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INVARIANTS OF LINKS IN 3-MANIFOLDS AND SPLITTING PROBLEM OF TEXTILE STRUCTURES

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ABSTRACT
An infinite family of invariants of multicomponent links in 3-manifolds is introduced and used to prove the non-splitting and non-equivalence of textile structures.

Keywords: Finite type link invariant, textile structure, split links

1. Introduction

We introduce and exploit new invariants of links in three-dimensional orientable connected manifolds with $\pi_1 \neq 0$, first of all of the form $M^2 \times \mathbf{R}^1$.

In §3 we define a large family of finite type invariants of such links, generalizing those introduced in [11] and [12] for the case of knots.

In §4 we apply our invariants to the splitting problem for textile structures, i.e. double periodic 1-dimensional submanifolds in $\mathbf{R}^3$, which can also be considered as (compact) links in $\mathbf{T}^2 \times \mathbf{R}^1$. A n-component link in $M^2 \times \mathbf{R}^1$ is a *split link* if for some $k = 1, \ldots, n-1$ it is isotopic to a link whose k components lie in $M^2 \times \mathbf{R}^1_-$ and remaining $n-k$ ones in $M^2 \times \mathbf{R}^1_+$. The problem of detecting split links occurs naturally in the automatic search of new textile designs discussed in [9]. Morton and Grishanov [20] considered this problem using factorization of the multivariable Alexander polynomial as an indicator for the fabric to be decomposable into layers. However, they noticed that there may be fabrics which do not split into layers but their polynomials do factorize non-trivially.

In §4 we prove non-splitting of many standard textile structures. In §5 we use our invariants to obtain some other results on non-equivalence and non-triviality of well-known textile structures.

We always assume that the ambient manifold M^3 is connected.

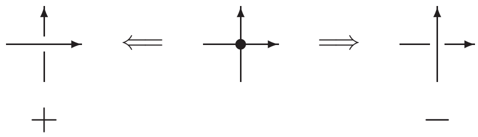


Figure 1: Resolutions of a transverse self-intersection

2. Finite type invariants of links in orientable 3-manifolds

We consider ordered links in orientable connected 3-manifolds, i.e. collections of smooth embeddings $S^1 \to M^3$ with non-intersecting images. An obvious invariant of such links is the collection of homotopy classes of all its components. Therefore we usually assume such a collection to be fixed, and consider the invariants distinguishing links within these fixed classes.

2.1. *General facts*

Finite type invariants of multi-component links in M^3 are defined in almost the same way as for the knots, see e.g. [14], [23], [26], and [11], with only a few natural modifications. In particular,

1. Let C_n be the disjoint collection of n oriented circles: $C_n \equiv S^1_1 \sqcup \ldots \sqcup S^1_n$. A n-component link in M^3 is a smooth embedding of C_n into M^3.

2. A self-intersection point $f(x) = f(y)$ of a map $f : C_n \to M^3$ is *transverse* if the images of $f'(x)$ and $f'(y)$ are linearly independent in the tangent space at this point. Such a point can be resolved in two different ways, see Fig. 1. If M^3 is oriented, then one of these resolutions, in an invariant way, can be called *positive*, and the other *negative*. We indicate these signs in Fig. 1 assuming that the orientation is specified by the frame of directions ((to the right margin); (to the top margin); (to us)).

3. Given an invariant I of n-component links in an orientable 3-manifold, and a map $f : C_n \to M^3$ having exactly k different transverse self-intersection points, the *k-th residue* of I at f is the alternate sum of values of our invariant at 2^k possible resolutions of this map into non-singular ones: these values at resolutions obtained by an even (respectively, odd) number of negative local resolutions should be summed up with sign $+$ (respectively, $-$). Our invariant is *of degree k* if its $(k+1)$-st residue is equal to zero at all singular links with exactly k transverse self-intersections. If I is an invariant of degree k but not $k-1$, then its *principal part* is its residue considered as a function on the space of singular maps with k simple double points. It is easy to see that this function is locally constant.

4. A k-chord diagram in the manifold C_n is a family of $2k$ different points in C_n,

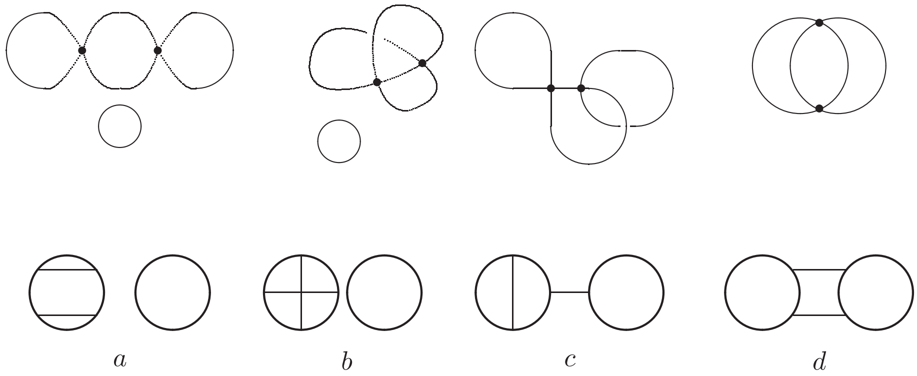


Figure 2: Degree two (self-)intersections of two circles and their chord diagrams

divided into k pairs. For example, all topologically different 2-chord diagrams in the 2-component collection C_2 are listed in the lower part of Fig. 2, where any pair of points is depicted by a thin segment connecting these points. A smooth map $f : C_n \to M^3$ *respects* some chord diagram if $f(x) = f(y)$ for any pair (x, y) of this diagram. For images of some maps, respecting 2-chord diagrams, see the upper part of Fig. 2. Two k-chord diagrams are *equivalent* if they can be transformed into one another by a diffeomorphism of C_n preserving all components and their orientations.

5. Given an equivalence class A of k-chord diagrams, the A-*routes* are equivalence classes of pairs $(f, \bar{A})$, consisting of a chord diagram $\bar{A} \in A$ and a map $f : C_n \to M^3$ respecting $\bar{A}$, under the equivalence relation generated by the following two classes of relations:

   a) If both f_1 and f_2 respect one and the same k-chord diagram $\bar{A}$ and are homotopic in the class of maps respecting $\bar{A}$, then the pairs $(f_1, \bar{A})$ and $(f_2, \bar{A})$ are equivalent.

   b) If f_1, f_2 can be reduced to one another by an orientation-preserving reparameterization of C_n, then the pair $(f_1, \bar{A}_1)$ is equivalent to $(f_2, \bar{A}_2)$ (in particular such a reparameterization should move $\bar{A}_1$ to $\bar{A}_2$).

6. A *degree k weight system* is a function on the space of all A-routes of degree k, satisfying some two conditions **1T** and **4T** motivated by the fact that these conditions surely are satisfied by the residues of degree k invariants of links $C_n \to M^3$ restricted to maps with exactly k intersection points. Namely, the condition **1T** follows from the consideration of neighborhoods of singular links $C_n \to M^3$ having $k - 1$ double points and a typical singular point with $f'(x) = 0$. It claims that our function takes zero value on any A-route of degree k such that

a) some (and then any) chord diagram of class A contains a chord, whose endpoints x_i, y_i belong to one and the same component of C_n and bound in it a segment containing no endpoints of other chords of this diagram, and

b) the loop $f : [x_i, y_i] \to M^3$ or $f : [y_i, x_i] \to M^3$, defined by the image of this segment under a map f from our A-route, is contractible in M^3.

The second series of restrictions (2.1), called **4T**-relations, is more complicated; it follows from the study of singular maps with $k-2$ double self-intersections and one triple point. Let us consider any such generic map, i.e. a map $f : C_n \to M^3$ with $k-2$ transverse double self-intersections, one triple point such that three derivatives of f at this point are linearly independent in T_*M^3, and no other self-intersection or singular points. The triple point of this map can be resolved in six different ways, splitting it into two double self-intersection points, see Fig. 3, so that f turns into singular knots with k self-intersections in six different ways. Let I be a degree k invariant, and $I(m)$, $m = 1, \ldots, 6$, be the value of its residue on the singular knot obtained from f as indicated in Fig. 3 in the sector labelled by m. Then necessarily

$$I(1) - I(4) = I(2) - I(5) = I(3) - I(6). \tag{2.1}$$

If at this triple point only one (respectively, two, three) components of C_n meet, then the chord diagrams corresponding to pictures 1, $\ldots$, 6 coincide outside of a neighborhood of some three points in C_n, and in this neighborhood they are as shown in Fig. 4 (respectively, 5, 6); in these pictures all circles are oriented in the same way, say counterclockwise.

The residue of any degree k invariant of links $C_n \to M^3$ should satisfy all these conditions **1T** and **4T**. For general M^3, these necessary conditions may be not sufficient: it can happen that they are satisfied for a function on the set of all A-routes of degree k in M^3 (in which case this function is by definition a *weight system*), but there is no degree k knot invariant with the principal part equal to this function; see [26]. In the case of $M^3 = M^2 \times \mathbf{R}^1$ the situation is much better.

Proposition 1 (see [2], [1]). *Suppose that $M^3 = M^2 \times \mathbf{R}^1$, M^2 an orientable surface (maybe with boundary), and I_k is a $\mathbf{R}$-valued function on the set of all A-routes of degree k in M^3. If I_k satisfies* **1T**- *and* **4T**-*relations, then there exists a $\mathbf{R}$-valued degree k invariant of knots in M^3, whose principal part coincides with this function I_k.* □

2.2. *Example: invariants of degree 1 of links in orientable 3-dimensional manifolds*

The degree 0 invariants of n-component links with ordered components are nothing else than the functions on the set of homotopy classes of maps $C_n \to M^3$. For

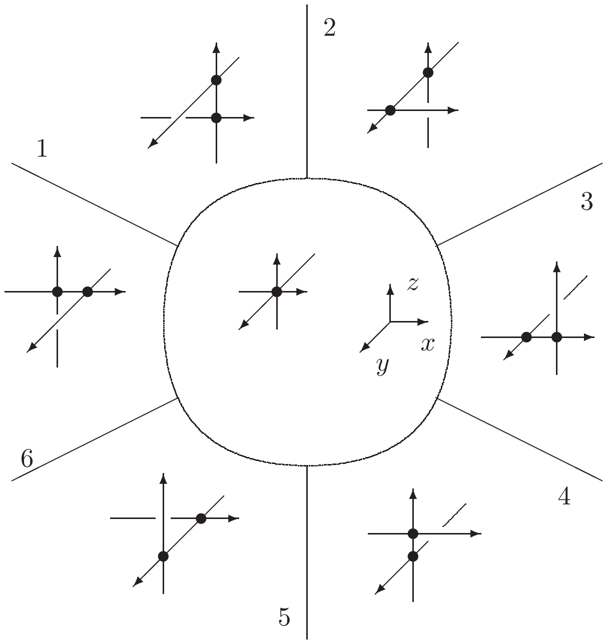


Figure 3: Resolutions of a triple point

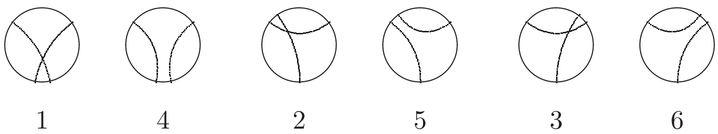


Figure 4: Chord diagrams for 3-term relations (a)

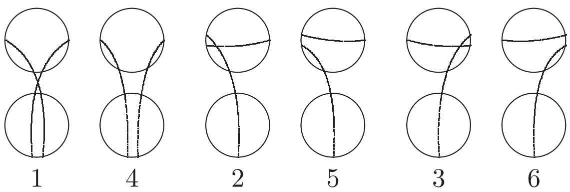


Figure 5: Chord diagrams for 3-term relations (b)

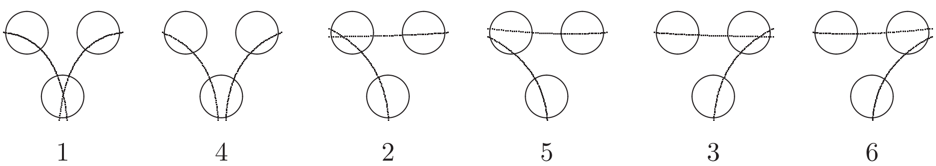


Figure 6: Chord diagrams for 3-term relations (c)

$n = 1$ this set is the set of conjugacy classes of $\pi_1(M^3)$, and for arbitrary n this set is just the n-th power of the latter.

There are two different groups of degree 1 invariants of links in M^3, generating together the space of all such invariants modulo degree 0 invariants.

The first group is composed by degree 1 invariants of components of the link, considered as particular knots, see [26]. It splits obviously into the direct sum of n groups, corresponding to these components. If M^3 is orientable then any of these n groups (reduced modulo constant invariants) is a subgroup G_0 of the group G freely generated by closed irreducible components of the discriminant subvariety in the space of smooth maps $S^1 \to M^3$ (i.e. the variety consisting of maps with self-intersections and singularities). Any such component is uniquely characterized by an unordered pair of non-unit elements (b, c) of the group $\pi_1(M^3)$ considered up to simultaneous conjugacy: $(b, c) \sim (b', c')$ if there is $s \in \pi_1(M^3)$ such that $b' = s^{-1} \cdot b \cdot s$, $c' = s^{-1} \cdot c \cdot s$. (In the special case when the group $\pi_1(M^3)$ is Abelian, these components are counted just by the unordered pairs of non-zero elements $b, c \in H_1(M^3)$.) Any such irreducible component defines a cycle of codimension 1 in the space $C^\infty(S^1, M^3)$. The intersection indices of such cycles with 1-homology classes in $C^\infty(S^1, M^3)$ are well-defined. By linearity we obtain also a homomorphism $G \to H^1(C^\infty(S^1, M^3))$. The subgroup $G_0 \subset G$ is the kernel of this map.

In a similar way, the second subgroup of degree 1 invariants is generated by generalized linking numbers of components of the link; it obviously splits into the sum of $\binom{n}{2}$ isomorphic groups corresponding to different pairs of components. Any of these groups is a subgroup $\mathcal{G}_0$ of the Abelian group $\mathcal{G}$ freely generated by ordered pairs of arbitrary (maybe unit) elements $\beta, \gamma \in \pi_1(M^3)$, considered up to the simultaneous conjugation: $(\beta, \gamma) \sim (\beta', \gamma')$, if $\beta = \sigma^{-1}\beta'\sigma, \gamma = \sigma^{-1}\gamma'\sigma$ for some $\sigma \in \pi_1(M^3)$. Again, every such pair defines an element of the group $H^1(C^\infty(C_2, M^3))$; by linearity we obtain a homomorphism $\mathcal{G} \to H^1(C^\infty(C_2, M^3))$. The subgroup $\mathcal{G}_0$ is the kernel of this homomorphism.

Both subgroups $G_0 \subset G$ and $\mathcal{G}_0 \subset \mathcal{G}$ can be proper: say, they are proper in the case $M^3 = S^2 \times S^1$. However, if $M^3 = M^2 \times \mathbf{R}^1$, M^2 a connected orientable surface, then all these 1-cohomology classes of components of the discriminant are trivial, so that $G_0 = G$, $\mathcal{G}_0 = \mathcal{G}$: this is a direct corollary of Proposition 1 but can also be deduced immediately from the (very simple) structure of the group $H^1(C^\infty(S^1, M^2))$.

For example, if $M^3 = \mathbf{T}^2 \times \mathbf{R}^1$ then for any homotopy class of maps $C_2 \to M^3$ the restriction of the group $\mathcal{G}$ to the space of links of this class is isomorphic to $\mathbf{Z}$.

For any $M^3 = M^2 \times \mathbf{R}^1$, M^2 orientable, there is a canonical epimorphism $\mathcal{G} \to \mathbf{Z}$: the linking number of two components. It can be defined as follows. Given two maps $f_1 : S^1 \to M^3$, $f_2 : S^1 \to M^3$, with $f_1(S^1) \cap f_2(S^1) = \emptyset$, we consider a generic homotopy of the map $(f_1, f_2) : (S^1 \sqcup S^1) \to M^2 \times \mathbf{R}^1$ deforming it into a pair of maps $(\tilde{f}_1, \tilde{f}_2)$ such that $\tilde{f}_1(S^1) \in M^2 \times \mathbf{R}^1_-$, $\tilde{f}_2(S^1) \in M^2 \times \mathbf{R}^1_+$. This homotopy of the map (f_1, f_2) crosses the discriminant variety several times. Counting all these intersections with signs according to Fig. 1 we obtain a number, which is

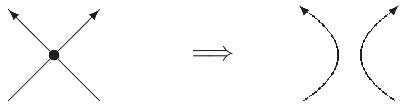


Figure 7: Natural resolution

Figure 8: Characteristic graphs for immersions of Fig. 2

independent on the choice of the homotopy.

Generally, the *linking number* $\mathrm{lk}(a,b)$ thus defined is not invariant under renumbering the components.

Proposition 2. $\mathrm{lk}(b,a) - \mathrm{lk}(a,b) = \langle a,b\rangle,$ where $\langle a,b\rangle$ is the intersection index of projections of a and b into M^2 with respect to the fixed orientation of M^2, inducing also the orientation of $M^2 \times \mathbf{R}^1$. $\square$

3. New link invariants

Definition 1. An immersion $C_n \to M^3$ is called *conventional*, if it has no other singularities than transverse self-intersections only. The *chord diagram of* a conventional immersion is the largest chord diagram respected by this immersion. The *natural resolution* of a transverse self-intersection of an immersion $C_n \to M^3$ is the local move shown in Fig. 7. The natural resolution $\tilde{\varphi}(C_n)$ of a conventionally immersed curve $\varphi(C_n)$ is the oriented smooth 1-submanifold in M^3 obtained from $\varphi(C_n)$ by the simultaneous natural resolution of all its intersection points. For any set of intersection points of $\varphi(C_n)$, the corresponding *partial resolution* is the (maybe singular) subvariety obtained by resolving only these intersections.

Theorem 1. Suppose that we have a conventional immersion $\varphi : C_n \to M^3$ with exactly k intersections and self-intersections, and the image of this immersion consists of t connected components. Then the natural resolution $\tilde{\varphi}(C_n)$ of this immersion consists of no more than $k + 2t - n$ embedded circles.

Proof. With our conventional immersion $\varphi : C_n \to M^3$ we associate a graph $\Gamma(\varphi)$, whose vertices correspond to components of the resolution $\tilde{\varphi}(C_n)$. The edges of this graph correspond to self-intersection points of φ and connect the vertices corresponding to components of this resolution participating in the right-hand part of the corresponding copy of Fig. 7; if these components coincide then we obtain a loop of the graph. E.g., for four immersions $C_2 \to M^3$ shown in Fig. 2 these graphs are as shown in Fig. 8. Further, it is sufficient to consider the case $t = 1$, so that this graph is connected and has k edges.

Lemma 1. If the graph $\Gamma(\varphi)$ is connected then it has at least $n-1$ independent cycles.

Proof of Lemma 1. Any component S^1_i of C_n defines a cycle in the graph $\Gamma(\varphi)$: it goes consecutively through the vertices corresponding to components, whose pieces come from pieces of our circle; any time when $\varphi(S^1_i)$ passes through a crossing point this cycle goes from a vertex to another (or maybe the same) one along the edge of $\Gamma(\varphi)$ corresponding to this crossing point; two different passages of one and the same crossing point by $\varphi(C_n)$ define opposite trips along the corresponding edge. For example, in four cases shown in Figs. 2 and 8 (counted from the left) these two cycles are as follows.

1) The trivial cycle inside the isolated point of the graph; a cycle going from one root of the A_3-graph to the other and back;

2) The trivial cycle inside the isolated point of the graph; the cycle $aba^{-1}b^{-1} \in \pi_1(8)$;

3) The cycle consisting of the unique loop of the graph; the cycle going from the root to the other vertex, then once along the loop, and coming back to the root;

4) Two equal cycles generating the fundamental group of the graph.

The homological sum of all these n cycles obtained from all initial circles S^1_i is equal to 0 because it goes exactly twice (in opposite directions) along any edge of $\Gamma(\varphi)$. If $\Gamma(\varphi)$ is connected then this is the unique non-trivial linear relation on these cycles. Indeed, if some linear combination of them defines the 0-cycle in the graph, then any two cycles, corresponding to two initial immersed circles, having at least one intersection point, should participate in this combination with equal coefficients. □

Proof of Theorem 1. Let x be the number of connected components of the resolved curve $\tilde{\varphi}(C_n)$. Then the Euler characteristic of $\Gamma(\varphi)$ is equal to $x-k$. By Lemma 1 it also does not exceed $1-(n-1)=2-n$; so we get $x \leq 2-n+k$. □

Obviously, the number of connected components of the resolution $\tilde{\varphi}(C_n)$ (φ conventional) depends only on the chord diagram of φ, but does not depend on M^3 at all. Therefore the following definition is valid.

Definition 2. A conventional immersion $\varphi : C_n \to M^3$ with exactly k self-intersection points, and the image $\varphi(C_n)$ consisting of t connected components, is called an M-*immersion*, if its natural resolution consists of exactly $k+2t-n$ circles. A chord diagram in C_n is an M-*diagram* if any conventional immersion with such diagram of intersections is an M-immersion.

Given a chord diagram in C_n, considered as a collection of n disjoint circles $\chi_i : S^1_i \to \mathbf{R}^2 \subset \mathbf{R}^3$, $i=1,\ldots,n$, $\chi_i(S^1_i) \cap \chi_j(S^1_j) = \emptyset$ for $i \neq j$, with chords which are smooth not intersecting paths connecting these circles in $\mathbf{R}^3$, the *canonical realization* of this chord diagram is the conventional immersion $C_n \to \mathbf{R}^3$ coinciding with the initial embeddings χ_i outside some small neighborhood of the union of endpoints of chords, and in these neighborhoods behaving as shown in the right-hand part of Fig. 9 (with arbitrary choice of undercrosses or overcrosses).

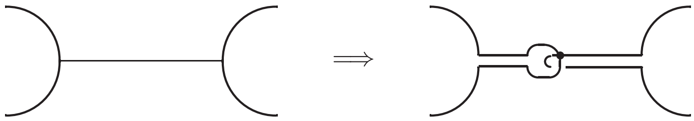
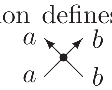
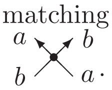
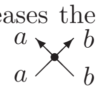
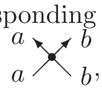


Figure 9: Canonical realization of a chord

By definition, the chord diagram of this canonical realization of a chord diagram is equivalent to the latter.

Proposition 3. If $\varphi : C_n \to M^3$ is a conventional M-immersion, then its partial resolution at an arbitrary set of intersection points also is an M-immersed curve in M^3.

Proof. It is sufficient to prove this statement for the partial resolution at only one intersection point like the left-hand part of Fig. 7. Four endpoints of this drawing are connected somehow by corresponding embedded components through the entire M^3. This connection defines some matching of these components, which can be described either by  or by . Suppose that the partial resolution $\varphi \mapsto \tilde{\varphi}$ at this crossing point increases the number t of connected components of $\varphi(\cdot)$. Then only the first possibility  can hold, so the number $\tilde{n}$ of immersed components of the partially resolved immersion $\tilde{\varphi}$ is equal to $n+1$, and $\tilde{k} = k-1$. In total, we have $\tilde{k} + 2\tilde{t} - \tilde{n} = k - 1 + 2t + 2 - n - 1 \equiv k + 2t - n$. By the definition of the M-immersion, the last number is equal to the number of components of the total resolution of $\varphi(C_n)$, and hence also of $\tilde{\varphi}(C_{n+1})$, so the characteristic equality of M-immersions is again satisfied.

If our partial resolution does not split the corresponding connected component of $\varphi(C_n)$, then the first matching type of endpoints, , is impossible: indeed, this matching implies that $\tilde{k} + 2\tilde{t} - \tilde{n} = k - 1 + 2t - n - 1 \equiv k + 2t - n - 2$, which contradicts Theorem 1. The second version of matching of endpoints gives us $\tilde{k} + 2\tilde{t} - \tilde{n} = k - 1 + 2t - n + 1 \equiv k + 2t - n$, i.e. the characteristic equality is again satisfied. $\square$

In cases $n = 1$ and $n = 2$ M-diagrams have especially easy characterizations.

Proposition 4 (see [11]**).** A chord diagram in C_1 is an M-diagram if and only if it does not have intersecting chords. Components of the normal resolution of any conventional M-immersion of S^1 correspond canonically to the pieces into which the chords of the corresponding M-diagram cut the disc bounded by its basic circle. $\square$

Proposition 5. A chord diagram in C_2 is an M-diagram if and only if it satisfies the following three conditions:

a) for any component of C_2, all chords connecting the points of this component do not cross each other, so that the union of these chords is an M-diagram in C_1;

b) given an arbitrary conventional immersion $\varphi : C_2 \to \mathbf{R}^3$ with this chord diagram, its partial resolution which smoothes the self-intersection points of components of C_2 only (and not the intersection points of distinct components) breaks $\varphi(C_2)$ in the union of embedded oriented circles, at most two of which can intersect each other; these two circles (if they exist) occur from different components of C_2;

c) the cyclic order of intersection points of these two circles with respect to the original orientation of one of these circles coincides with the cyclic order of the same intersection points defined by the *reversed* orientation of the other circle.

Proof. It is easy to see that these three conditions are sufficient; let us prove that any M-immersion of C_2 should satisfy them.

(a) Suppose that there are two crossing chords, connecting some four points of one and the same component of C_2. The partial resolution of $\varphi(C_2)$ at these two points is illustrated by the second from the left pictures in Figs. 2 and 8; namely, it reduces the number k by 2 and does not change n and t. Then by Theorem 1, applied to this partially resolved curve, the number of components of the (total) normal resolution of $\varphi(C_2)$ does not exceed $k + 2t - n - 2$, in particular φ is not an M-immersion. This implies condition a).

(b) Suppose that there is a chord of our chord diagram, connecting two points of one component (say S^1_1) of C_2 in such a way that both parts into which the endpoints of this chord cut this component S^1_1 are connected by chords with some points of S^1_2. The partial resolution of $\varphi(C_2)$ at the crossing point corresponding to this chord reduces k by 1, increases n by 1, and does not change t. Therefore by Theorem 1 the number of components of the normal resolution of $\varphi(C_2)$ does not exceed $k + 2t - n - 2$, so that φ is not an M-immersion. This implies condition b).

(c) Suppose that both conditions a) and b) are satisfied, then the partial resolution of $\varphi(C_2)$ along all chords connecting the points of one and the same component reduces it to an immersion of two circles having only mutual intersections, but no self-intersections. It remains to prove that if such an immersion of C_2 is an M-immersion then condition c) is satisfied. Let us prove it. The partial resolution of such an M-immersion φ at an arbitrary crossing point turns φ into an M-immersion $\tilde{\varphi} : S^1 \to M^3$ with $k - 1$ self-intersections. Moreover, this circle S^1 is split by the pre-images of our resolved crossing point into two parts, and any chord of the chord diagram of $\tilde{\varphi}$ joins a point of one of these parts to a point of the other. There is only one topological picture of the system of such *non-intersecting* chords of $\tilde{\varphi}$; recovering from it the chord diagram of φ we obtain the picture described in condition c). $\square$

Definition 3. For any natural k and n, $k \geq n - 1$, and any conventional M-immersion $\varphi : C_n \to M^3$, having exactly k intersection and self-intersection points and connected image $\varphi(C_n)$, the *passport* of this immersion is the unordered collection of classes in $H_1(M^3)$ of all $k + 2 - n$ naturally oriented components of the normal resolution of $\varphi(C_n)$.

For any such numbers k, n, and unordered collection $\Gamma = (\gamma_1, \ldots, \gamma_{k+2-n})$ of

$k+2-n$ elements of $H_1(M^3)$, we denote by I_Γ the function on the set of conventional immersions $C_n \to M^3$ with k (self-)intersections and connected image, which takes value 1 on all M-immersions whose passports are equal to Γ, and value 0 on all other conventional immersions (in particular on immersions that are not M-immersions or whose images are not connected).

Theorem 2. If M^3 is orientable, then for any natural n, k and any collection $\Gamma = (\gamma_1, \dots, \gamma_{k+2-n})$, $\gamma_i \in H_1(M^3)$, such that all classes γ_i are not equal to 0, the function I_Γ is a weight system of degree k.

Proof. If some conventional immersion $\varphi : C_n \to M^3$ has a self-intersection point $\varphi(x) = \varphi(y)$ of one of components, such as in the definition of the **1T**-relation (i.e. the loop $\varphi([x, y])$ is contractible in M^3 and does not contain other intersection points), then one of components of the normal resolution $\tilde{\varphi}(C_n)$ is 0-homologous. In particular, the homology class of this component cannot coincide with any of elements γ_i. Therefore the function I_Γ takes zero value on φ, i.e. satisfies the condition **1T**; let us prove that **4T**-condition also is satisfied. Consider an immersion $\varphi : C_n \to M^3$ with $k-2$ conventional double points and unique generic triple point as in the middle part of Fig. 3. Six deformations of this triple point into two double points, shown in this picture, turn φ into conventional immersions φ_i, $i = 1, \dots, 6$, coinciding outside of our picture. Let $\check{\varphi}$ and $\check{\varphi}_i, i = 1, \dots, 6$ be the partial resolutions of the immersions φ and φ_i at all $k-2$ double points outside of our picture. Any such resolution $\check{\varphi}_i$ splits the variety $\varphi(C_n)$ into several (say, p) irreducible components, of which some $q = 1, 2$ or 3 meet at the two intersection points in question; all the remaining components do not depend on i. Depending on q, we have one of three topological types of the connection of the endpoints of singular links of Fig. 3 through the entire M^3, see Figs. 10, 11 and 12. By Theorem 1, the numbers p and q satisfy the inequality $p \le k + 2q - n - 2$. If this inequality is strict, then none of our six immersions φ_i is an M-immersion, and I_Γ takes zero value on all of them. Suppose now that we have the equality $p = k + 2q - n - 2$, and consider our three cases.

q=1 (see Fig. 10). In this case exactly three of our six immersions $\check{\varphi}_i$ are M-immersions, namely, they correspond to $i = 4, 5, 6$. The collection of homology classes of all three components of the resolution of this immersion is the same for all these i. Thus **4T**-relation is satisfied.

q=2 (see Fig. 11). Denote by α, β the homology classes of two loops of the self-intersecting component of $\check{\varphi}$ participating in this surgery, and by ω the class of the other component. Then for $i = 1, 2, 3$ the collection of homology classes of components of the resolution $\check{\varphi}_i$ will be equal to $(\alpha + \omega; \beta)$, while for $i = 4, 5, 6$ it will be equal to $(\alpha, \beta + \omega)$. So all the minuends (respectively, subtrahends) in the **4T**-relation (2.1), see also Fig. 5, are equal to each other, and this relation is again satisfied.

q=3 (see Fig. 12). In this case resolutions of all 6 immersions $\varphi_i : C_3 \to M^3$ provide one and the same homology class equal to the sum of three participating

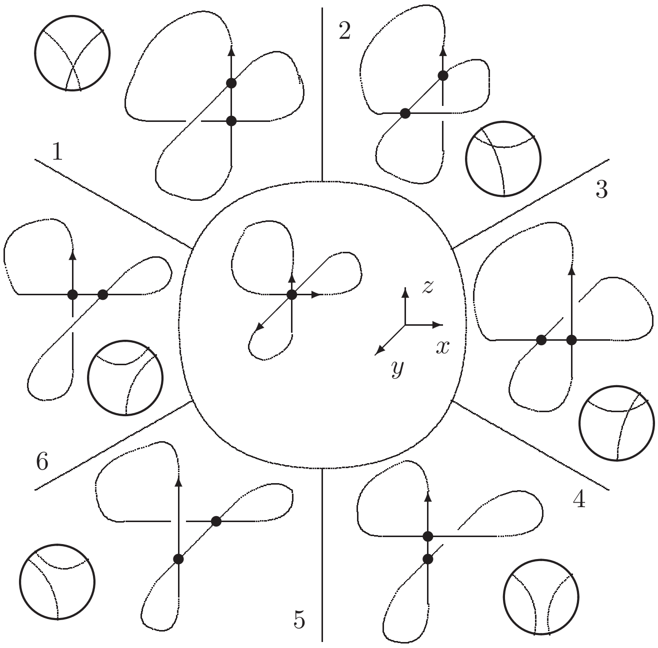


Figure 10: Inner topology of curves $\check{\varphi}_i$ in the case $q = 1$...

components, so that all three numbers compared in the **4T**-relation (2.1) are equal to 0. □

Corollary 1. If $M^3 = M^2 \times \mathbf{R}^1$, M^2 orientable, and Γ is as in the condition of Theorem 2, then there exists a degree k invariant of n-component links in M^3, whose principal part is equal to I_Γ. All these invariants, corresponding to different collections Γ, are linearly independent.

Indeed, by Proposition 1 any weight system in $M^2 \times \mathbf{R}^1$, M^2 orientable, can be integrated to a link invariant. The independence of these invariants follows from the independence of their principal parts. □

Remark 1. In the particular case of $n = 1$, $k = 1$ this invariant was in other terms introduced by T. Fiedler [6]; for $n = 1$, $k = 2$ these invariants are a minor extension of his invariant introduced in [7]. The general case of $n = 1$ was introduced and studied in [11], [12].

Remark 2. The construction of invariants I_Γ relies very much on the orientations of components of the link. For any subset $\sigma \subset \{1, \dots, n\}$, an additional invariant of ordered oriented links $\varphi : C_n \to M^3$ can be defined as the same invariant I_Γ applied to the link φ^σ, coinciding with φ up to orientation-reversing diffeomorphisms of components S^1_j, $j \in \sigma$, of C_n.

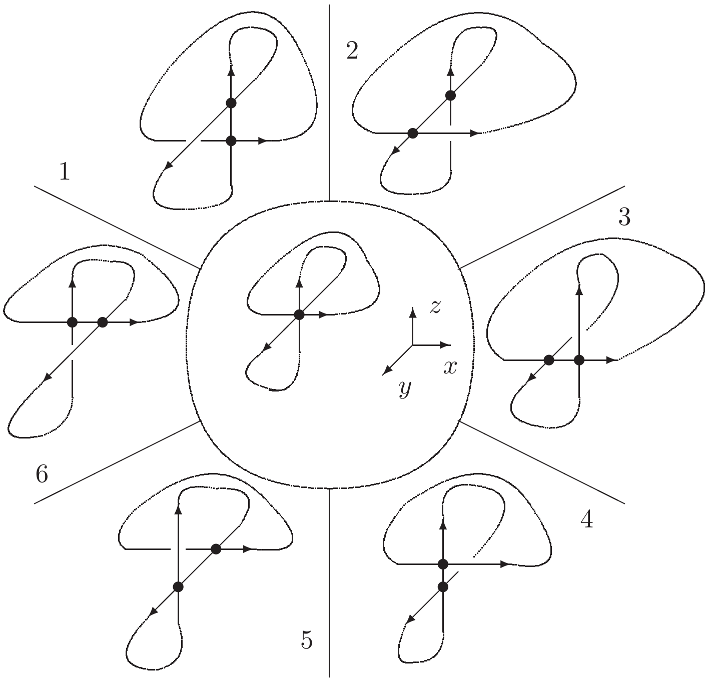


Figure 11: ...in the case of two components, $q = 2$...

Remark 3. For some different constructions of link invariants in manifolds $M^2 \times \mathbf{R}^1$ and closely related virtual link invariants, mainly developing the Kauffman and Khovanov constructions, see [18], [19] and references therein, and also [10].

4. Splitting of textile structures

Definition 4. Given a link in $M^2 \times \mathbf{R}^1$, its *splitting* is an isotopy of $M^2 \times \mathbf{R}^1$ moving a non-empty subset of components of this link into the domain $M^2 \times \mathbf{R}^1_+$, and the complementary, also non-empty, subset to $M^2 \times \mathbf{R}^1_-$.

4.1. *A criterion for links to admit splittings*

Proposition 6. An ordered link in $M^2 \times \mathbf{R}^1$, consisting of components $s_1, \dots, s_n$, admits a splitting which moves the components $s_1, \dots, s_i$ to $M^2 \times \mathbf{R}^1_+$ and $s_{i+1}, \dots, s_n$ to $M^2 \times \mathbf{R}^1_-$ if and only if this link it isotopic to one, obtained from it by the simple shift of $s_1 \cup \ldots \cup s_i$ along the axis $\mathbf{R}^1$ into the domain $M^2 \times \mathbf{R}^1_+$, and the simultaneous shift of $s_{i+1} \cup \ldots \cup s_n$ along the same axis into $M^2 \times \mathbf{R}^1_-$.

Proof. The assertion “if” is obvious, let us prove “only if”. Let J be the isotopy realizing such a splitting. By Theorem VIII.1.3 of [13], if our link in $M^2 \times \mathbf{R}^1$ can

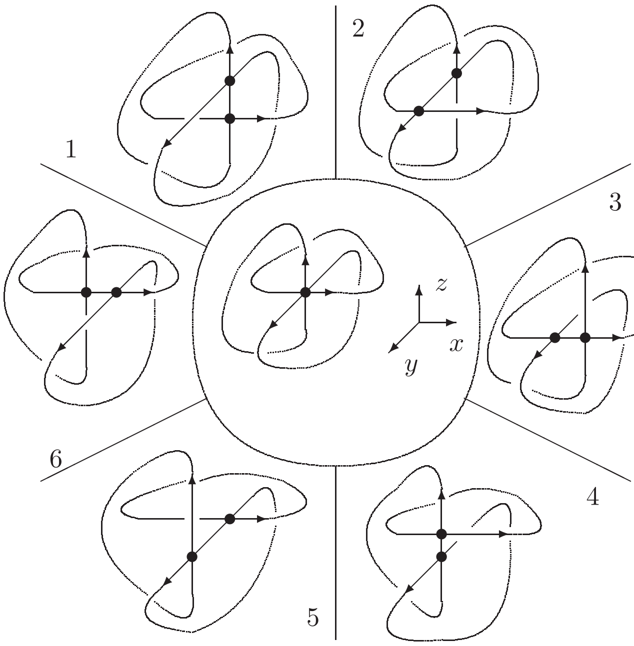


Figure 12: ... and in the case of three components, $q = 3$

be moved to some other link by an isotopy of $M^2 \times \mathbf{R}^1$, then the same move can be realized by an isotopy which is constant outside some compact layer $M^2 \times [-T, T]$. Therefore we will assume that our isotopy J satisfies this condition, also we can and will assume that the projection of the initial link $s = s_1 \cup \ldots \cup s_n$ to the factor $\mathbf{R}^1$ lies in the interval $(-T, T)$. Let $s_+ \equiv s_1 \cup \ldots \cup s_i$ and $s_- \equiv s_{i+1} \cup \ldots \cup s_n$. Denote by s'_+ and s'_- the results of our isotopy J applied to s_+ and s_- respectively, in particular $s'_+ \subset M^2 \times \mathbf{R}^1_+, s'_- \subset M^2 \times \mathbf{R}^1_-$. Let $m_- \in \mathbf{R}^1$ be the maximal point in the projection of the link s'_- to this factor $\mathbf{R}^1$; by the above assumption $m_- \in (-T, 0)$. Consider an arbitrary smooth isotopy λ of $\mathbf{R}^1$, $\lambda : \mathbf{R}^1 \times [0, 1] \to \mathbf{R}^1$, constant in $\mathbf{R}^1_+$ and shifting the interval $(-\infty, m_-]$ isometrically to $(-\infty, -2T]$; let Id be the constant isotopy in M^2. Then the isotopy $\mathrm{Id} \times \lambda$ in $M^2 \times \mathbf{R}^1$ moves s'_- into $M^2 \times (-\infty, -2T]$. The isotopy $J^{-1} \circ (\mathrm{Id} \times \lambda) \circ J$ moves s_+ identically to itself and s_- into $M^2 \times (-\infty, -2T)$. Then the desired isotopy is defined by the composition

$$J^{-1} \circ (\mathrm{Id} \times (2T + m_-)) \circ J^{-1} \circ (\mathrm{Id} \times \lambda) \circ J,$$

where $2T + m_-$ is the shift of $\mathbf{R}^1$ defined by adding this number. $\square$

4.2. *Several examples*

All our examples come from the theory of textile structures, i.e. one-dimensional

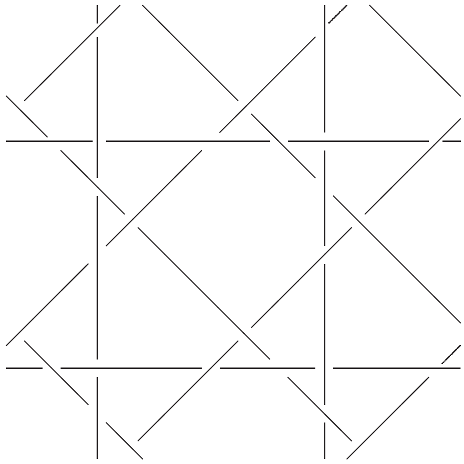
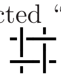


Figure 13: Four-axial woven fabric with plain weave and homogeneous net

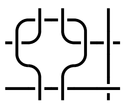
submanifolds in $\mathbf{R}^3$ invariant under some lattice $\mathbf{Z}^2$ of translations, see [10], [11]. All such structures can be obviously represented by links in the corresponding *basic cells*, i.e. the fundamental domains $\mathbf{T}^2 \times \mathbf{R}^1$ of this action.

So, in all these examples we have $M^2 = \mathbf{T}^2$. In particular, for any two homotopy classes of maps $S^1 \to M^2 \times \mathbf{R}^1$ the group of generalized linking numbers of knots from these classes (see §2.2) is one-dimensional. Drawing the pictures, we shall assume that the factor $\mathbf{R}^1$ of $\mathbf{T}^2 \times \mathbf{R}^1$ is directed "to us".

For many fabrics, e.g. for the simplest web  or for the four-axial web of Fig. 13 (see [8]), their non-splitting can be proved already by means of linking numbers of distinct components. We consider below several more complicated cases.

4.2.1. *Single jersey + weft + warp interlaced*

The basic cell of this fabric is as follows:

 . (4.2)

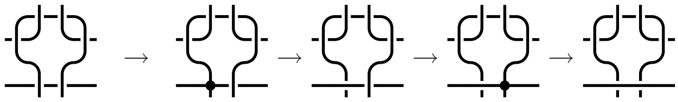
This web consists of three families of threads, two of which have one and the same asymptotic direction (horizontal in our picture), and the third a different (vertical) one. One of the first two families defines a non-trivial knot in the basic cell (we denote this knot by s_1), and the second splits into parallel lines, represented in the basic cell by unique horizontal line s_2. Let s_3 be the vertical component representing in the same cell the third family. These conditions define completely these families of threads, and also the choice of the basic cell. We orient s_3 from the bottom to the top, while s_1 and s_2 from the left to the right. Thus the class of s_3 in $H_1(\mathbf{T}^2 \times \mathbf{R}^1)$

is equal to $(0,1)$, and that of s_1 and s_2 is equal to $(1,0)$.

Proposition 7. The link (4.2) cannot be split in any way.

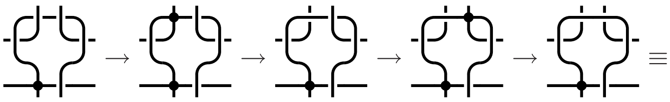
Proof. The linking numbers of s_3 with other two components (see §2.2) exclude all possibilities of splittings, except for two, in both of which s_1 is moved down and s_2 up. Thus it is enough to consider the sublink composed by s_1 and s_2 only and to prove that it is not isotopic to the similar link obtained from it by shifting s_1 to $\mathbf{T}^2 \times \mathbf{R}^1_-$ and s_2 to $\mathbf{T}^2 \times \mathbf{R}^1_+$.

To do it, let us consider the simplest path connecting these two links:

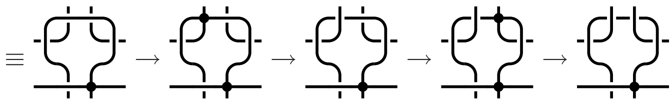


(4.3)

This path crosses the discriminant variety twice, each time at points corresponding to singular links with one intersection of different threads. These crossings have opposite signs with respect to the canonical transversal orientation of the discriminant variety, see Fig. 1; therefore all degree 1 invariants take equal values on the first and the last links of this path. To calculate the difference of degree 2 invariants on these links, let us connect the two surgeries occurred in the path (4.3) by a generic path inside the discriminant variety:



(4.4)

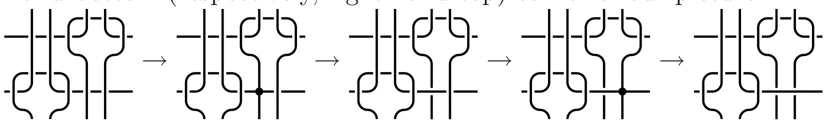


(4.5)

We get four surgeries of degree 2, with signs equal to $+, -, -$, and $+$ respectively. At any of them we have an oriented singular link intrinsically homeomorphic to the upper part of Fig. 2c. Namely, the corresponding component $\tilde{s}_1$, obtained by deforming s_1, has one self-intersection (and is therefore homeomorphic to the figure 8), and the component $\tilde{s}_2$ is non-singular and has one intersection with $\tilde{s}_1$ at one of its non-singular points. The passport (see p. 10) of such a figure consists of a) the homology class of the non-intersected loop of the 8-like component, and b) the sum of homology classes of the other loop and of $s_2 \sim (1,0)$. Namely, the passports of these four surgeries are equal to $((1,-1)(1,1))$, $((0,-1)(2,1))$, $((0,1)(2,-1))$, and $((1,1)(1,-1))$ respectively. Thus, all degree 2 invariants, whose principal parts I_Γ are described in Theorem 2 and correspond to passports $((1,-1)(1,1))$, $((0,-1)(2,1))$, or $((0,1)(2,-1))$, take non-zero values on our path and prove that the first and the last links in (4.3) are not isotopic; such invariants do exist by Corollary 1. $\square$

4.2.2. *Plain lock-knit*

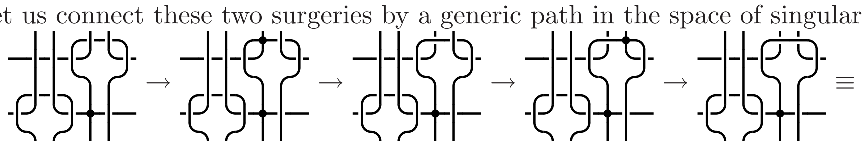
The basic link of this textile structure is drawn in the left-hand part of (4.6). It consists of two components, both with homology class $(1, 0)$. Denote by s_1 (respectively, s_2) the component, whose crossing points are concentrated in the left-hand bottom (respectively, right-hand top) corner of our picture.



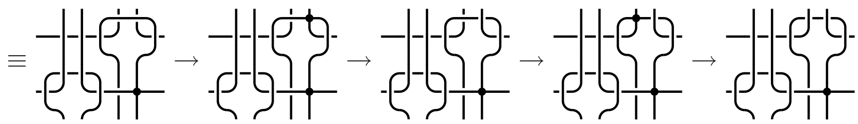
(4.6)

Proposition 8. The basic link of the plain lock-knit cannot be split in any way.

Proof. Suppose that this link can be split in such a way that s_1 goes up. By Proposition 6 this implies that our link is isotopic to the link shown in the right-hand part of (4.6). It remains to separate these two links by some link invariant. To do it, we connect our two links by the simplest possible path (4.6). It contains two surgeries of one and the same topological type but with opposite signs, so the degree one invariants do not prevent the required isotopy. To apply degree 2 invariants, let us connect these two surgeries by a generic path in the space of singular links:



(4.7)



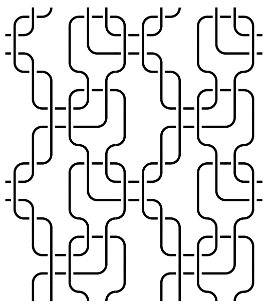
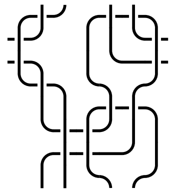
(4.8)

We have four second degree surgeries with signs $+, -, +, -$, and passports equal to $((1, -1)(1, 1))$, $((0, -1)(2, 1))$, $((1, 1)(1, -1))$, $((0, 1)(2, -1))$ respectively. Therefore any of degree 2 invariants, whose principal parts are described in Theorem 2 and correspond to pairs of homology classes $((1, -1)(1, 1))$, $((0, -1)(2, 1))$, or $((0, 1)(2, -1))$, proves that our web does not admit a splitting moving s_1 up and s_2 down. The proof of the similar statement concerning moving s_2 up and s_1 down is absolutely symmetric to the above. $\square$

Remark 4. The link in $\mathbf{T}^2 \times \mathbf{R}^1$ studied in this section *is not* a minimal representative of this textile structure: the latter admits one additional symmetry turning it into a one-component link in some reduced cell.

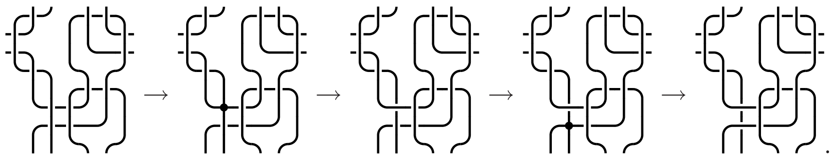
4.2.3. *Tricot opened + weft inlay tied-up*

This textile structure (4.9) contains two components, both with homology class $(0, 1) \in H_1(\mathbf{T}^2, \mathbf{Z})$: the (highly knotted) component s_1 representing the *tricot* structure (see [11]) and the unknotted component s_2.

 with elementary cell  (4.9)

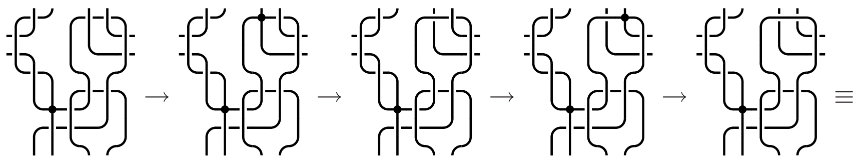
Proposition 9. The link (4.9) cannot be split in such a way that the component s_2 moves up and s_1 down.

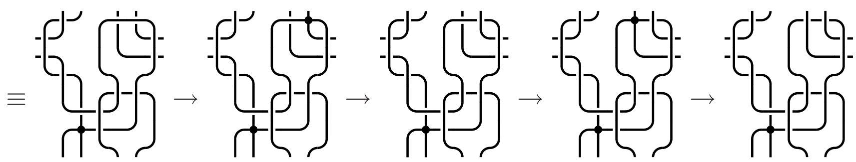
Proof. By Proposition 6, such a splitting exists if and only if the first and the last links in (4.10) are isotopic to one another; indeed, the right-hand link of (4.10) obviously can be deformed in the required way.

 (4.10)

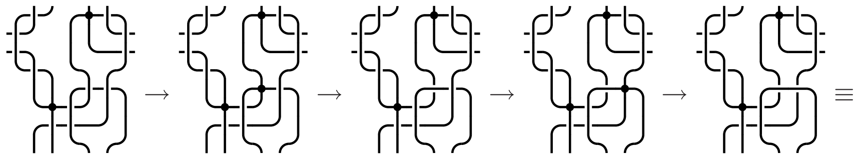
Let us separate these links by our invariants. The sequence (4.10) represents a path between these links in the space of (possibly singular) links in $\mathbf{T}^2 \times \mathbf{R}^1$. This path passes two surgeries, at any of which our two components, s_1 and s_2, intersect each other. The signs of these surgeries are equal to $-$ and $+$. Therefore the generalized linking numbers of these components (and all other degree 1 invariants) do not prevent the existence of the isotopy between our links.

To use degree 2 invariants, let us connect these two surgeries by a generic path inside the discriminant:

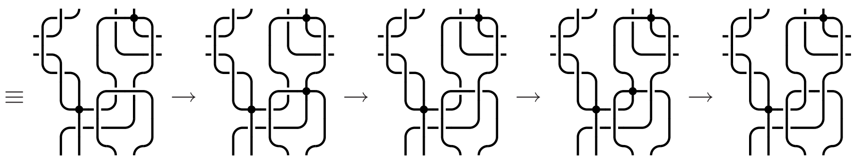
 (4.11)

 (4.12)

Here we have four surgeries of second order, with signs $-, +, -$, and $+$ respectively. At any of them the intrinsic topology of the singular link is as in Fig. 2c; namely, the component s_1 has a self-intersection (thus being homeomorphic to figure 8) and a single intersection with s_2. The homology classes of two loops of the component s_1 are as follows (with the class of the loop intersected by component s_2 underlined): $\underline{(-1,0)}(1,1)$, $\underline{(-1,0)}(1,1)$, $(-1,0)\underline{(1,1)}$, and $(-1,0)\underline{(1,1)}$ respectively. So, they are matched into pairs represented by homologically (and in fact also homotopically) equivalent figures in $\mathbf{T}^2 \times \mathbf{R}^1$, with opposite signs of corresponding singularities. Indeed, we can connect two surgeries in (4.11) by a generic path inside the self-intersection set of the discriminant:

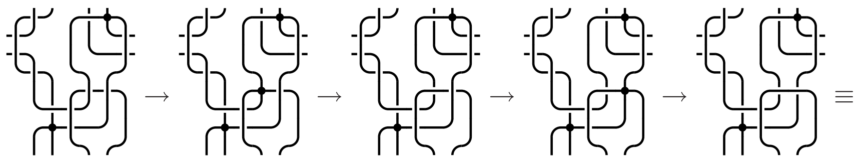


(4.13)

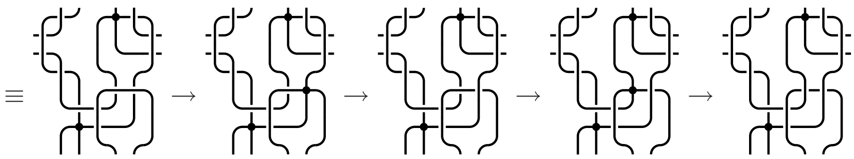


(4.14)

and equivalently for (4.12):



(4.15)



(4.16)

This implies that all possible degree 2 invariants take zero value on the path (4.10). Let us calculate the values of degree 3 invariants on it. All eight degree 3 surgeries in (4.13)–(4.16) represent singular links, with one component having two self-intersections, and the other (non-singular) component intersecting it once.

By Proposition 5 those of these surgeries, for which the chord diagram of self-intersections of the first component is crossed, do not define M-immersions, and hence are not interesting for us from the point of view of the invariants I_Γ. Namely, they are the first surgeries in (4.13) and (4.14), and the second surgeries in (4.15) and (4.16). The remaining four surgeries of degree 3 have non-crossed chord diagrams of these components, the homology classes of three loops of this component being as follows (with the class of the loop intersected by the second component underlined, and the sign of the surgery put before the entire collection).

The second surgery in (4.13): $+\underline{(-1,0)}(0,1)(1,0)$.

The second surgery in (4.14): $+\underline{(-1,1)}(0,-1)(1,1)$.

The first surgery in (4.15): $-(-1,1)(0,-1)\underline{(1,1)}$.

The first surgery in (4.16): $-(-1,0)(0,1)\underline{(1,0)}$.

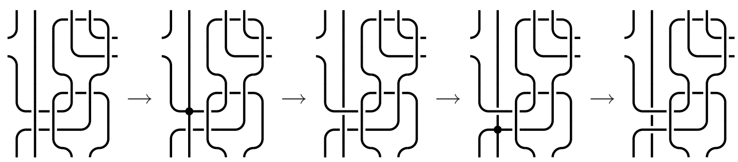
Finally, the passports of these four surgeries are as follows:

$$(-1,1)(0,1)(1,0);\ (-1,2)(0,-1)(1,1);\ (-1,1)(0,-1)(1,2);\ (-1,0)(0,1)(1,1). \qquad (4.17)$$

Therefore any degree three invariant, whose principal part coincides with either of four degree three weight systems I_Γ, corresponding to these four multi-indices Γ, takes value 1 or -1 on the path (4.10); thus any such invariant (existing by Corollary 1) proves Proposition 9. □

Proposition 10. The link (4.9) cannot be split in such a way that the component s_1 moves up, and s_2 moves down.

Proof. By Proposition 6, the existence of such a splitting is equivalent to the existence of an isotopy of the link (4.9) to the right-hand picture of the following line:

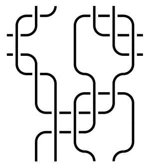


(4.18)

The left-hand picture in this line is obviously isotopic to the right-hand one in (4.10). Therefore a path in the space of immersions $C_2 \to \mathbf{T}^2 \times \mathbf{R}^1$, connecting (4.9) to the right-hand picture of (4.18), can be composed of paths (4.10), (4.18), and (non-singular) isotopies. So, the value of any of our invariants on this composition path is equal to the sum of its values on these two paths. For (4.10) the calculation of such values of some degree three invariants is already done. For the path (4.18) this calculation consists in considering literally the same deformations of the component s_1, as in (4.11)–(4.16), while the component s_2 remains the same as previously in the whole domain where these deformations are performed. Therefore all values of degree 3 invariants on our path (4.18) are the same as for (4.10); respectively, the values of four invariants (4.17) on the path (4.10)+(4.18) are just twice the values of the same invariants on the path (4.10). □

4.2.4. *Tricot closed + weft inlay tied-up*

Its elementary cell is as follows:

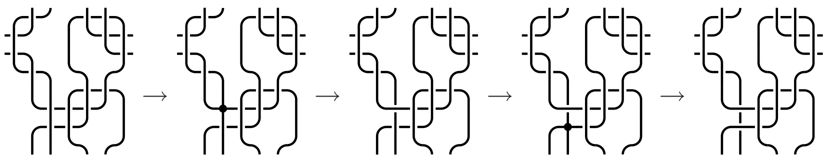


(4.19)

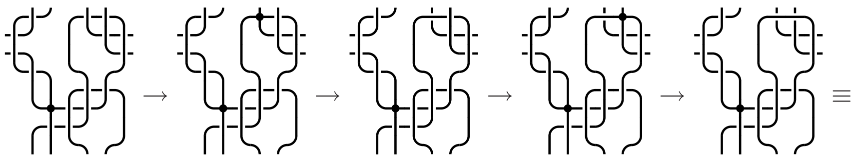
Again, denote by s_1 its knotted component, and by s_2 the unknotted one.

Proposition 11. The link (4.19) cannot be split in any way.

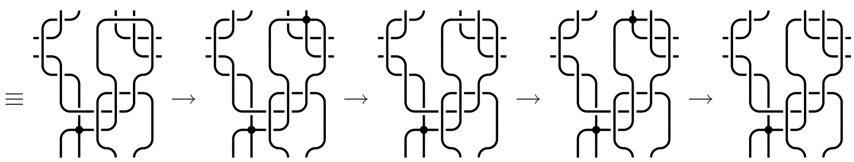
Proof. We will prove only the fact that it cannot be split so that the component s_1 moves down; the other possibility can be reduced to it in exactly the same way as Proposition 10 was reduced to Proposition 9. The first steps of the proof repeat those for Proposition 9. Namely, the following surgeries (4.20), (4.21)–(4.22), (4.23)–(4.26) are exact analogues of (4.10), (4.11)–(4.12), (4.13)–(4.16).



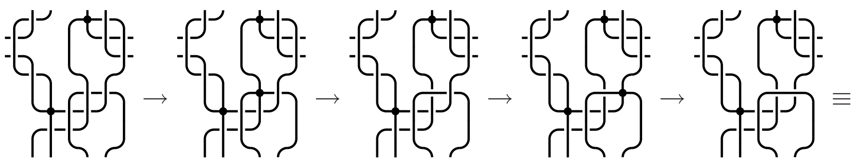
(4.20)



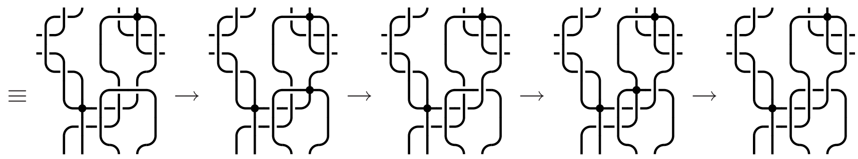
(4.21)



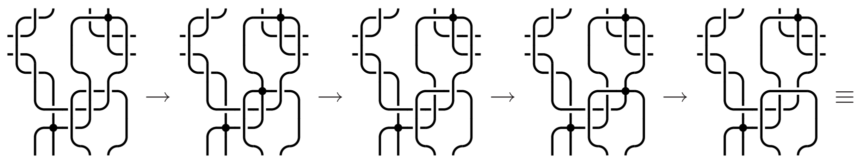
(4.22)



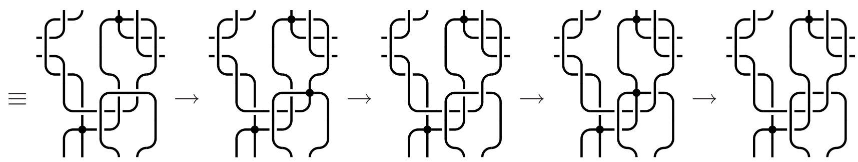
(4.23)



(4.24)



(4.25)



(4.26)

Four of eight degree 3 surgeries in (4.23)–(4.26) do not define M-diagrams: namely, it are the first surgeries in (4.23), (4.24) and the second ones in (4.25) and (4.26). The passports of remaining four surgeries are respectively as follows:

$$(-1,2)(0,-1)(1,1);\ (-1,1)(0,1)(1,0);\ (-1,0)(0,1)(1,1);\ (-1,1)(0,-1)(1,2). \qquad (4.27)$$

Therefore any of the corresponding four invariants I_Γ separates two marginal links of (4.20), the right-hand one of which is a split link. □

On the other hand, we obtain the following negative result.

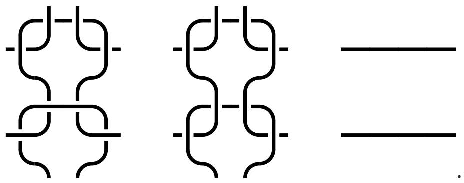
Proposition 12. The links (4.9) and (4.19) cannot be separated by any invariants I_Γ of degree up to 3.

Proof. A path connecting these two links can be composed of the path (4.10), a path connecting the right-hand parts of (4.10) and (4.20), and the path (4.20) passed in inverse direction. The collections of invariants (4.27) and (4.17) coincide up to some permutation; it is easy to calculate that equal invariants from these lists take opposite values on the first and the third pieces (4.10) and $(4.20)^{-1}$ of our path. The endpoints of the second piece are split links, whose unknotted components can be moved far up. Therefore this piece of our path can be realized by a path moving somehow only one (knotted) component of the link. It was proved in [11] that the endpoints of this path cannot be separated by any invariants of degree 3. □

5. Some other applications

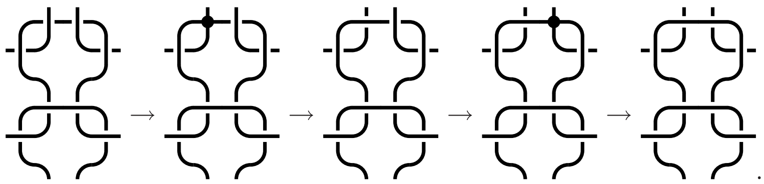
In this section we prove non-equivalence of several standard multi-component links, first of all of the purl and vertically doubled jersey.

Proposition 13. No two of the following three links are equivalent to one another (see (5.28)) : the *purl*, two copies of elementary cells of *jersey* placed one over the other, and the trivial 2-component link of the same homotopy class.



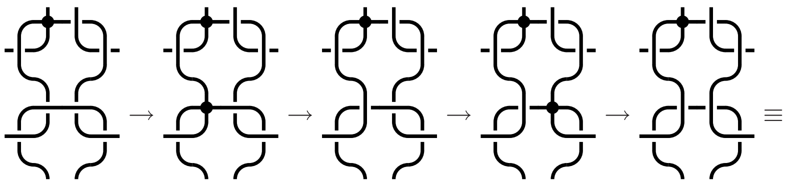
(5.28)

Proof. We orient all components of these links from the left to the right, so that their homology classes are equal to $(1,0) \in H_1(\mathbf{T}^2 \times \mathbf{R}^1)$. First, we separate the first and the third links in (5.28) by our invariants. To do this, we connect these links by the path

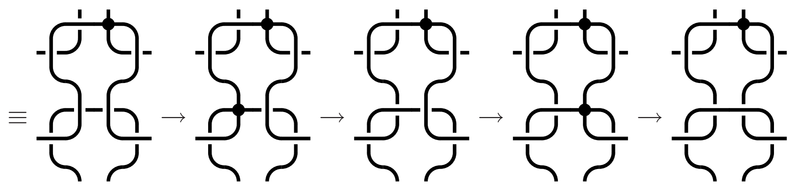


(5.29)

This path contains two surgeries, both corresponding to intersections of different components; their signs are equal to + and − respectively. Thus our links are not separated by degree 1 invariants. To calculate the difference of values of degree 2 invariants on these surgeries (and hence also on the original two links) we connect these surgeries by a generic path inside the discriminant:



(5.30)



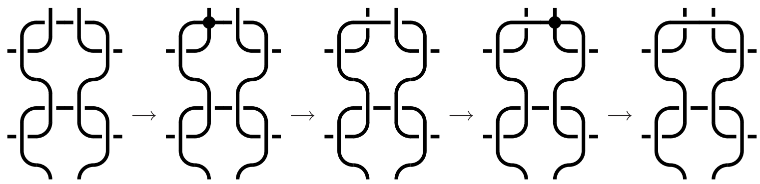
(5.31)

This path contains four degree 2 surgeries, with signs equal to $-, +, +$, and $-$ respectively. All of them have no self-intersecting components. Their common combinatorial type is as shown in Fig. 2d, and their passports are equal to

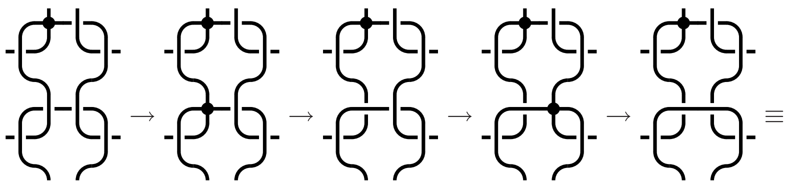
$$((2,-1)(0,1)), ((1,-1)(1,1)), ((1,1)(1,-1)), \text{ and } ((0,-1)(2,1)) \tag{5.32}$$

respectively; in particular they define three independent link invariants I_Γ. Thus any of these three invariants separates the first and the last links in (5.28).

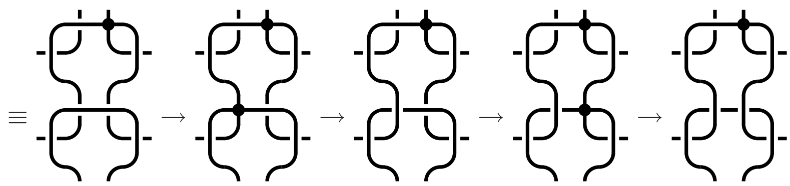
Now, let us compare the second link in (5.28) with the trivial one. The following formulas (5.33)–(5.35) are exact analogues of (5.29)–(5.31).



(5.33)



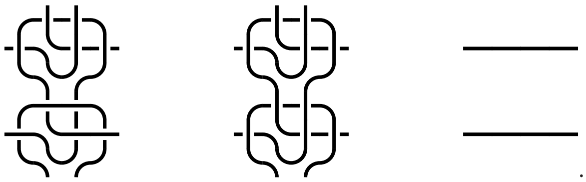
(5.34)



(5.35)

There are four surgeries of second degree in (5.34–5.35), with signs $+, -, -$, and $+$ respectively. The passports of all these four surgeries are respectively the same as in (5.32), because the projections of corresponding singular knots to $\mathbf{T}^2$ coincide; on the other hand, their indices are opposite. Therefore, the same three invariants I_Γ separate also the middle link of (5.28) from the other two. $\square$

Proposition 14. None two of the following two-component links in $\mathbf{T}^2 \times \mathbf{R}^1$ are equivalent to one another (see (5.36)) : the *purl with closed loops*; two neighboring strings of *jersey with closed loops*, and the trivial two-component link within the same homotopy class. Moreover, all these links can be separated by degree 2 invariants I_Γ.



(5.36)

Proof repeats that of Proposition 13. □

On the other hand, the degree 2 invariants I_Γ proving both these propositions take equal values on both left-hand (respectively, middle) links in (5.28) and (5.36), in particular cannot separate these pairs of links.

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