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GAUSS DIAGRAM INVARIANTS OF LINKS IN $M^2 \times \mathbf{R}^1$

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ABSTRACT

We construct an infinite series of invariants of Fiedler type (i.e. composed of oriented arrow diagrams arranged by elements of $H_1(M^3)$) for multicomponent links in $M^3 = M^2 \times \mathbf{R}^1$, M^2 orientable with $\pi_1(M^2) \neq \{\mathbf{1}\}$.

Keywords: Link invariant, combinatorial formula

1. Introduction

In [5, 6] we have constructed an infinite series of weight systems (i.e. potential knot invariants) of all degrees for knots in orientable 3-manifolds M^3. By [1, 2] in the case $M^3 = M^2 \times \mathbf{R}^1$ all weight systems can be integrated to finite type knot invariants of the same degree, so we obtain also a series of such invariants (up to the choice of integration). These weight systems (and corresponding knot invariants) of degree d are indexed by collections of $d+1$ non-zero elements of the group $H_1(M^3)$. For degrees 1 and 2 these invariants in $M^2 \times \mathbf{R}^1$ essentially coincide with ones introduced by T. Fiedler [3, 4] in the terms of arrow diagrams arranged by homology classes of M^2. In fact (see [7]) the indexation by non-zero 1-homology classes can be replaced by the indexation by homotopy classes of non-contractible maps $S^1 \to M^3$; this gives us a more ample family of weight systems and knot invariants if $\pi_1(M^3)$ is not commutative.

In [7] we give explicit combinatorial formulas, similar to the Fiedler's construction, for all knot invariants from [5, 6] indexed by collections of 1-homotopy classes, no more than two of which can coincide. In [8] we extend the construction from [5, 6] to the case of multi-component links in orientable 3-manifolds: for any unordered collection of $d+2-n$ elements of $H_1(M^3) \setminus \{0\}$ we construct a degree d weight system of n-component links (with ordered components) in M^3. Again, the 1-homology classes in this construction can be almost literally replaced by free

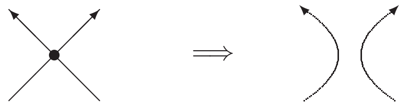


Figure 1: Natural resolution

1-homotopy classes. In the case $M^3 = M^2 \times \mathbf{R}^1$ this also gives us (up to the integration of weight systems) invariants of such links in M^3.

Below, we complete the commutative diagram of papers $\begin{array}{ccc} [6] & \rightarrow & [7] \\ \downarrow & & \downarrow \\ [8] & \rightarrow & ? \end{array}$, i.e., in the case $M^3 = M^2 \times \mathbf{R}^1$, M^2 orientable, we construct explicit combinatorial formulas for link invariants from [8] defined by any collections of different 1-homology (and in the case of ≤ 2-component links even 1-homotopy) classes.

Also, we prove an easy general characterization for the class of chord M-diagrams, essential for the construction of [8], and give new proofs of Theorem 1 and Proposition 5 from [8].

We always assume that the factor M^2 of the ambient manifold $M^2 \times \mathbf{R}^1$ is a connected and oriented smooth surface.

2. Main construction and statement

2.1. *H-diagrams, h-diagrams and universal chains*

We use notions and notation from [8], in particular the following ones.

Definition 1. Denote by C_n the ordered collection of n oriented circles: $C_n \equiv S^1_1 \sqcup \ldots \sqcup S^1_n$. An *n-component link* in M^3 is a smooth embedding of the manifold C_n into M^3.

A *k-chord diagram* over C_n is any graph obtained from C_n by adding k segments (called *chords*) connecting some points of C_n, such that all their $2k$ endpoints in C_n are pairwise distinct. A smooth map $F : C_n \to M^3$ *respects* some chord diagram if $F(x) = F(y)$ for the endpoints x, y of any its chord. Obviously, the set $F(C_n)$ is connected if and only if some chord diagram over C_n respected by F is a connected graph.

Two k-chord diagrams are *equivalent* if they can be transformed one into another by a diffeomorphism of C_n preserving all components and their orientations.

An intersection point $F(x) = F(y)$, $x \neq y \in C_n$, of the curve $F(C_n)$ is *transverse* if the derivatives $F'(x)$ and $F'(y)$ are linearly independent in $T_{F(x)}M^3$. Given a smooth map $F : C_n \to M^3$ with k different transverse intersection points and no other more complicated singular points of $F(C_n)$, the *natural resolution* of the subvariety $F(C_n) \subset M^3$ is the oriented curve in M^3 obtained from $F(C_n)$ by the surgery shown in Fig. 1 at all intersection points.

For any such map F, the number of connected components of the natural resolution of $F(C_n)$ is determined by the equivalence class of the k-chord diagram respected by F. Denote by $m(G)$ this number defined by the k-chord diagram G.

Theorem 1 (see [8]). If the k-chord diagram G over C_n is a connected graph, then $m(G) \leq k - n + 2$.

Proof. Any chord diagram G over C_n can be embedded into a compact surface $K(G)$, obtained from G by gluing some $n + m(G)$ discs according to some maps of their boundary circles to G, see [9]. Namely, the boundaries of first n of these discs should be mapped homeomorphically to the components of C_n. The boundary of any of the remaining $m(G)$ discs starts from any regular point of a component of C_n, goes along this *oriented* component of C_n to the nearest endpoint of a chord, moves along this chord to the other component of C_n, goes along this component (again respecting its orientation) to the next endpoint of a chord, and so on until the return to the starting point. It is easy to see that the union of all these added discs is homeomorphic to a closed surface, which can be oriented in such a way that the initial orientation of all components of C_n coincides with the boundary orientation of the first n discs bounded by these components. The number of connected components of $K(G)$ is equal to that of the graph G. The Euler characteristic of $K(G)$ is equal to $n + m(G) - k$. If G is connected, then this number cannot exceed 2, and theorem is proved. $\square$

Definition 2. A k-*arrow diagram* is a k-chord diagram with some orientation of all its chords. A connected k-chord (or arrow) diagram over C_n satisfying the equality $m(G) = k - n + 2$ is called an M-*diagram*.

In all our drawings of chord M-diagrams we shall assume that the plane of the drawings coincides with the sphere $K(G)$ with one point removed; the components of C_n bounding the discs in $K(G) \setminus G$ not containing the removed point will be oriented counterclockwise, and the "exterior" component surrounding the entire picture will be oriented clockwise.

Let us fix once and forever the sphere S^2 with n disjoint ordered discs $D_1, \dots, D_n$ in it, let $\mathbf{D} \equiv D_1 \sqcup \dots \sqcup D_n$.

Proposition 1. 1. A connected k-chord diagram G over C_n is an M-diagram if and only if the graph G can be embedded into S^2 in such a way that the image of any component S^1_i of C_n bounds the disc D_i, its orientation induced by this embedding from S^1_i coincides with the boundary orientation induced from that of D_i, and these discs D_i do not contain images of chords.

2. All embeddings of a M-diagram to $(S^2, \mathbf{D})$, satisfying these conditions, are isotopic in the class of embeddings sending any S^1_i to ∂D_i.

3. The equivalence classes of k-chord (respectively, arrow) M-diagrams over C_n are in a natural one-to-one correspondence with isotopy classes of embedded non-oriented (respectively, oriented) graphs in $S^2 \setminus \mathbf{D}$ with k edges and $k - n + 2$ vertices, such that all n discs of $\mathbf{D}$ are separated in S^2 from one another by these graphs.

Proof. The first assertion follows from the construction of the surface $K(G)$. The isotopy uniqueness of the embedding (statement 2) can be proved easily by induction over the number n of components of C_n. The correspondence from statement 3 is just the duality of embedded graphs: with any M-diagram G canonically embedded into $(S^2, \mathbf{D})$ we associate a graph in $S^2 \setminus \mathbf{D}$ having exactly one vertex in any connected component of $S^2 \setminus (\mathbf{D} \cup G)$ and exactly one edge crossing any chord of G; if this chord is oriented then also the corresponding edge inherits an obvious orientation. □

Definition 3. A *marking* of an M-diagram G is any labelling of all $k - n + 2$ connected components of the complement of G in $S^2 \setminus \mathbf{D}$ (or, equivalently, of the vertices of the dual graph) by the numbers $1, \dots, k - n + 2$.

Given a marked M-diagram, the *natural orientation* of its chord separating two different components of the complement of G in $S^2 \setminus \mathbf{D}$ is such one that the component with the smaller number lies to the left of this chord with respect to this orientation. If the chord is approached by one and the same component from both sides, then it necessarily connects two different discs D_i, D_j, and its natural orientation is defined as that from the disc with smaller number to that with the greater one.

Equivalence of marked M-diagrams is any isotopy in the class of such diagrams in $(S^2, \mathbf{D})$ preserving the numbers of components of their complement in $S^2 \setminus \mathbf{D}$.

The *universal* (n, k)*-chain* is the formal sum of marked k-chord M-diagrams in $(C_n, \mathbf{D})$, taken one from each equivalence class of such diagrams, and supplied with the natural orientation of all their chords (so that the summands of this chain are some marked arrow diagrams).

Given an oriented connected surface M^2, we denote by $h_1(M^2)$ the set of homotopy classes of maps $S^1 \to M^2$; it naturally coincides with the set of conjugacy classes of the group $\pi_1(M^2)$.

An *H-diagram* (respectively, *h-diagram*) of type (n, k) in M^2 is any k-arrow M-diagram over C_n, with all $k - n + 2$ components of its complement in $S^2 \setminus \mathbf{D}$ labelled by some elements of the group $H_1(M^2)$ (respectively, of the set $h_1(M^2)$).

Given any ordered collection $\Gamma = (\gamma_1, \dots, \gamma_{k-n+2})$ of $k - n + 2$ pairwise different non-zero elements $\gamma_i \in H_1(M^2) \setminus \{0\}$ (respectively, of non-contractible elements $\gamma_i \in h_1(M^2) \setminus \{\mathbf{1}\}$), the chain Φ_Γ is defined as the formal sum of H-diagrams (respectively, h-diagrams) of type (n, k), obtained from the universal (n, k)-chain by replacing any labelling index $i \in \{1, \dots, k - n + 2\}$ by the corresponding class $\gamma_i \in H_1(M^2)$ (respectively, $\gamma_i \in h_1(M^2)$).

Example 1. If $n = 1$, then $k - n + 2 = k + 1$ and $S^2 \setminus \mathbf{D}$ is homeomorphic to $\mathbf{R}^1$, so the k-chord M-diagrams over C_n are dual to the planar trees, and the universal (n, k)-chain is equal (via the duality correspondence from statement 3 of Proposition 1) to the formal sum of all equivalence classes of naturally oriented planar trees with $k + 1$ vertices labelled by the numbers $1, \dots, k + 1$. The first three such chains (with $k = 1, 2$ and 3) are shown in Figs. 1, 2 and 3 of [7].

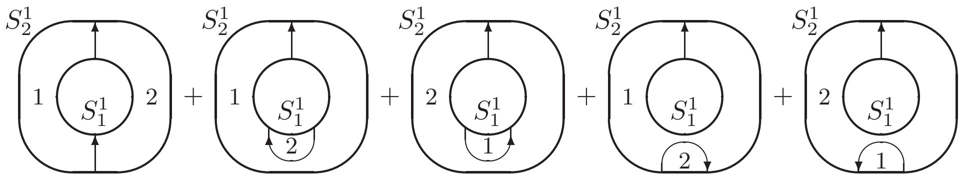


Figure 2: The universal (2,2)-chain

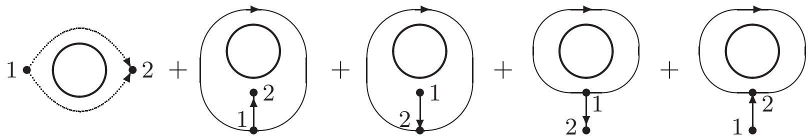


Figure 3: The universal (2,2)-chain (graph presentation)

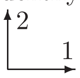
For $(n,k) = (2,1)$ the universal chain consists of the unique arrow in S^2 directed from D_1 to D_2; the corresponding invariants of 2-component links in M^2 (which we shall define below) are the generalized linking numbers discussed in §2.2 of [8].

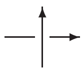
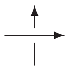
The first non-trivial new case is that of $(n,k) = (2,2)$. In this case the universal (n,k)-chain is shown in Fig. 2, its dual graph presentation is given in Fig. 3.

2.2. *H-diagrams as functions on generic links*

This subsection is an almost literal translation of §2.1 of [7] to the case of multi-component links, see also [10], [4].

We distinguish the source manifold C'_n of the links $F : C'_n \to M^2 \times \mathbf{R}^1$ from the basis C_n of chord and arrow diagrams, although both of them are equal to the disjoint union of n ordered circles.

Let M^2 be a connected oriented 2-dimensional manifold. Let $F : C'_n \to M^2 \times \mathbf{R}^1$ be a link in $M^2 \times \mathbf{R}^1$, generic with respect to the standard projection $p : M^2 \times \mathbf{R}^1 \to M^2$, i.e. the composition $p \circ F$ is an immersion with transverse crossing points only. In our pictures, we identify (a piece of) the oriented factor M^2 with the "blackboard plane", oriented by 

, and the factor $\mathbf{R}^1$ with the orthogonal line oriented "to us". Recall that the *local writhe* of a crossing point looking like  (respectively, ) is equal to -1 (respectively, $+1$).

Let G be an H- or h-diagram of type (n,k), i.e. a k-arrow M-diagram over C_n, with $k - n + 2$ domains of its complement in $K(G) \setminus \mathbf{D}$ labelled by elements $\gamma_i \in H_1(M^2) \setminus \{0\}$ (respectively, by elements $\gamma_i \in h_1(M^2) \setminus \{\mathbf{1}\}$), $i = 1, 2, \ldots, k - n + 2$.

Definition 4 (cf. [3], [10], [4]). A *representation* of the diagram G in the generic link $F : C'_n \to M^2 \times \mathbf{R}^1$ is any homeomorphism $r : C_n \to C'_n$, preserving the numbers and orientations of all components, such that

a) for any chord of G, connecting some points $x, y \in C_n$ and oriented from y to x, the images of x and y under the map $p \circ F \circ r : C_n \to M^2$ coincide, and moreover the point $F \circ r(x) \in M^2 \times \mathbf{R}^1$ lies above $F \circ r(y)$ in the sense of the standard orientation of the factor $\mathbf{R}^1$; in particular for any component of $K(G) \setminus (G \cup \mathbf{D})$, the class in $H_1(M^2)$ (and even in $h_1(M^2)$) of the boundary of this component under the map $p \circ F \circ r$ is well-defined;

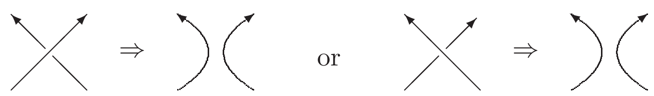
b) for any component of $K(G) \setminus (G \cup \mathbf{D})$, the latter class in $H_1(M^2)$ (or in $h_1(M^2)$) is equal to the element γ_i marking this component.

The *sign* of such a representation is equal to the product of local writhes of our link over all k crossing points of $p \circ F(C'_n)$ corresponding to the chords of G via this representation.

Two representations of G in one and the same link F are *equivalent* if they coincide on all endpoints of chords. Obviously, equivalent representations have equal signs, so the *sign of* any *equivalence class of representations of G in F* is well-defined.

The *value* $G(F)$ of the H- or h-diagram G on the generic link F is equal to the sum of signs over all equivalence classes of representations of G in F. The value on F of a formal linear combination of H- or h-diagrams (like Φ_Γ) is defined by linearity.

Given a representation of the arrow M-diagram G in a link, the homology and homotopy classes of boundaries of components of $K(G)$ can be realized as follows: we replace in the diagram of the link all crossing points corresponding to arrows of this diagram by the rule



(2.1)

and consider the homology and homotopy classes of oriented curves into which these surgeries split our link.

Theorem 2. For any ordered collection $\Gamma = \{\gamma_1, \dots, \gamma_{k-n+2}\}$ of $k-n+2$ pairwise different elements of $H_1(M^2) \setminus \{0\}$, the value of the corresponding chain Φ_Γ is an invariant of links in $M^2 \times \mathbf{R}^1$. In the case of ≤ 2-component links, the similar statement holds, in which instead of elements γ_i of the set $H_1(M^2) \setminus \{0\}$ we can take arbitrary different elements of the set $h_1(M^2) \setminus \{\mathbf{1}\}$ of homotopy classes of non-contractible continuous maps $S^1 \to M^2$.

If $n = 1$ and $k = 2$, then the invariants Φ_Γ defined in this theorem coincide with some invariants I_3^K from Theorem 2.10 of [4], for $n = k = 1$ they coincide with invariants from [3]; for $n = 1$ and arbitrary natural k they were introduced in [7].

3. Proof of Theorem 2

We need to prove that the value $\Phi_\Gamma(F)$ of the sum Φ_Γ is invariant under all Reidemeister moves of the generic link F, cf. [3], [7].

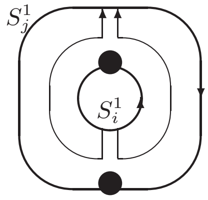


Figure 4: Arrow diagram for the second Reidemeister move with coinciding directions

First move. The crossing point arising/perishing at this move cannot contribute to the calculation of any function Φ_Γ because the smoothing (2.1) at such a point provides a contractible loop in M^2.

Second move. Let us consider separately the contributions to the sums $\Phi_\Gamma(F)$ provided by the representations in which only one (I) or both (II) crossing points arising/perishing at this move are counted. In the case (I) the proof is exactly the same as in [7], see also [3]: to any representation of this type of an H- or h-diagram, sending an arrow to some crossing point in the standard picture of this move, there corresponds another representation of exactly the same diagram, sending the same arrow in the other crossing point and having the opposite sign.

In the case (II), if the projections to M^2 of branches of the link at the instant of surgery are tangent with opposite directions, then the simultaneous splitting (2.1) at both these crossing points provides a contractible component of the split link. Therefore the summands of the chain Φ_Γ cannot have such representations.

In the remaining case (II) with coinciding directions, we need to consider two subcases: two local branches of the link participating in the move can belong to one and the same or to two different components of C_n. In the first subcase, the pre-images of these two crossing points are the alternating pairs of points in this component of C_n, therefore these arrows cannot participate simultaneously in an M-diagram, since otherwise two corresponding arrows of the H- or h-diagram G would intersect in $K(G)$.

In the second subcase, the represented diagram $G \subset K(G)$ can have only the topological form shown in Fig. 4 or the same with both arrows reversed; here the black circles can cover other arrows or/and components of G. However, each of these two arrows separates the same two different domains of $K(G) \setminus \mathbf{D}$, hence by the definition of the natural orientation exactly one of them (depending on the order of these domains) should be oriented in the opposite direction.

Third move. As in [7], we need to consider six different cases (any of which has a further subdivision into eight subcases related with our new situation). Namely, there are (up to reflections) exactly two different arrangements of orientations of participating branches of a link: Δ and ∇ shown respectively in Figs. 5 and 6. Fortunately, the second of them obviously is a composition of the first one and two

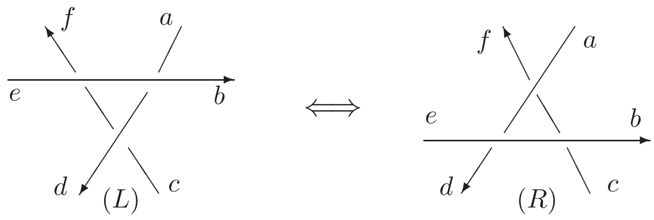


Figure 5: Third Reidemeister move Δ

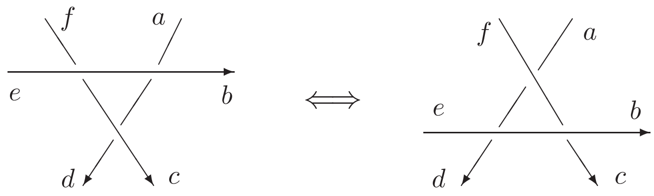


Figure 6: Third Reidemeister move ∇

second Reidemeister moves, therefore it is enough to consider the moves of type Δ only. Also, we need to consider and compare the representations of diagrams G in which exactly one, two or three different chords correspond to the crossing points shown in our picture. This number of chords is indicated by the lower index I, II or III under the notation Δ. Further, we distinguish the cases when three local branches of the link, participating in the picture of the move, belong 1) to one and the same, 2) to exactly two, or 3) to three different global components of C'_n. These numbers will be indicated by the upper indices over the notation Δ. Finally, the cases $\Delta^{(2)}$ should be subdivided into six subcases depending on the choice of the local branch that belongs to a component of C'_n other than that for other two branches, and also on whether the number of this distinguished component is smaller (m) or bigger (p) than that of the other component. Depending on the position of this selected branch in Fig. 5, the corresponding signs $-$, $\backslash$ and $/$ will occur in the notation of the case; so, a typical case to be considered is denoted like $\Delta^{(2)}_{II}(-; p)$ or $\Delta^{(2)}_{III}(\backslash; m)$.

All cases $\Delta^{(\cdot)}_I$ can be considered in exactly the same way as in [7]: in these cases the move does not affect the set of corresponding representations.

All eight cases $\Delta^{(1)}_{III}$, $\Delta^{(2)}_{III}(\cdot\ ;\ \cdot)$, and $\Delta^{(3)}_{III}$ are void, because the simultaneous splitting (2.1) at all three crossing points of either part of Fig. 5 provides a contractible component of the curve.

So, it remains to consider only the (different versions of) the case Δ_{II}.

3.1. *The case* $\Delta^{(1)}_{II}$

Consider a link $F : C'_n \to M^2 \times \mathbf{R}^1$, some component of which experiences the move shown in Fig. 5. The six endpoints of the left- or right-hand part of

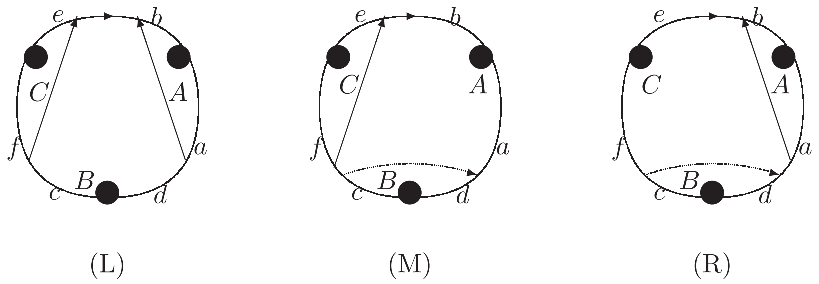


Figure 7: Arrow diagrams for $\Delta^{(1)}_{II}$

this figure have a natural pairing defined by the continuation of this component through the exterior of this picture. There is essentially unique possibility for this pairing: $(a-b)$, $(c-d)$ and $(e-f)$ (the other possibility can be reduced to this one by the permutation of parts of our figure, reversion of the orientation of the factor $\mathbf{R}^1$ and a rotation of the plane). Suppose that the H- or h-diagram G is a summand of the chain Φ_Γ, and we have a representation of it in our link F, sending one of components of C_n to the knot shown in Fig. 5(L) or 5(R), such that exactly two arrows go into crossing points of this picture. If the pairing of endpoints of this picture is as above, then such a representation can exist only for Fig. 5(L): indeed, any two chords corresponding to the crossing points of Fig. 5(R) intersect one another in the disc bounded by the corresponding component of C_n.

Suppose first that the arrows of our representation correspond to two upper crossing points of Fig. 5(L). Then our arrow diagram G (embedded into the corresponding sphere $K(G)$) looks as is shown in Fig. 7(L); here the black circles can cover additional chords and/or components of C_n. Consider the resolution (2.1) of our link at all crossing points corresponding to the arrows of G by this representation. Denote by A (respectively, B, respectively, C) the 1-homology (or 1-homotopy) class of the component of this resolved link containing the points a and b (respectively, c and d, respectively, e and f). Then in our ordering of homology or homotopy classes (see Definition 3) we have $A < B < C$: otherwise the orientation of our H-diagram is not natural. The H-diagram, obtained from G by the flip replacing two arrows explicitly indicated in Fig. 7(L) by the arrows shown in Fig. 7(R), also is a summand of Φ_Γ having a representation in our link: its two arrows correspond to the upper right-hand and the lower crossing points in Fig. 5(L). The sign of this representation is opposite to that of the initial representation of G, and the collection of 1-homology (homotopy) classes associated with this H-(h-)diagram is the same as for G; so the contributions of these two representations to the value $\Phi_\Gamma(F)$ annihilate.

Almost the same reasonings hold if we suppose that the arrows of a given representation of G correspond to the lower and right-hand crossing points in Fig. 5(L). In this case we get Fig. 7(M) and ordering $B < A < C$; again, the contribution of

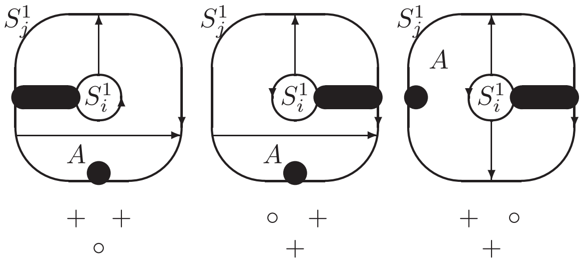


Figure 8: Case $\Delta_{II}(\backslash\ ; m)$, the left-hand part

this representation is annihilated by that of a representation of the diagram shown in Fig. 7(R).

Finally, if we start from the representation reflected in Fig. 7(R) (which assumes the order relations $A < C > B$), then its contribution is killed as previously by exactly one of representations considered above (see Figs. 7(L), 7(M)), depending on whether $A < B$ or $A > B$.

3.2. *The case* $\Delta_{II}^{(2)}(\backslash\ ; m)$

In this case the pairing of endpoints of either part of Fig. 5 is as follows: the pairs of points $(c - f)$, $(a - b)$, and $(d - e)$ are connected somehow through the rest of $M^2 \times \mathbf{R}^1$. Suppose that there is a representation of the H-diagram G in the link shown in Fig. 5(L), sending a segment of one component (S_i^1) of C_n to the \-like local branch and two segments of some other component (S_j^1) to the other two local branches, and such that exactly two arrows of the diagram G go to the crossing points of this part. Depending on the choice of these two points (whose position is indicated by pluses in the bottom of Fig. 8) this diagram G (as it occurs in the sphere $K(G)$) looks as is shown in the corresponding picture of the upper part of the same Fig. 8. Again, here black domains can cover other chords and/or components of C_n; A is the homology or homotopy class of the component of the split link containing points a and b. In the present and two next subsections we consider the case (m), i.e. we assume that $i < j$.

There are two main cases. In the first one, the bigger black spot in our picture indeed should be necessarily drawn as a continuous one, connecting our two components of C_n: this means that there is a chain of arrows and (maybe) other components of C_n in G connecting some points of S_i^1 and S_j^1, besides the arrows explicitly drawn in our picture. In this case we can perform the partial splitting (2.1) of our link F at the crossing points corresponding to the arrows of this chain only, which turns all the components of C_n participating in this chain (including S_i^1 and S_j^1) into a single component, and we fall into the conditions of Subsection 3.1. In particular, the diagram G can be matched by a different diagram, obtained from it by a single flip, together with a representation in the link F, having the same collection Γ of 1-homology (homotopy) classes and the opposite sign.

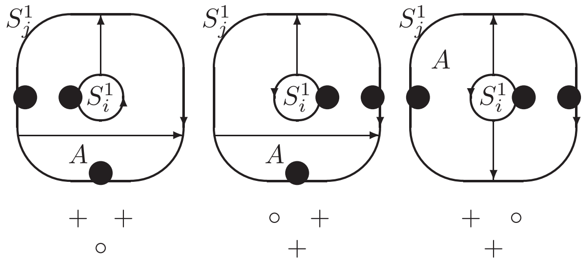


Figure 9: Case $\Delta_{II}(\backslash\ ; m)$, the left-hand part (reduced)

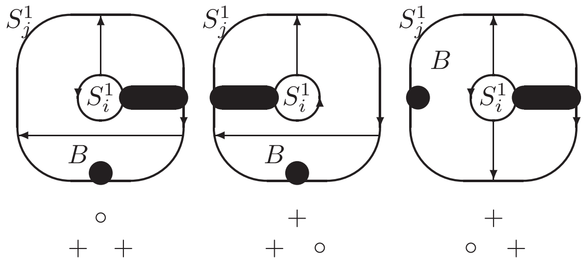


Figure 10: Case $\Delta_{II}(\backslash\ ; m)$, the right-hand part

In the second case such a chain does not exist, and the corresponding part of Fig. 8 can be redrawn in the refined way shown in one of three parts of Fig. 9. The right-hand part of the latter figure cannot correspond to an M-diagram by the orientation rule, see Definition 3. The remaining two diagrams (whose parts covered by black circles are assumed to be the same up to an isotopy preserving the numbers of all hidden components of C_n) match each other: to any representation of one of them in F there corresponds a representation of the other, with the same set of represented crossing points outside of Fig. 5(L), and the unique change of these crossing points inside this figure. The signs of these matched representations are always opposite, and the sets Γ of 1-homology or 1-homotopy classes are the same, so their contributions to the value $\Phi_\Gamma(F)$ annihilate.

The possible representations sending two arrows to crossing points shown in Fig. 5(R) can be considered in exactly the same way, see Figs. 10, 11; here B is the homology or homotopy class of the component of the split link containing the points d and e.

3.3. *The case* $\Delta_{II}^{(2)}(/; m)$

In this case the pairing of endpoints of Fig. 5 is as follows: $(a-d)$, $(b-c)$, $(e-f)$. Possible arrow M-diagrams, having representations in Fig. 5(L) (respectively, 5(R)) with exactly two arrows sent to its crossing points, are shown in Fig. 12 (respectively, 13). In any of these pictures the letter B (respectively, A) denotes the 1-homology or 1-homotopy class of the component of our link, split (see 2.1))

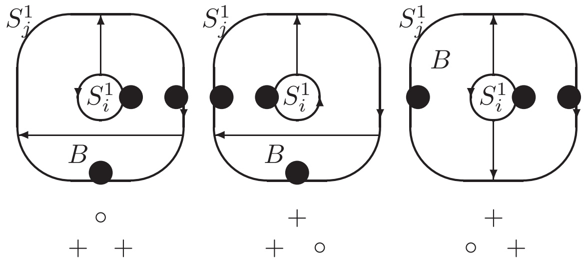


Figure 11: Case $\Delta_{II}(\backslash\ ; m)$, the right-hand part (reduced)

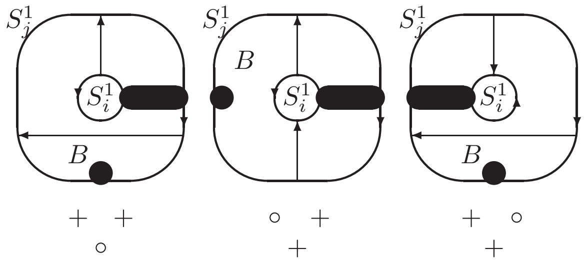


Figure 12: Case $\Delta_{II}(/; m)$, the left-hand part

at crossing points indicated by the arrows of the corresponding representation and containing the points e and f (respectively, b and c). As previously, it is enough to consider only the case when the long black bars can be broken. Then the right-hand diagrams in both Figs. 12, 13 contain arrows prohibited by the orientation rule or Definition 3 and in fact are impossible. The remaining two diagrams of Fig. 12 (respectively, 13) (with isotopic parts covered by black domains) have opposite signs and equal sets Γ of 1-homotopy (and hence also 1-homology) classes; moreover, depending on the orders of these classes in the list Γ they either both are or both are not naturally oriented. Therefore their contributions to $\Phi_\Gamma(F)$ are opposite.

3.4. *The case* $\Delta^{(2)}_{II}(-; m)$

In this case the pairing of endpoints of Fig. 5 is as follows: $(a - f)$, $(b - e)$,

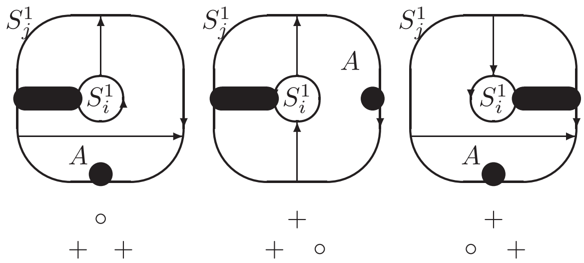


Figure 13: Case $\Delta_{II}(/; m)$, the right-hand part

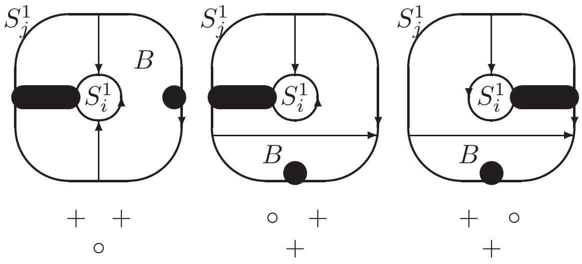


Figure 14: Case $\Delta_{II}(-;m)$, the left-hand part

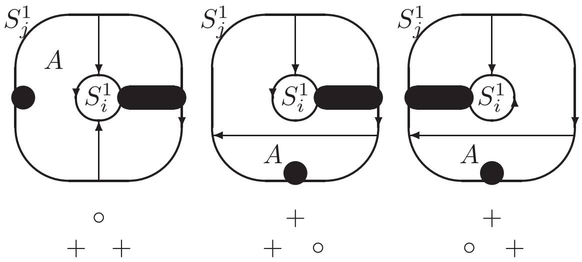


Figure 15: Case $\Delta_{II}(-;m)$, the right-hand part

$(c-d)$. The diagrams similar to the ones considered above are shown in Figs. 14 and 15. Again, we consider only the case of broken black bars in these pictures. In this assumption, *all* six diagrams of these figures are prohibited by the orientation rule of Definition 3.

3.5. *The cases* $\Delta_{II}^{(3)}$

Let F_1 and F_2 be two links coinciding outside of some domain in $M^2 \times \mathbf{R}^1$, and in this domain situated as shown in Figs. 5(L) and 5(R) respectively. Suppose that all three local branches of either of these links belong to different global components of this link, namely, the component of C_n going into the horizontal segment of Fig. 5 is called S_k^1, the component situated as / is called S_j^1, and the \ − like component is called S_i^1, $k \neq j \neq i \neq k$. Let G be an H-diagram participating as a summand in the chain Φ_Γ and represented in the link F_1 (respectively, F_2) in such a way that exactly two arrows of G go to crossing points of this representation.
Then the diagram G is isotopic to that shown in one of three pictures of Fig. 16 (respectively, 17); the choice of one of three pictures in this figure depends on the choice of two crossing points of Fig. 5(L) (respectively, 5(R)) represented by the arrows of this diagram: their positions are indicated by pluses in the bottom part of our figures 16, 17. The black spot in this picture covers all the pieces of these three components of C_n whose images in the link $F_1(C'_n)$ (respectively, $F_2(C'_n)$) lie outside the area shown in Fig. 5; also they can cover additional arrows and/or components of C_n in G. Again, we consider two main cases. In the first one, this

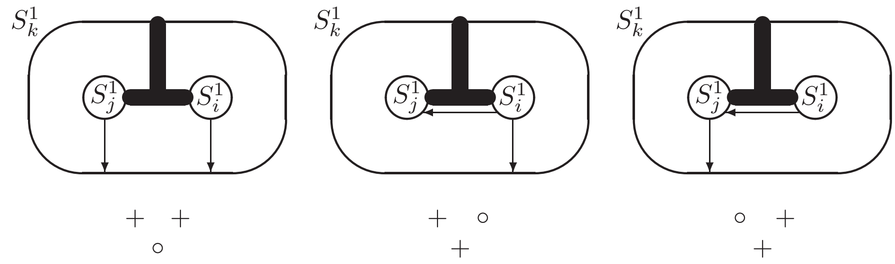


Figure 16: Case $\Delta_{II}^{(3)}$, the left-hand part

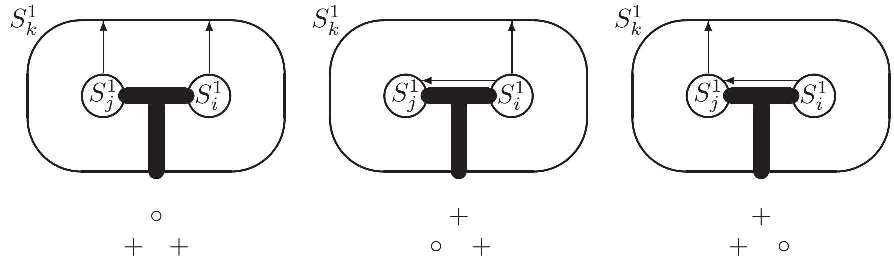


Figure 17: Case $\Delta_{II}^{(3)}$, the right-hand part

black spot for our picture cannot be broken into three separated pieces, each of which touches only one of our circles S_i^1, S_j^1, S_k^1. This means that there exists a chain of arrows and (maybe) other components of G connecting some two of these three circles, besides the arrows explicitly drawn in our picture. In this case the further consideration can be reduced to the above-considered case $\Delta_{II}^{(2)}$ in the same way in which in subsections 3.2–3.4 we have reduced that case $\Delta_{II}^{(2)}$ to $\Delta_{II}^{(1)}$. So it remains to consider the second case when our picture can be reduced to one of pictures shown in Fig. 18 or 19.

Let us perform all the splittings (2.1) of our link $F_1(C'_n)$ or $F_2(C'_n)$ corresponding to the arrows of the diagram G except for two ones shown in our picture; let x, y and z be the components of the resulting split link containing respectively some points of images of circles S_i^1, S_j^1, S_k^1 in our picture and oriented in accordance with the

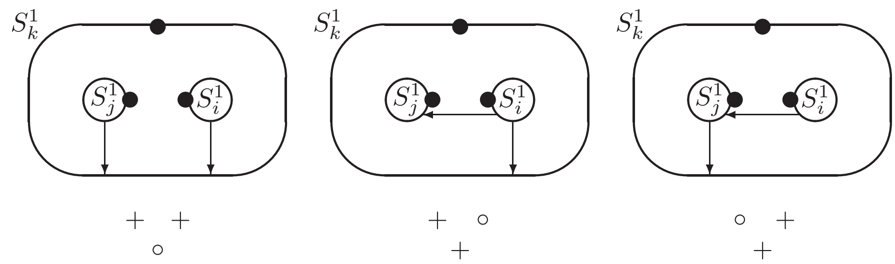


Figure 18: Case $\Delta_{II}^{(3)}$, the left-hand part (reduced)

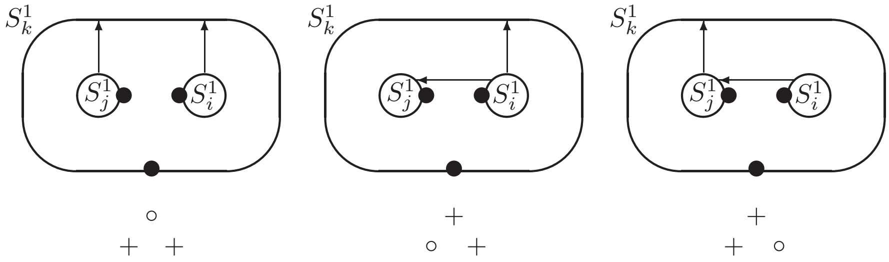


Figure 19: Case $\Delta^{(3)}_{II}$, the right-hand part (reduced)

orientation of the original link. Let x', y' and z' be the classes of these components in the group $\pi_1(M^2)$ with the basepoint somewhere in the area shown in Fig. 5. Then the homotopy class of the single component obtained by the concluding splitting at two arrows shown in our picture is equal to $x'z'y'$ for all diagrams of Fig. 18 and to $x'y'z'$ for all diagrams of Fig. 19. In particular, the homology classes of these last components are the same in all six cases.

The further possibilities depend on the order of components S^1_i, S^1_j, S^1_k. If $i > k$ then such representations of summands of Φ_Γ in our links F_1, F_2 simply do not exist. Indeed, our two arrows do not separate any components of $S^2 \setminus G$, hence in any summannd of Φ_Γ they should be oriented from components of C_n with smaller numbers to these with greater ones.

If $j < i < k$ then by the same reason only the left-hand pictures of Figs. 18 and 19 can be realized by naturally oriented H-diagrams having such a representation. Let us call two such H-diagrams the *siblings* if their parts covered by spots are pairwise isotopic in $K(G)$, and these isotopies preserve the numbers of all touched components of C_n. There is an obvious one-to-one correspondence between the representations of any left-hand diagram of Fig. 18 in F_1 and representations of its sibling in the left-hand diagram of Fig. 19 in F_2; these representations have one and the same set of crossing points outside Fig. 5, into which the arrows of these diagrams are sent. In particular, almost all curves into which the splittings (2.1) corresponding to these arrows break the links $F_1(C'_n)$, $F_2(C'_n)$ are the same (and in particular have the same 1-homotopy and 1-homology classes); the unique possible exception is provided by the curves obtained from the entire non-spotted parts of our diagrams, because one of them (corresponding to the left-hand part of Fig. 18) has the homotopy class $x'z'y'$, and the other one the class $x'y'z'$. Nevertheless, the homology classes of these two curves also do coincide, and the signs of these representations coincide too, so they make the same contributions to the values $\Phi_\Gamma(F_1)$ and $\Phi_\Gamma(F_2)$ if Γ is an ordered collection of classes of $H_1(M^2)$. However, if the elements $x'z'y'$ and $x'y'z'$ are not conjugate in the group $\pi_1(M^2)$ then we cannot replace in the construction of our invariants the collections of 1-homology classes by collections of 1-homotopy classes. Of course, this obstruction is void in the case of $\leq$ 2-component links, where the entire case $\Delta^{(3)}_{III}$ cannot happen.

If $i < k < j$ then only the middle pictures in Figs. 18, 19 can be realized; they can be compared in the same way as in the previous paragraph.

Finally, if $i < j < k$ then all six pictures of these two figures are allowed; with an representation of any of them the representations of all its five siblings are associated such that all collections of homology classes of curves, into which their arrows split these links, are the same. The signs of all four representations corresponding to the left-hand and middle parts of Figs. 18, 19 coincide with one another and are opposite to these of the right-hand ones; so again their sums give equal contributions to the values of $\Phi_\Gamma(F_1)$ and $\Phi_\Gamma(F_2)$. □

Remark 1. Although for the case of ≥ 3 components we do not have combinatorial formulas, related with collections of the 1-homotopy (and not just 1-homology) classes, the construction [8] of weight systems associated with collections of (not necessarily different) homology classes can be generalized to the similar construction based on collections of 1-homotopy classes in the case of arbitrary n. The unique place in [8] that needs an additional care for this generalization is the proof of the **4T**-relation in the situation of Fig. 12 of [8]. If we replace homology classes by homotopy ones, then this relation in this situation is no more the equality of zero terms (as in the proof of Theorem 2 of [8]), but is the equality of differences, both whose minuends and subtrahends are respectively the same.

Problem. In the special case of one-component links, considered in [7], the condition that all elements of the collection $\Gamma = (\gamma_1, \dots, \gamma_{k+1})$ should be different, was slightly reduced: some two of these classes can coincide. Is it possible to reduce the similar restriction also in the case of multi-component links?

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