b+c,c) \sim \qquad (1)$$

$$\sim (a+c,b-c,c) \sim (a-b,b,c+b) \sim (a+b,b,c-b).$$

Definition 3 A singular knot in M^3 with exactly two transverse self-intersections *respects* element Θ of the structure $\clubsuit(H_1(M^3))$, if

1) the chord diagram of this singular knot is crossed, and

2) the contraction of an arbitrary segment of this knot, joining its two singular points, gives a singular knot with a triple point, such that three loops of this knot (taken in the cyclic order defined by its orientation) define a triplet of elements of $H_1(M^3)$ belonging to the element Θ.

According to (1), contracting a different segment of the singular knot we obtain the same element of $\clubsuit(H_1(M^3))$.

Proposition 4 *Let M^3 be an orientable 3-manifold. Given an element Θ of the trefoil structure $\clubsuit(H_1(M^3))$, there exists a weight system in M^3 taking value 1 on all singular knots in M^3 respecting Θ and value 0 on all other singular knots with two transverse self-intersections.* $\square$

An obvious invariant of elements of trefoil structures is the subgroup in π generated by elements a, b and c. However, there are many other invariants. For instance, if $\pi \equiv \mathbf{Z}^k$, $k \geq 2$, and $a+b+c=0$ in $\mathbf{Z}^k$, then the integer area of the oriented triangle with vertices a, b, c is the same for all triplets (a,b,c) defining one and the same element of $\clubsuit(\pi)$.

3 Invariants defined by non-crossed chord diagrams

Every unordered collection of $k+1$ non-zero elements of the group $H_1(M^3)$ (M^3 orientable) defines well a degree k weight system in M^3.

Given such a collection Γ of elements $\gamma_0, \dots, \gamma_k \in H_1(M^3) \setminus 0$, the corresponding function I_Γ on the space of all A-routes of degree k is defined as follows. If the chord diagram A has at least one pair of crossing chords (i.e. chords whose four endpoints alternate in S^1, as e.g. in diagrams 2_2, 3_3, 3_4, 3_5 of Fig. 1), then the value of I_Γ on any A-route is equal to 0. If A has no such crossing chords, then for any generic immersion $f : S^1 \to M^3$, respecting this chord diagram and having no other self-intersections, the variety $f(S^1)$ defines naturally $k+1$ elements of $H_1(M^3)$; to obtain these elements, we smooth any self-intersection of $f(S^1)$ by the rule $\Longrightarrow$ and take the classes of $k+1$ separate circles, into which this smoothing splits our curve. The value of the desired function I_Γ on an A-route is equal to 1 (respectively, to 0) if the obtained unordered collection of elements of $H_1(M^3)$ coincides (respectively, does not coincide) with the given collection $(\gamma_0, \dots, \gamma_k)$.

Theorem 1 *For any collection Γ of non-zero elements $\gamma_0, \dots, \gamma_k$ of $H_1(M^3)$, this function I_Γ on the space of A-routes satisfies the 1T- and 4T-relations.*

Proof. Consider a generic singular knot $f : S^1 \to M^3$ with one triple point and $k-2$ double points, see Fig. 2. If one of its six decompositions into singular knots with k double points defines a chord diagram without crossing chords, then exactly two other decompositions also have chord diagrams with this property; in Fig. 2 they are decompositions 4, 5 and 6. The collections of $k+1$ homology classes, corresponding to these three decompositions, also coincide, therefore our function satisfies the 4T-relations; 1T-relation follows now from the condition that none of elements γ_i is trivial. □

Remark 1 If M^3 has the form $M^2 \times \mathbf{R}^1$, then the initial (of degree 1) weight systems of this series coincide with residues of Fiedler's invariant of [4], and a majority of our degree 2 weight systems coincide with residues of certain invariants $I_3^K(a,b)$ from Theorem 2.10 of [5]. However, the construction of [5] requires some additional restrictions (in our terms, difference of all elements γ_i), which are unnecessary for us, and misses one necessary restriction ($a \neq K$ in notation of [5]).

3.1 Non-Abelian version

As in the previous section, we can consider also the non-Abelian version of these weight systems, replacing homology types of loops by their homotopy types. Let us describe explicitly the corresponding weight systems of degree 1 and 2.

Proposition 5 *Let M^3 be a connected orientable manifold. Then*

A) (see [8]) degree 1 weight systems in M^3 are in the natural one-to-one correspondence with the functions on the set of unordered pairs of non-unit elements of $\pi_1(M^3)$ considered up to the simultaneous conjugation: a pair (b, c) of such elements is considered equal to $(g^{-1}bg, g^{-1}cg)$ for any $g \in \pi_1(M^3)$.

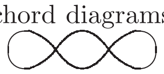
B) degree 2 weight systems in M^3, taking zero value on all $\oplus$-routes, are in the natural one-to-one correspondence with the functions on the set of cyclically ordered triplets of non-unit elements of $\pi_1(M^3)$ taking equal values on triplets related by the following elementary equivalence relations:

a) definition of the cyclic ordering: $(b, c, d) \sim (c, d, b) \sim (d, b, c)$;

b) simultaneous conjugations: the triplet (b, c, d) of such elements is equivalent to the triplet $(g^{-1}bg, g^{-1}cg, g^{-1}dg)$ for any $g \in \pi_1(M^3)$;

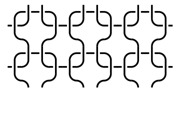
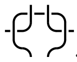
c) the relation $(b, c, d) \sim (b, cdc^{-1}, c)$.

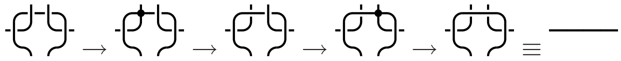
Proof. A. The irreducible components of the discriminant variety in $C^\infty(S^1, M^3)$ (i.e. the 1_1-routes) are in the one-to-one correspondence with such equivalence classes of pairs (b, c) of (maybe unit) elements of $\pi_1(M^3)$. 4T-relation is void in this case, and 1T-relation coincides with the restriction $b \neq 1 \neq c$.

B. Let A be the class of non-crossed two-chord diagrams. Any A-route in M^3 consists of parameterized singular knots  mapped somehow into M^3. Contracting the image of one of two segments connecting the self-intersection points of this singular knot, we obtain a singular knot with a triple point in M^3. We can assume that this triple point is the basepoint in M^3, then three loops of our knot define three cyclically ordered elements of $\pi_1(M^3)$. Exactly three of six partial resolutions of this triple self-intersection (see Fig. 2) correspond to singular knots respecting non-crossed 2-chord diagrams. By 4T-relation our weight system should take equal values on all such singular knots, and hence determines a function on the set of cyclically ordered triplets of elements of $\pi_1(M^3)$ defined by either of them. Travelling inside the stratum of singular knots, we can arbitrarily move the common basepoint of our three loops; this implies restriction b) in Proposition 5 B). If we contract a different segment between two self-intersection points, then we arrive at a similar stratum characterized by the triplet (b', c', d') with $b' = b, c' = cdc^{-1}, d' = c$, which implies restriction c) in our proposition. It is easy to see that this set of restrictions is complete. □

4 Examples of practical calculations

A wealth of non-trivial knots in $\mathbf{T}^2 \times \mathbf{R}^1$, $\mathbf{T}^2 \equiv S^1 \times S^1$, is provided by the textile structures. These structures define 1-dimensional submanifolds in $\mathbf{R}^3$ invariant under a lattice of parallel shifts $\mathbf{Z}^2$, and hence knots or links in the quotient space $\mathbf{R}^3/\mathbf{Z}^2 \sim \mathbf{T}^2 \times \mathbf{R}^1$.

For example, the *single jersey* structure  [7] can the depicted in the standard rectangular chart of $\mathbf{T}^2$ by the picture . To distinguish it from the trivial structure in the same homotopy class, we join these two structures by a generic path in the space $C^\infty(S^1, \mathbf{T}^2 \times \mathbf{R}^1)$:

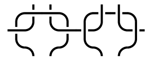
 (1)

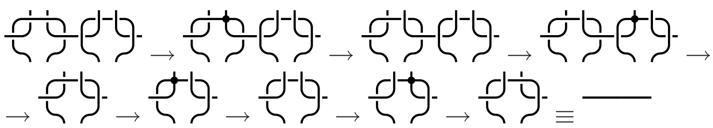
This path crosses the discriminant variety twice. These crossing points belong to irreducible components of the discriminant, characterized (in accordance with Proposition 5A) by two pairs of homotopy classes

$$((0,1)(1,-1)) \ \text{ and } \ ((0,-1)(1,1)) \ ; \tag{2}$$

here $(1,-1)$ denotes the class in the Abelian group $\pi_1(\mathbf{T}^2)$ equal to the difference of the horizontal generator of this group oriented "to the right" in our picture and the vertical generator oriented "to the top of the page". Any of these components defines a dual weight system of degree one. Therefore any of two degree one invariants, corresponding by Proposition 1 to these weight systems, separates single jersey from the trivial structure.

Remark 2 Adding a constant function to a degree k knot invariant, $k \geq 1$, we obtain an invariant of the same degree and the same weight system, therefore we can and will assume that all our invariants take zero value on the trivial knot indicated in the right-hand part of (1).

The next example is more complicated. The $1+1$ rib structure  [7] is the connected sum of the single jersey structure with its "mirror image". To separate $1+1$ rib from the trivial structure, we join them by the generic path

 (3)

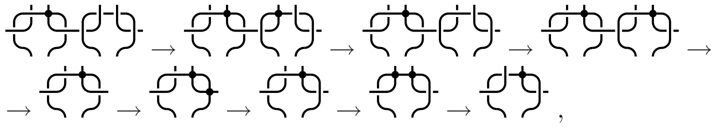
There are four surgeries in this path; they belong to components of the discriminant characterized by pairs

$$((0,-1)(1,1)), \ ((0,1)(1,-1)), \ ((0,1)(1,-1)), \ \text{ and } \ ((0,-1)(1,1)) \tag{4}$$

respectively. It is easy to calculate that at these four points our path crosses the discriminant respectively in the positive, positive, negative and negative directions. Thus, this path crosses exactly two different irreducible components of the discriminant; each of these components is crossed twice in different directions. Therefore the first and the last structure in the sequence (3) are not separated by first degree invariants. Let us try to separate them by second degree invariants. According to the general theory (see [8]), the same sequence

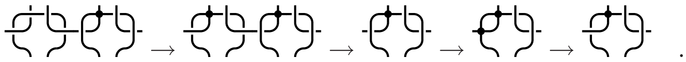
(3) can be used for this, however, different surgeries of this sequence should be taken not only with their signs, but also with certain weights defined by these second degree invariants. Moreover, for our calculation the exact values of these weights are not necessary since it is sufficient to know the differences of such weights for surgeries within one and the same component of the discriminant.

To calculate these differences, we join the first and the fourth surgery in (3) by a generic path inside the discriminant:



(5)

and then the second and the third surgery:



(6)

The first and the last surgeries in (5) have non-crossed chord diagrams. Their characteristic triplets of elements of $H_1(\mathbf{T}^2)$ (see Theorem 1) are equal to $((0,-1)(1,0)(0,1))$, but their signs are opposite (and equal to $+$ and $-$ respectively). The second surgery also has a non-crossed chord diagram; its characteristic triplet is equal to $((0,-1)(1,2)(0,-1))$, and the sign is equal to $-$. The third surgery has a crossed chord diagram and sign $+$, the corresponding element of the trefoil structure $\clubsuit(\pi_1(\mathbf{T}^2))$ is represented by the triplet

$$((0,-1)(0,-1)(1,2)). \tag{7}$$

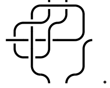
The first surgery of (6) has a non-crossed chord diagram, sign $-$, and characteristic triplet $((0,1)(1,-2)(0,1))$. The second surgery has the crossed chord diagram and sign $+$; the corresponding trefoil element is defined by the triplet

$$((0,1)(0,1)(1,-2)). \tag{8}$$

Thus, in total we have in (5) and (6) six crossings of the set of singular knots with two self-intersections. Considering all these surgeries, we arrive at the following statement.

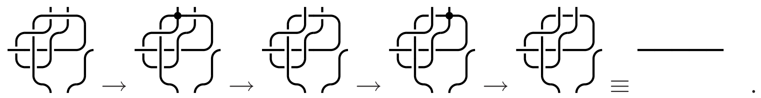
Proposition 6 *Both degree two invariants defined by non-crossed chord diagrams and characteristic triplets* $((0,1)(1,-2)(0,1))$ *and* $((0,-1)(1,2)(0,-1))$ *take on the* $1+1$ *rib knot values equal to* -1*. All other basic invariants with non-crossed 2-chord diagrams take zero value on this knot. The cumulative degree 2 knot invariant defined by the crossed 2-chord diagram* $\oplus$ *takes value* $+2$ *on the same knot. In particular, any of these three invariants separates the* $1+1$ *rib knot from the unknit.* □

Further, consider the *fake weaver's knot* in $\mathbf{T}^2 \times \mathbf{R}^1$ given by the picture



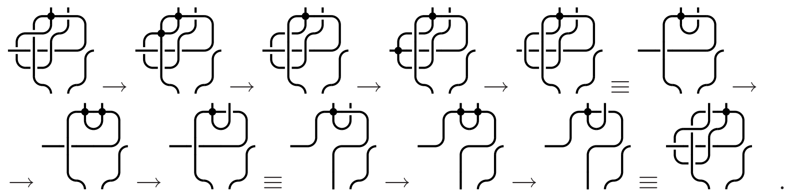
(9)

It can be reduced to the trivial knot by the sequence of two surgeries



(10)

Characteristic pairs of these surgeries are both equal to $((0,-1)(1,1))$, their signs are equal to $-$ and $+$ respectively. Therefore the initial knot (9) cannot be separated from the trivial knot by invariants of degree one. Now, let us try to separate these knots by degree two invariants. To do this, we connect two singular knots, occurring in (10) at surgery points, by a generic path inside the discriminant variety:



(11)

The first and the last surgeries in this path have crossed chord diagrams corresponding to the trefoil element $((0,-1)(0,0)(1,1))$; signs of both of these surgeries are equal to $-$. Two other surgeries have non-crossed chord diagrams, both with characteristic triplets $((0,-1)(0,0)(1,1))$, are useless for the calculation of degree two invariants. In the same way as in the previous subsection, this implies the following statement.

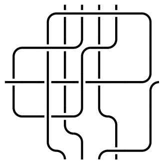
Proposition 7 *The second degree invariant defined by the crossed chord diagram and element* $((0,-1)(0,0)(1,1))$ *of the structure* $\clubsuit(\pi_1(\mathbf{T}^2))$ *takes value* $+2$ *on the knot (9). All second degree invariants corresponding to other trefoil elements or to non-crossed chord diagrams take zero value on the same knot.* $\square$

It is easy to calculate that this triplet $((0,-1)(0,0)(1,1))$ defines the same element of the trefoil structure $\clubsuit(\pi_1(\mathbf{T}^2))$ as (7) and (8). Thus, comparing propositions 6 and 7 we obtain the following statement.

Corollary 1 *1) The* $1+1$ *rib structure and the fake weaver's knot (9) are separated by the second degree invariants* I_Γ *defined by non-crossed chord diagrams and characteristic triplets* $((0,1)(1,-2)(0,1))$ *and* $((0,-1)(1,2)(0,-1))$.

2) These two structures are not separated by any other second degree invariants (including invariants $I_3^k(a,b)$ *of [5]).*

Now we consider a more complicated knot in $\mathbf{T}^2 \times \mathbf{R}^1$ obtained from (9) by the following operation. We cut our knot (9) at both of its points placed over the top/bottom margin of the quadrilateral chart in $\mathbf{T}^2 \times \mathbf{R}^1$, and replace these cutting points by loops such that 1) their projections to $\mathbf{T}^2$ are (almost) vertical segments in our picture, and 2) all crossing points of these vertical segments with the initial knot diagram are undercrosses only, see (12).



(12)

We can calculate the first and second degree invariants of this knot in exactly the same way as for the knot (9). The inserted strings do not participate in the surgeries; their only contribution is that all second coordinates in the formulae for characteristic triplets and trefoil elements become multiplied by two. Therefore we obtain the following statement.

Proposition 8 *The second degree invariant defined by the crossed chord diagram and the element* $((0,-2)(0,0)(1,2))$ *of the trefoil structure* $\clubsuit(\pi_1(T^2))$ *takes value* $+2$ *on the knot* (*12*). *All second degree invariants corresponding to other trefoil elements or to non-crossed chord diagrams take zero value on the same knot.* □

Corollary 2 *Knots (9) and (12) are separated by second degree invariants corresponding to different elements of the trefoil structure* $\clubsuit(\pi_1(\mathbf{T}^2))$*, although they are not separated by degree 2 invariants corresponding to non-crossed chord diagrams, as well as by the cumulative invariant corresponding to the Casson invariant.*

Proof. Comparing Propositions 7 and 8, it remains to show that the triplets $((0,-1)(0,0)(1,1))$ and $((0,-2)(0,0)(1,2))$ belong to different elements of the trefoil structure $\clubsuit(\pi_1(\mathbf{T}^2))$. But the subgroup in $\mathbf{Z}$ generated by the second coordinates of vectors a, b and $c \in \mathbf{Z}^2$ obviously is an invariant of the trefoil structure. □

In the next proposition we show that the entire system of invariants of all degrees in $T^2 \times \mathbf{R}^1$, whose weight systems are induced from weight systems in $\mathbf{R}^3$, cannot separate the knots (9) and (12). We need to be careful with the statement, because any weight system of degree k determines an invariant only up to adding the invariants of lower degrees, some of which can distinguish our knots.

Proposition 9 *For any weight system* W *in* $\mathbf{R}^3$ *there exists an invariant of knots in* $\mathbf{T}^2 \times \mathbf{R}^1$*, whose weight system is induced from* W *as described in the beginning of* §*2, but which takes equal values on the knots* (*9*) *and* (*12*).

Proof. Let I be an arbitrary invariant of knots in $\mathbf{R}^3$ (or, equivalently, in S^3) with the weight system W; such an invariant I exists according to the Kontsevich's theorem. The manifold $\mathbf{T}^2 \times \mathbf{R}^1$ can be identified with the complement of the Hopf link (i.e. two unknotted linked circles) $C_1 \sqcup C_2 \subset S^3$ in such a way that

1) any line $x \times \mathbf{R}^1$, $x \in \mathbf{T}^2$, tends to C_1 (respectively, to C_2) when the parameter in $\mathbf{R}^1$ tends to $+\infty$ (i.e., "to the reader" in our pictures) (respectively, to $-\infty$);

2) the horizontal (respectively, vertical) generator of $H_1(\mathbf{T}^2) \sim \mathbf{Z}^2$ in our pictures generates the kernel of the induced homomorphism $H_1(\mathbf{T}^2) \to H_1(S^3 \setminus C_2)$ (respectively, $H_1(\mathbf{T}^2) \to H_1(S^3 \setminus C_1)$).

The images of our knots (9) and (12) under this embedding $\mathbf{T}^2 \times \mathbf{R}^1 \to S^3$ are isotopic in S^3 (and even in $S^3 \setminus C_1$), therefore they cannot be separated by the invariant induced from I by this embedding. $\square$.

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