

RESOLUTIONS OF DISCRIMINANTS AND TOPOLOGY OF THEIR COMPLEMENTS

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ABSTRACT. We study topological invariants of spaces of nonsingular geometrical objects (such as knots, operators, functions, varieties) defined by the linking numbers with appropriate cycles in the complementary *discriminant* sets of degenerate objects. We describe the main construction of such classes (based on the *conical resolutions* of discriminants) and list the results for a number of examples.

The *discriminant* subsets of spaces of geometric objects are the sets of all objects with singularities of some chosen type. The important examples are: spaces of polynomials with multiple roots, *resultant* sets of polynomial systems having common roots, spaces of functions with degenerate singular points, of non-smooth algebraic varieties, of linear operators with zero or multiple eigenvalues, of smooth maps $S^1 \rightarrow M^n$ ($n \geq 3$) having singular or self-intersection points, of non-generic plane curves, and many others.

The discriminants are usually singular varieties, whose stratifications correspond to the classification of degenerations of the corresponding objects. E.g., the discriminant subset in the space of polynomials $x^3 + ax + b$ is the semicubical parabola $(a/3)^3 + (b/2)^2 = 0$: its regular points correspond to polynomials with a root of multiplicity exactly 2, and the vertex to the polynomial x^3 . The discriminant in the space of polynomials $x^4 + ax^2 + bx + c$ is the *swallowtail*, i.e. the surface shown in the right-hand part of Fig. 1: its self-intersection curve consists of polynomials having two double roots, and the semicubical edges correspond to the polynomials with one triple root; the most singular point is the polynomial x^4 with a root of multiplicity 4. Similar stratifications hold for polynomials of all higher degrees: their strata are indexed by the multiplicities and orders in \mathbb{R}^1 of all corresponding multiple roots.

Usually one is interested in the space of non-singular objects which is the complement of the discriminant Σ , e.g. in the space of polynomials without multiple roots, of smooth varieties, of non-degenerate operators, or of knots, i.e. maps $S^1 \rightarrow \mathbb{R}^3$ having no self-intersection or singular points.

If the total space \mathcal{F} of geometric objects is an N -dimensional vector space then the homology groups of these complementary spaces are related by the Alexander duality formula

$$(1) \quad H^i(\mathcal{F} \setminus \Sigma) \simeq \bar{H}_{N-i-1}(\Sigma),$$

where \bar{H}_* denotes the *Borel-Moore homology group*, i.e. the homology group of the one-point compactification relative to the added point. It was Arnold [3] who first used this reduction in the case of the space of complex polynomials with(out)

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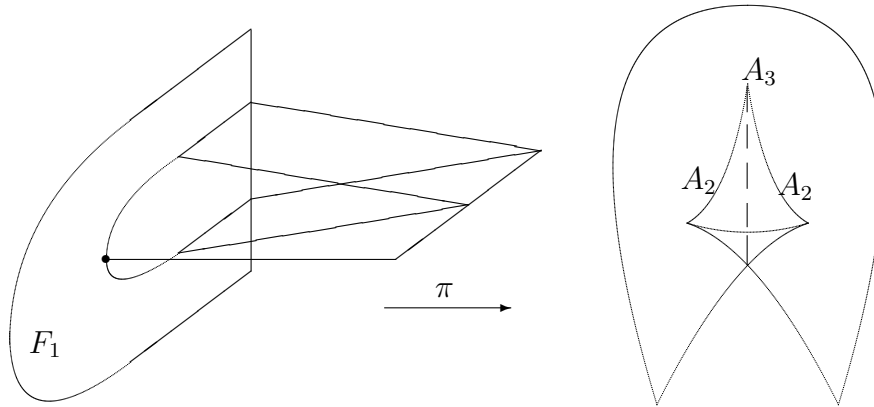


FIGURE 1. A swallowtail and its resolution

multiple roots, see also his works [4], [5], [7], [10] where this reduction is applied for some other discriminant spaces. This reduction is always very useful, because the space $\mathcal{F} \setminus \Sigma$ is an open manifold without any clear geometrical structure; on the other hand a lot of topological properties of the singular space Σ can be expressed in terms of its stratification.

The method of computing the groups $\bar{H}_*(\Sigma)$ invented in [57] is based on the notion of the simplicial resolution of the discriminant variety. One of its advantages consists in the fact that all its ingredients behave properly under stabilizations of \mathcal{F} (e.g. if we consider the sequence of spaces of polynomials of increasing degree, or of operators of increasing order), and therefore allows us to calculate the left-hand groups in (1) even for infinite-dimensional spaces \mathcal{F} , when the right-hand part of (1) has, formally speaking, no sense.

Some first results of this method are described in [64], [55], [56]. Below we describe a more general version of it, based on the notions of *topological order complexes* and *conical resolutions*, and extending similar calculations to many new situations, especially related to *non-normal* discriminants. For a more extended description of this construction, see [73].

1. ORDER COMPLEXES OF DISCRETE POSETS AND SIMPLICIAL RESOLUTIONS OF SUBSPACE ARRANGEMENTS

In this section we demonstrate the method of simplicial resolutions in a simple ‘discrete’ case: that of *plane arrangements*, cf. [24], [28], [62], [77]. The ‘continuous’ version of the method will be demonstrated in the next section.

DEFINITION. Let (A, \geq) be a discrete *poset* (=partially ordered set). The corresponding *order complex* $\Delta(A)$ is the simplicial complex whose vertices are the elements of A , and whose simplices span all strictly monotone finite sequences $\{a_1 < \dots < a_m\}$, $a_i \in A$.

Consider any *affine plane arrangement* \mathcal{L} , i.e. a finite collection of affine subspaces L_1, \dots, L_k of arbitrary dimensions in \mathbb{R}^N . Set $L = \cup L_i$, and, for any set of indices $I \subset \{1, \dots, k\}$, $L_I \equiv \cap_{i \in I} L_i$. Then all possible nonempty planes L_I form a partially

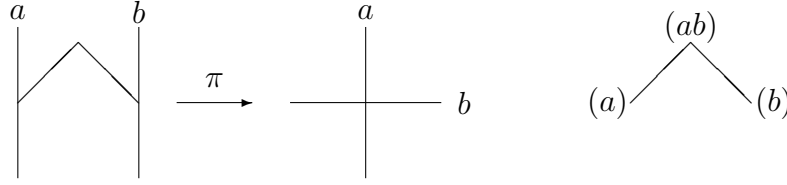


FIGURE 2. Resolution of a cross

ordered set (by inclusion). Denote by $\Delta(\mathcal{L})$ the corresponding order complex. The simplicial resolution of the variety L can be constructed as a subset of the Cartesian product $\Delta(\mathcal{L}) \times \mathbb{R}^N$.

For any plane L_I the corresponding order subcomplex $\Delta(L_I) \subset \Delta(\mathcal{L})$ is defined as the union of the simplices all of whose vertices are subordinate to $\{L_I\}$, i.e. correspond to planes L_J containing L_I . This is a cone with vertex $\{L_I\}$. Denote by $\partial\Delta(L_I)$ its *link*, i.e. the union of all simplices in $\Delta(L_I)$ not containing the vertex $\{L_I\}$.

The resolution space $L' \subset \Delta(\mathcal{L}) \times \mathbb{R}^N$ is defined as the union of all spaces of the form $\Delta(L_I) \times L_I$ over all geometrically distinct planes L_I . The obvious projection $\Delta(\mathcal{L}) \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ induces a map $\pi : L' \rightarrow L$. This map is proper, and all its fibers are contractible finite complexes of the form $\Delta(L_I)$. It follows easily that this map is a homotopy equivalence, and its extension to the map of one-point compactifications, $\bar{\pi} : \bar{L}' \rightarrow \bar{L}$, is also a homotopy equivalence.

EXAMPLE. Let L be the union of two intersecting lines a and b in \mathbb{R}^2 , see the middle part of Fig. 2. The corresponding order complex $\Delta(\mathcal{L})$ consists of two segments (see the right-hand part of Fig. 2) joining the vertices (a) and (b) (corresponding to these lines) to the vertex (ab) (corresponding to the point of intersection). The resolution space L' consists of three complexes: the line $(a) \times a$, the line $(b) \times b$, and the complex $\Delta(\mathcal{L}) \times (a \cap b)$, see the left part of the picture.

In the general case, the resolution space L' has a natural increasing filtration $F_1 \subset F_2 \subset \dots \subset F_{N-1} = L'$: the term F_m is the union of the spaces $\Delta(L_I) \times L_I$ over all planes L_I of *codimension* $\leq m$ in \mathbb{R}^N . The difference $F_m \setminus F_{m-1}$ is the disjoint union of the spaces $(\Delta(L_I) \setminus \partial\Delta(L_I)) \times L_I$ over all planes L_I of dimension exactly $N - m$. Also we get a filtration $\bar{F}_0 \subset \bar{F}_1 \subset \dots \subset \bar{F}_{N-1} = \bar{L}'$ of the one-point compactification \bar{L}' of the space L' : the term \bar{F}_0 is the added point, and each space \bar{F}_i , $i > 0$, is the closure of the corresponding subspace $F_i \subset L'$.

The results of [77] imply in particular that this filtration is homotopically split: there is a homotopy equivalence

$$(2) \quad \bar{L}' \sim \bar{F}_1 \vee (\bar{F}_2/\bar{F}_1) \vee \dots \vee (\bar{F}_{N-1}/\bar{F}_{N-2}),$$

where \vee denotes the *wedge* (\sim *bouquet*). An equivalent result was obtained simultaneously in [62].

This formula implies the Goresky–MacPherson formula for the cohomology of the complementary space $\mathbb{R}^N \setminus L$ (see [28]), and also the fact that the stable homotopy type of this space is determined by the dimensions of the planes L_I .

2. CONICAL RESOLUTIONS OF DETERMINANT SETS

Let \mathbb{K} be any of the fields \mathbb{R} , \mathbb{C} or \mathbb{H} . The *determinant* variety $Det(\mathbb{K}^n) \subset End(\mathbb{K}^n)$ consists of all degenerate operators $\mathbb{K}^n \rightarrow \mathbb{K}^n$.

Its *tautological resolution* is defined by elimination of quantifiers. Namely, an operator A belongs to $Det(\mathbb{K}^n)$ if \exists a point $x \in \mathbb{K}\mathbb{P}^{n-1}$ such that $\{x\} \subset \ker A$. Define the resolution space $det_1(\mathbb{K}^n)$ as the space of all pairs $(x, A) \in \mathbb{K}\mathbb{P}^{n-1} \times End(\mathbb{K}^n)$ such that $\{x\} \in \ker A$. This space admits the (tautological) structure of a $(n^2 - n)$ -dimensional \mathbb{K} -vector bundle over $\mathbb{K}\mathbb{P}^{n-1}$, whose fiber $L(x)$ consists of all A such that $x \in \ker A$. The obvious projection $\pi : det_1(\mathbb{K}^n) \rightarrow Det(\mathbb{K}^n)$ is regular over operators with 1-dimensional kernels, but the pre-image of an operator with $\dim \ker = l$ is isomorphic to $\mathbb{K}\mathbb{P}^{l-1}$.

The situation is very similar to the one in the previous subsection: the variety $Det(\mathbb{K}, n)$ is the union of planes $L(x)$ in the same way as the space L was the union of planes L_i . Keeping the analogy, we construct the order complex of all intersections of these spaces $L(x)$. However we have two important new difficulties: the family of planes $L(x)$ is not discrete, and moreover the set of such planes passing through one and the same point of $Det(\mathbb{K}^n)$ can be continuous. Indeed, the possible intersections of several planes $L(x_j) \subset End(\mathbb{K}^n)$ are just the planes of the form $L(X)$ where X is a subspace of a certain dimension in \mathbb{K}^n (i.e. a point of a certain Grassmannian manifold $G_i(\mathbb{K}^n)$, $i \in [1, n]$), and $L(X)$ consists of all operators whose kernels contain X .

Thus the set of parameters indexing the planes in our poset is the *disjoint union of all Grassmann manifolds* $G_1(\mathbb{K}^n), \dots, G_{n-1}(\mathbb{K}^n), G_n(\mathbb{K}^n)$. The *continuous order complex* of all these Grassmannians is defined as follows. Consider the join $G_1(\mathbb{K}^n) * \dots * G_n(\mathbb{K}^n)$, i.e., roughly speaking, the union of all simplices whose vertices correspond to points of different Grassmannians. Such a simplex is *coherent* if the planes corresponding to its vertices form a flag. The desired order complex $\Theta(\mathbb{K}^n)$ is the union of all coherent simplices, with topology induced from that of the join. This is a cone with vertex $\{\mathbb{K}^n\} \in G_n(\mathbb{K}^n)$. Its *link* $\partial\Theta(\mathbb{K}^n)$ is the union of the coherent simplices not containing this vertex $\{\mathbb{K}^n\}$.

This link $\partial\Theta(\mathbb{K}^n)$ is homeomorphic to the sphere S^M , $M = \frac{1}{2}n(n-1)(\dim_{\mathbb{R}} \mathbb{K}) + n - 2$. (Probably this fact is assumed in the remark 1.4 of [20], see also [61], [56].) Hence $\Theta(\mathbb{K}^n)$ is homeomorphic to a ball.

The conical resolution of $Det(\mathbb{K}^n)$ is constructed as a subset of the direct product $\Theta(\mathbb{K}^n) \times Det(\mathbb{K}^n)$. For example, let $\mathbb{K} = \mathbb{R}$, $n = 2$. The space $End(\mathbb{R}^2)$ of all operators $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ is 4-dimensional, and $Det(\mathbb{R}^2)$ is a 3-dimensional conical subvariety in it. There is a single point in $Det(\mathbb{R}^2)$, over which the tautological resolution is not a homeomorphism: the zero operator. Its preimage is the line $\mathbb{R}\mathbb{P}^1$. In order to get from this resolution a space homotopy equivalent to $Det(\mathbb{R}^2)$, we need to insert a disc whose boundary coincides with this preimage. It is useful to take this disc as the space $\Theta(\mathbb{R}^2)$.

We proceed in a similar way for any n and \mathbb{K} . To any plane $X \subset \mathbb{K}^n$ there corresponds a subspace $\Theta(X) \subset \Theta(\mathbb{K}^n)$, namely, the union of all coherent simplices whose vertices correspond to planes lying in X . This is a cone with vertex $\{X\}$, and is homeomorphic to a closed ball. Define the conical resolution $\delta(\mathbb{K}^n) \subset \Theta(\mathbb{K}^n) \times$

$Det(\mathbb{K}^n)$ as the union of the products $\Theta(X) \times L(X)$ over all planes X of dimensions $1, \dots, n$. It is easy to see that the obvious projection $\delta(\mathbb{K}^n) \rightarrow Det(\mathbb{K}^n)$ induces a homotopy equivalence of one-point compactifications of these spaces (indeed, this projection is proper and semialgebraic, and all its fibers are contractible cones of the form $\Theta(X)$). On the other hand, the space $\delta(\mathbb{K}^n)$ has a nice filtration: its term F_i is the union of products $\Theta(X) \times L(X)$ over planes X of dimensions $\leq i$. The term $F_i \setminus F_{i-1}$ of this filtration is the total space of a fibre bundle over $G_i(\mathbb{K}^n)$. Its fiber over a point $\{X\}$ is the space $(\Theta(X) \setminus \partial\Theta(X)) \times L(X)$, and is homeomorphic to an Euclidean space. Thus the Borel–Moore homology group of this term can be reduced to that of the base. The spectral sequence, generated by this filtration and converging to the Borel–Moore homology group of $Det(\mathbb{K}^n)$ (or, equivalently, to the cohomology group of the complementary space $GL(\mathbb{K}^n)$), degenerates at the first term (i.e. $E_1^{p,q} \equiv E_\infty^{p,q}$) and gives, in particular, the homological *Miller splitting*

$$(3) \quad H_m(GL(\mathbb{C}^n)) = \bigoplus_{k=0}^n H_{m-k^2}(G_k(\mathbb{C}^n))$$

and similar splittings over \mathbb{R} and \mathbb{H} .

3. SOME MORE EXAMPLES AND DIFFICULTIES

All the other discriminant spaces can be resolved in a similar way. However, in some examples we meet two further difficulties: 1) the families of spaces L_I forming our posets may not be closed, and 2) the function space can be infinite-dimensional, so that the Alexander duality (1) formally does not work in it. We discuss these difficulties in the next two subsections and show in the easiest examples how to overcome them.

3.1. All the families of planes should be closed. Following [3], [7], consider the space \mathcal{F}_d of polynomials $x^d + a_1x^{d-1} + \dots + a_d$ over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , and the discriminant space Σ_k consisting of polynomials having at least one root of multiplicity $\geq k$. It can be swept out by a family of planes $L(x)$, $x \in \mathbb{K}^1$, of codimension k in \mathcal{F}_d : any such plane consists of polynomials with a k -fold root at x . Nonempty intersections of such planes are parametrized by the points of configuration spaces $B(\mathbb{K}^1, i)$, i.e. by unordered collections $X = (x_1, \dots, x_i)$ of i distinct points in \mathbb{K}^1 , $i = 1, \dots, [d/k]$. Unfortunately, if $i > 1$ then the set of all such planes $L(X)$ is not closed in the manifold of all planes of codimension ik in \mathcal{F}_d . For instance, if all the i points of a configuration X (depending on a parameter) tend smoothly to one and the same point $x \in \mathbb{K}^1$, then the corresponding planes $L(X)$ tend to some limit position, which is not of the form $L(X')$ for any $X' \in B(\mathbb{K}^1, i)$ (but lies in the plane $L(x)$; this limit position depends on the relative velocities with which our points tend to x). Therefore, if we formally apply the construction of §§1, 2 to the poset formed by spaces $B(\mathbb{K}^1, i)$, then we get a non-closed resolution space, whose projection onto Σ_k is not proper and does not preserve its local homotopy type. The space $B(\mathbb{K}^1, i)$ of such planes needs to be completed. In our 1-dimensional case such a completion $\overline{B(\mathbb{K}^1, i)}$ is obvious: it is the space $S^i(\mathbb{K}^1)$ of all collections of i not necessarily distinct points (or, equivalently, the space of all ideals of codimension i in $\mathbb{K}[t]$). In more general cases, when we resolve discriminants in $C^\infty(M^n, \mathbb{R})$, we can take

any reasonable completion of the configuration space, e.g. just its closure in the space of all affine planes of codimension k in $C^\infty(M^n, \mathbb{R})$ with respect to the natural Grassmannian topology.

The disjoint union of these closures forms a poset (by the inclusions of corresponding ideals), and we can construct its continuous order complex and the resolution of the discriminant exactly as previously; for subtleties see [55], [56], [73], [68].

This situation is shown in the left part of Fig. 1. The right-hand segment in it symbolizes the space $\overline{B}(\mathbb{R}^1, 2)$. Its interior points are regular configurations $(x, y), x \neq y$; they are connected by two coherent segments with the points $\{x\}$ and $\{y\}$ of the space $B(\mathbb{R}^1, 1) \sim \mathbb{R}^1$ shown by the parabola on the left. The endpoint of the segment corresponds to a degenerate configuration (x, x) and is joined with only one point $\{x\} \in B(\mathbb{R}^1, 1)$.

3.1.1. Geometrization. This construction can be slightly simplified: all the ‘nongeometrical’ coherent simplices arising in the construction of the resolution (i.e. the simplices containing ‘nongeometrical’ vertices corresponding to the added boundary points of configuration spaces) can be contracted onto their maximal ‘geometrical’ faces: this contraction does not change the homotopy type of the one-point compactification of the resolution space. For instance, in Fig. 1 we can contract the segment $[(x, x), \{x\}]$ into a point. (For a direct construction of such a ‘geometrical’ resolution in many cases see [55], [56].) The quotient space obtained is in obvious set-theoretical bijection with the subset of the resolution space consisting of coherent simplices all whose vertices are ‘geometric’, i.e. belong to spaces $B(\mathbb{R}^1, i)$ and not to their boundaries. However as topological spaces they are different.

In our case of the discriminant $\Sigma_k \subset \mathcal{F}_d$ the spectral sequence again degenerates at the first term, $E^1 \equiv E^\infty$ (see [55], [58]). Moreover, in this case we also have homotopy splittings of discriminants similar to (2). This is especially obvious if $\mathbb{K} = \mathbb{R}$: indeed, in this case any term $F_i \setminus F_{i-1}, i \leq [d/k]$, is fibered over the trivial space $B(\mathbb{R}^1, i) \sim \mathbb{R}^i$ with fiber equal to the product of the affine space of dimension $d - ik$ and an open $(i - 1)$ -dimensional simplex. So these terms are open cells of decreasing dimensions, and the summands of the wedge will be the spheres of the same dimensions.

3.2. Stabilization and resolution of infinite dimensional discriminants. The power of the above-described construction of resolutions is shown by its perfect functoriality under embeddings of functional spaces and their discriminants. For instance, let us consider the space \mathcal{F}_d from the previous section, and a polynomial $f \in \Sigma_k \subset \mathcal{F}_d$ having exactly one root of multiplicity $c \in [k, d]$ and no other multiple roots. In a neighborhood of f , the variety $\Sigma_k \subset \mathcal{F}_d$ is ambient diffeomorphic to the direct product of the variety $\Sigma_k \subset \mathcal{F}_c$ and the space \mathbb{K}^{d-c} . (E.g. the strata $\{A_2\}$ of the swallowtail in Fig. 1 are locally direct products of \mathbb{R}^1 and the semicubical parabola.) In the restriction to this neighborhood, our resolution of the discriminant $\Sigma \subset \mathcal{F}_d$ coincides with the resolution of $\Sigma_k \subset \mathcal{F}_c$ multiplied by \mathbb{R}^{d-c} . Therefore we get a morphism of spectral sequences converging to the Borel–Moore homology groups of these discriminants: it maps any cell $E_{p,q}^r(d)$ to $E_{p,q-(d-c)}^r(c)$. Further, we can

formally replace these homological sequences by the cohomological ones (converging to the Alexander dual cohomology groups of complements of discriminants),

$$(4) \quad E_r^{p,q}(d) \equiv E_{-p, D-q-1}^r(d)$$

(where D is the dimension of the functional space and in our particular case is equal to d). These ‘inverted’ spectral sequences lie in the second quadrant, $\{p \leq 0, q \geq 0\}$. The induced morphisms of them preserve both gradings p and q ; moreover their final action on the groups E_∞ is compatible with the cohomology map induced by the corresponding embedding $\mathcal{F}_c \setminus \Sigma_k \subset \mathcal{F}_d \setminus \Sigma_k$.

This allows us to define a *stable* spectral sequence $E_r^{p,q}(\infty)$, converging to the cohomology of some limit space: ‘ $\mathcal{F}_d \setminus \Sigma_k$ with infinitely large d ’. For $\mathbb{K} = \mathbb{R}$, the last space can be realized as the space of all smooth functions $\mathbb{R}^1 \rightarrow \mathbb{R}^1$ with some standard behavior at infinity (say, equal identically to 1 outside some compact) and having no zeros of multiplicity k . This is the simplest manifestation of a general method of computing cohomology groups of complements of discriminants in infinite-dimensional functional spaces: we consider an increasing sequence of finite-dimensional approximations, consider resolutions of their intersections with the discriminant set, and then prove a stabilization theorem for the corresponding spectral sequences converging to the cohomology groups of the complements of these intersections. These theorems can be of different strength in different situations: we will discuss some of them in items G (convergence) and I (stabilization) in the next section.

4. LIST OF EXAMPLES

4.1. What can be said on a discriminant and its resolution. In this section we outline in a uniform way resolutions of different discriminant spaces and results on the topology of their complements. The description of each case consists of the following items (some of which may be omitted):

- A. FUNCTIONAL SPACE.
- B. THE DISCRIMINANT.
- C. THE TAUTOLOGICAL RESOLUTION (the set of maximal planes sweeping out the discriminant.)
- D. DESCRIPTION OF THE POSET.
- E. TOPOLOGY OF THE CORRESPONDING CONTINUOUS ORDER COMPLEX (or of its link if the poset has a unique maximal element).
- F. THE SUPPORT OF THE COHOMOLOGICAL SPECTRAL SEQUENCE.
- G. CONVERGENCE OF THE SPECTRAL SEQUENCE: does it converge (in some sense) to the whole cohomology group of the complement of Σ ? Or perhaps to an important subgroup of it? Of course this question is trivial if the function space is finite-dimensional, or if on any line $\{p + q = \text{const}\}$ we have only a finite number of nonzero groups $E_1^{p,q}$ and all these groups are finitely generated.
- H. DEGENERATION OF THE SPECTRAL SEQUENCE: what is the least r for which we have $E_\infty \equiv E_r$? If $r = 1$, then maybe even a homotopical splitting (2) of discriminants holds?
- I. STABILIZATION OF SPECTRAL SEQUENCES. Often our problems form a directed family: e.g. the discriminants $\Sigma_k \subset \mathcal{F}_d$ with different d , or determinant varieties

in $End(\mathbb{K}^n)$ with different n . Then their resolutions and the corresponding spectral sequences can stabilize to interesting limit objects.

J. COMPARISON THEOREMS AND SMALE–HIRSCH PRINCIPLE. Often we have isomorphic spectral sequences for related but different discriminant spaces. This proves the isomorphism of the cohomology groups of their complements, and often reflects the fact that these complements are homotopy equivalent (at least stably). Such comparison theorems often provide versions of the generalized *Smale–Hirsch principle* relating the space of smooth maps without singularities of certain types and the space of continuous sections of the jet bundle not intersecting the corresponding singular set. Example: the space of functions $S^1 \rightarrow \mathbb{R}^1$ without zeros of multiplicity k is homotopy equivalent to the space of all maps of S^1 into $\mathbb{R}^k \setminus 0$ (or, equivalently, to S^{k-1}): this equivalence is induced by the jet extension map sending any function f to the collection of k functions f, f', \dots, f^{k-1} .

K. EXPLICIT FORMULAS expressing cohomology classes obtained from the spectral sequence (i.e. in terms of the Alexander dual cycles) in terms intrinsic to the complement of the discriminant.

L. MULTIPLICATION IN THE SPECTRAL SEQUENCE.

4.2. Monic polynomials without k -fold roots. **A. THE SPACE \mathcal{F}_d** of all polynomials $x^d + a_1x^{d-1} + \dots + a_d$, $a_i \in \mathbb{K} = \mathbb{C}$ or \mathbb{R} .

B. THE DISCRIMINANT is the set Σ_k of polynomials having at least one root of multiplicity k or more in \mathbb{K}^1 . For $\mathbb{K} = \mathbb{C}$ and $k = 2$, the space $\mathcal{F}_d \setminus \Sigma_k$ is the classifying space for the group of braids with d strings.

C. THE TAUTOLOGICAL RESOLUTION is fibered over \mathbb{K}^1 with fiber \mathbb{K}^{d-k} , see §3.1.

D. THE POSET consists of $[d/k]$ terms $\overline{B}(\mathbb{K}^1, i)$, $i = 1, \dots, [d/k]$.

E. THE ORDER COMPLEX is contractible, as \mathbb{K}^1 is.

F. THE SUPPORT. For $\mathbb{K} = \mathbb{R}$, it consists of $[d/k] + 1$ nonzero terms $E_1^{p,q} \sim \mathbb{Z}$ with $p = 0, -1, \dots, -[d/k]$ and $q + (k-1)p = 0$. For $\mathbb{K} = \mathbb{C}$ it lies in the domain $\{(p, q) | p = 0, -1, \dots, -[d/k], q \in [-p(2k-3), -p(2k-2) - 1]\}$.

G. CONVERGENCE. Yes, as $\dim \mathcal{F}_d$ is finite.

H. DEGENERATION. Yes, $E_\infty \equiv E_1$, and for $\mathbb{K} = \mathbb{R}$ even the homotopy splitting (2) holds: probably also for $\mathbb{K} = \mathbb{C}$.

I. STABILIZATION. For $\mathbb{K} = \mathbb{R}$ the spectral sequences stabilize to one calculating the cohomology groups of the space $\mathcal{F}_\infty \setminus \Sigma_k$ of functions $\mathbb{R}^1 \rightarrow \mathbb{R}^1$ with fixed behavior at infinity and without k -fold zeros, and also the cohomology group of the loop space $\Omega(\mathbb{R}^k \setminus 0) \sim \Omega S^{k-1}$, see [58], [55], [56]. For $\mathbb{K} = \mathbb{C}$ and $k = 2$ the stabilized spectral sequence calculates the cohomology group of the stable braid group (with infinitely many strings). For any k it also calculates the cohomology group of the double loop space $\Omega^2(\mathbb{C}^k \setminus 0) \sim \Omega^2 S^{2k-1}$.

J. COMPARISON AND SMALE–HIRSCH PRINCIPLE. For $\mathbb{K} = \mathbb{R}$, the limit space $\mathcal{F}_\infty \setminus \Sigma_k$ is homotopy equivalent (via the k -jet extension map) to the loop space $\Omega(\mathbb{R}^k \setminus 0)$. Any particular term $\mathcal{F}_d \setminus \Sigma_k$ is homotopy equivalent (via the same embedding) to the space of loops of length $< 2\pi([d/k] + 1)$ in the unit sphere S^{k-1} .

For $\mathbb{K} = \mathbb{C}$, the comparison of stable spectral sequences allows us to establish a stable homotopy equivalence between the limit space $\lim_{d \rightarrow \infty} \mathcal{F}_d \setminus \Sigma_k$ and $\Omega^2 S^{2k-1}$.

In particular, these spaces have the same cohomology groups. For $k = 2$ this stable homotopy equivalence was first proved by J.-P. May and G. Segal, see [39], [47].

K. EXPLICIT FORMULAS. These formulas can be induced via the same embedding from the known expressions for the space ΩS^{k-1} . For $\mathbb{K} = \mathbb{R}, k = 3$, the generator of the group $H_1(\mathcal{F}_d \setminus \Sigma_3) \simeq \pi_1(\Omega S^2) \simeq \pi_2(S^2)$ (with index equal to 1) is identified in [3], the generator of $\pi_2(\mathcal{F}_d \setminus \Sigma_3) \simeq_{d \geq 9} \pi_3(S^2)$ (with Hopf invariant 1) in [58], [55], [56].

L. MULTIPLICATION. In this case, the spectral sequence coincides with the Adams-Eilenberg-Moore-Anderson spectral sequence for loop spaces (see e.g. [1]), whose ‘de Rhamization’ is also known as the theory of iterated path integrals, and admits a natural multiplicative structure compatible with the multiplication in the limit cohomology group.

For $\mathbb{K} = \mathbb{C}$ these spaces were studied by V. Arnold from 1968: in his work [3] he invented the seminal reduction (1) and essentially started the topological study of discriminant sets. For $k = 2$, the complete calculation of cohomology rings was then obtained (by different methods) by D. B. Fuchs (for \mathbb{Z}_2 -coefficients) and F. R. Cohen (over \mathbb{Z}). Similar problems for $\mathbb{K} = \mathbb{R}$ were also considered by Arnold in [7], where in particular the cohomology groups of $\mathcal{F}_d \setminus \Sigma_k$ were calculated. The Smale-Hirsch principle for stabilizations of these spaces was found in [58] in answering Arnold’s question on the multiplicative structure in these groups.

4.3. Resultants. **A. THE SPACE** $Syst(k, m)$ of all systems of k polynomials of the form $x^m + b_1 x^{m-1} + \dots + b_m$ over $\mathbb{K} = \mathbb{C}$ or \mathbb{R} .

B. THE DISCRIMINANT is the resultant set $Res(k, m)$ of all systems having a repeated root in \mathbb{K}^1 .

C. THE TAUTOLOGICAL RESOLUTION is fibered over \mathbb{K}^1 with fiber over $x \in \mathbb{K}^1$ equal to the space of all systems having a repeated root at x .

D, E, F, G, H, K, L are exactly the same as in the previous subsection with the same k and $d = km$.

The unique difference for items I, J is that for $\mathbb{K} = \mathbb{C}$ the limit of spaces $Syst(k, m) \setminus Res(k, m)$ as $m \rightarrow \infty$ is homotopy equivalent to $\Omega^2 S^{2k-1}$ (and not just stably homotopy equivalent as in the previous case); this is a theorem of G. Segal [48].

Moreover, a comparison of resolved discriminants and resultants allows us to prove that for any finite k and m the space $Syst(k, m) \setminus Res(k, m)$ is stably homotopy equivalent to $\mathcal{F}_{km} \setminus \Sigma_k$. For $k = 2$ this fact was proved in [23], for arbitrary k in [55]. As usual, it is easier here to prove stable homotopy equivalence of one-point compactifications of resultants and discriminants, so that the stable homotopy equivalence of their complements follows by *Spanier-Whitehead duality*, see [55].

In [35] an even stronger comparison theorem for *real* discriminants and resultants was proved: the resultant variety in the space of pairs of two monic polynomials of degree m is homeomorphic to the discriminant subset of the space of such polynomials of degree $2m$.

4.4. Homogeneous polynomials $\mathbb{R}^2 \rightarrow \mathbb{R}$ without multiple roots. **A. THE SPACE** \mathcal{H}_d of all homogeneous polynomials $\mathbb{R}^2 \rightarrow \mathbb{R}$ of degree d .

B. THE DISCRIMINANT Σ_k consists of polynomials divisible by the k -th power of some linear function.

C. THE TAUTOLOGICAL RESOLUTION is fibered over \mathbb{RP}^1 with fiber equal to the space of polynomials having a zero of multiplicity k along the corresponding line.

D. THE POSET is the disjoint union of all compactified configuration spaces $\overline{B}(\mathbb{RP}^1, i)$, $i = 1, \dots, [d/k]$, plus the one-point set $\{\mathbb{RP}^1\}$ (corresponding to the identically zero polynomial).

E. THE LINK is homotopy equivalent to $S^{2[d/k]-1}$ (theorem of C. Caratheodory).

F. THE SUPPORT belongs to the union $2[d/k] + 3$ cells (p, q) : for any $p = 0, -1, \dots, -[d/k]$, the number q can be equal to either $-p(k-1)$ or $-p(k-1) + 1$, and there is one cell more, $(p, q) = (-[d/k] - 1, d - [d/k] + 1)$. For a description of the corresponding groups $E_r^{p,q}$ see [67], [56].

G. CONVERGENCE. Yes, as $\dim \mathcal{H}_d < \infty$.

H. DEGENERATION. $E_\infty \equiv E_1$ except for the case when k is odd and d is a multiple of k : in this case $E_\infty \equiv E_2 \neq E_1$.

I. STABILIZATION. For any fixed $k > 2$ and $d \rightarrow \infty$ the corresponding spectral sequences stabilize to one calculating the cohomology group of one of two homotopy equivalent spaces of smooth functions $S^1 \rightarrow \mathbb{R}^1$, those even or odd under the involution of S^1 , and having no zeros of multiplicity k . For $k = 2$ the stable spectral sequence is also well-defined but does not give the entire cohomology group of the space of functions without double roots. Indeed, there is an obvious 0-dimensional cohomology class (i.e. an invariant of such functions): the number of their simple zeros. It turns out that all other invariants arising from our spectral sequence depend polynomially on this one. In particular they cannot distinguish the identically positive and negative functions. These cohomology classes are exact analogues of finite-type knot invariants, and we get a picture showing *how the space of such invariants may not be complete*.

J. COMPARISON AND SMALE–HIRSCH PRINCIPLE. The spectral sequence allows us to prove the homotopy equivalence of these spaces with the free loop space $\Omega_f S^{k-1} \equiv \{S^1 \rightarrow S^{k-1}\}$ (or, in the case of odd functions, with the other component of the space of free loops $\Omega_f(\mathbb{RP}^{k-1})$ which consists of loops not liftable to closed loops in S^{n-1}).

K. EXPLICIT FORMULAS. Again, can be inferred from these for $H^*(\Omega_f(\mathbb{RP}^{k-1}))$.

L. MULTIPLICATION. Again, this is induced from the Eilenberg-Moore spectral sequence for the loop space.

4.5. Spaces of smooth functions $M^m \rightarrow \mathbb{R}^n$ without complicated singularities. A. THE SPACE of all smooth functions $M^m \rightarrow \mathbb{R}^n$ (with some fixed behavior close to ∂M^m , if the latter is nonempty).

B. THE DISCRIMINANT is the space of functions having singular points of some class of codimension $\sigma \geq 2$ in the function space (i.e. defined by any $\text{Diff}(\mathbb{R}^m)$ -invariant closed subvariety of codimension $m + \sigma \geq m + 2$ in the jet space $J^T(\mathbb{R}^m, \mathbb{R}^n)$ for some T).

C. THE TAUTOLOGICAL RESOLUTION is fibered over the space of pairs $(x \in M^m; \varphi \in J_x^T(M^m, \mathbb{R}^n))$ such that φ belongs to our singularity class, with the fiber equal to the space of functions $M^m \rightarrow \mathbb{R}^n$ with this T -jet at the point x .

D. THE POSET is the disjoint union of appropriate compactifications of configuration spaces $B(M^m, i)$ over all natural numbers i .

F. THE SUPPORT is in the wedge $\{(p, q) : p \leq 0, q + \sigma p \geq 0\}$.

G. CONVERGENCE. Yes, because there are only finitely many nonzero terms $E_1^{p,q}$ on any line $\{p + q = \text{const}\}$.

H. DEGENERATION. The entire spectral sequences (especially their *higher differentials*) for different singularity classes seem to be strong invariants of smooth manifolds. To what extent can they be derived from topological invariants?

J. COMPARISON AND SMALE–HIRSCH PRINCIPLE. The spectral sequence coincides with one calculating the cohomology group of the corresponding space of all continuous sections of the jet bundle not intersecting our singular subset. This allows us to prove the *homological* Smale-Hirsch principle stating the homology (and even stable homotopy) equivalence of these spaces (and hence, if $\sigma \geq 3$, even the usual homotopy equivalence), see [58], [55], [56]. Moreover, in the most classical case of codimension 2, when $n = 1$ and the forbidden singularity class consists of all germs more complicated than Morse and A_2 , ordinary homotopy equivalence also holds: it was proved up to dimension $m - 1$ in [32] and in all dimensions in [25].

4.6. Spaces of continuous maps of m -dimensional topological spaces into $(m - 1)$ -connected ones. This is essentially the special case of the calculation given in the previous subsection when the forbidden singularity class is defined in the terms of the 0-jets of maps. In this case the source space does not need to be a smooth manifold.

Any finite $(m - 1)$ -connected cell complex Y is homotopy equivalent to the complement of a closed subset $\Lambda(Y)$ of codimension $\geq m + 1$ in some space \mathbb{R}^N . If the topological space X is $\leq m$ -dimensional, then the maps $X \rightarrow \mathbb{R}^N$ not intersecting $\Lambda(Y)$ are dense in the space of all maps. Therefore we get the following way to study the homotopical properties of the space of maps $X \rightarrow Y$ (maybe fixed on some subcomplex $Z \subset X$). (If X is a manifold, then the spectral sequence obtained in this way coincides with one constructed by D. Anderson, see [1], but our approach allows us to remove this restriction.)

A. THE SPACE of all continuous maps $X \rightarrow \mathbb{R}^N$ (maybe coinciding with a fixed map $X \rightarrow \mathbb{R}^N \setminus \Lambda(Y)$ on some subcomplex $Z \subset X$.)

B. THE DISCRIMINANT consists of maps whose images meet $\Lambda(Y)$.

C. THE TAUTOLOGICAL RESOLUTION is fibered over the space of pairs $(x \in X \setminus Z, y \in \Lambda(Y))$ with fiber equal to the space of maps sending x to y .

D. THE POSET is the disjoint union of suitably completed configuration spaces $B(X \setminus Z, i)$ with all natural numbers i .

F. THE SUPPORT is in the wedge $\{(p, q) : p \leq 0, q + \sigma p \geq 0\}$, where $\sigma = n - \dim X - \dim \Lambda(Y)$.

G. CONVERGENCE. If $\sigma \geq 2$ and both X and Y are finite cell complexes (or, more generally, finite type cell complexes, i.e. have finitely many cells in any given dimension) then the spectral sequence obviously converges to the cohomology group of the space of continuous maps $X \rightarrow Y$.

H. DEGENERATION. Often we have $E_\infty \equiv E_1$. For instance this holds if $X = S^m, Y = S^n, m < n$ and $Z = (\text{one point})$, so that the space of maps considered is

the iterated loop space $\Omega^m S^n$; in that case the degeneration theorem provides the *Snaith splitting formula* (more precisely, its homological version)

$$(5) \quad H_t(\Omega^m S^n) \simeq \bigoplus_{i=0}^{\infty} H_{t-i(n-m)}(B(\mathbb{R}^m, i), \pm \mathbb{Z}^{\otimes(n-m)})$$

for the homology group of this space of maps, where $\pm \mathbb{Z}$ is the ‘sign’ local system over the configuration space, cf. [62], [56]. Many other cases when such a decomposition holds were found by C.-F. Bödigheimer, F. Cohen, L. Taylor and others, see the references in [56].

J. COMPARISON AND SMALE–HIRSCH PRINCIPLE. Conversely, the spaces of such maps are the ultimate objects to which one tries to reduce function spaces defined in terms of jet extensions, see e.g. all the previous items of this subsection.

K. EXPLICIT FORMULAS. For instance, the splitting (5) can be realized as follows. Let us fix any standard $(n - m)$ -spheroid Ξ generating the group $\pi_{n-m}(\Omega^m S^n) \sim \pi_m(S^m) \simeq \mathbb{Z}$ as a family of maps $\mathbb{R}^m \rightarrow S^n$ equal to the constant map outside a ball of small radius ρ centered at 0 and depending on a parameter running over the sphere S^{n-m} . For any i -configuration $X = (x_1, \dots, x_i) \in B(\mathbb{R}^m, i)$ such that the ρ -neighborhoods of all its points are disjoint, we can consider the $i(n - m)$ -parameter family Ξ^X of maps $\mathbb{R}^m \rightarrow S^n$ constant outside these neighborhoods and in the neighborhood of any point $x_j \in X$ coinciding up to a parallel translation $\{0 \rightarrow x_j\}$ with maps of the family Ξ . Any homology class in $B(\mathbb{R}^m, i)$ can be realized by a compact cycle in the set of configurations X all whose points are 2ρ -separated. Associating to any point of such a cycle the corresponding cycle Ξ^X we sweep out a cycle in $\Omega^m S^n$. Homology splittings for other spaces of maps $X \rightarrow Y$ as above usually can be realized in a similar way.

L. MULTIPLICATION. If X is a manifold, then this spectral sequence coincides with that of [1], and also admits a natural multiplicative structure. How can this be extended to the most general situation?

4.7. Determinants. A. THE SPACE is $End(\mathbb{K}^n)$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} .

B. THE DISCRIMINANT is the set $Det(\mathbb{K}^n)$ of degenerate operators.

C. THE TAUTOLOGICAL RESOLUTION is fibered over $\mathbb{K}\mathbb{P}^{n-1}$ with fiber over x equal to the space of operators whose kernel contains the line $\{x\}$.

D. THE POSET is the disjoint union of Grassmannians $G_i(\mathbb{K}^n)$, $i = 1, \dots, n - 1, n$.

E. THE LINK is PL -homeomorphic to S^M , $M = \dim_{\mathbb{R}} \mathbb{K} \cdot n(n - 1)/2 + n - 2$.

F. THE SUPPORT is in the wedge

$$\{(p, q) : p \in [-n, 0], q \in [\dim_{\mathbb{R}} \mathbb{K}(p(p - 1)/2) + p, \dim_{\mathbb{R}} \mathbb{K}(-np - p(p + 1)/2) + p]\}.$$

G. CONVERGENCE. Yes, as the function space is finite-dimensional.

H. DEGENERATION. $E_{\infty} \equiv E_1$. See the end of §2.

I. STABILIZATION. For $n \rightarrow \infty$, the spectral sequences stabilize to one calculating the cohomology group of the stable group $GL(\mathbb{K}, \infty)$.

K. EXPLICIT FORMULAS. The Miller splitting (3) for $H_*(GL(\mathbb{C}^n)) \sim H_*(U(n))$ has the following realization (see [61], [56]). For any $i = 1, \dots, n$ and any point $L \in G_i(\mathbb{C}^n)$ we imbed a copy of the group $U(i)$ into $U(n)$ as the set of all unitary operators acting trivially on the orthogonal complement of the i -plane $\{L\}$. When

L runs over a cycle in the Grassmann manifold, such copies of $U(i)$ sweep out a cycle in $U(n)$. For $\mathbb{K} = \mathbb{R}$ or \mathbb{H} realizations of homology groups of $O(n)$ and $Sp(n)$ are exactly the same (although in the real case we need to take care of orientations and use homology with twisted coefficients).

L. MULTIPLICATION. The ring $H^*(U(n))$ is the exterior algebra with canonical generators $\alpha_1, \alpha_3, \dots, \alpha_{2n-1}$ of corresponding dimensions. For any $i = 1, \dots, n$, the term F_i of its filtration induced by the above spectral sequence (i.e. consisting of linking numbers with cycles in the i -th term of the filtration of the resolved determinant) is spanned by all monomials with $\leq i$ factors α_{2j-1} .

4.8. **Knots and links in \mathbb{R}^n , $n \geq 3$, and in other manifolds.** The 0-dimensional cohomology classes of the space of knots in \mathbb{R}^3 are exactly the numerical knot invariants. The invariants arising from the resolutions of discriminants are exactly the *finite-type* knot invariants, see [14]. However the study of the whole cohomology rings of spaces of knots is a more natural problem, leading to equally beautiful algebraic structures, of which the algebraic theory of invariants can be obtained by easy factorization, see [52].

A. THE SPACE of all smooth maps $S^1 \rightarrow \mathbb{R}^n$, $n \geq 3$. (Variants: all smooth maps of S^1 to any manifold M^n , all smooth maps of a finite collection of circles to \mathbb{R}^n or M^n ; all smooth maps $\mathbb{R}^1 \rightarrow \mathbb{R}^n$ coinciding with a standard embedding outside a compact subset in \mathbb{R}^1 ('long knots').

B. THE DISCRIMINANT is the set of maps $S^1 \rightarrow M^n$ (or $\mathbb{R}^1 \rightarrow M^n$ etc.) which are not smooth embeddings, i.e. have either self-intersections or singular points with vanishing derivative.

C. THE TAUTOLOGICAL RESOLUTION is fibered over the completed configuration space $\overline{B(S^1, 2)}$ with fiber over the configuration $(x, y) \subset S^1$ equal to the space of maps $f : S^1 \rightarrow M^n$ such that $f(x) = f(y)$ if $x \neq y$ or $f'(x) = 0$ if $x = y$. If $M^n \neq \mathbb{R}^n$ then this space can be not a plane.

D. THE POSET is a model example of a *poset of multi-configurations* in the same way as the ones in subsections 3.1, 4.4 were the typical examples of posets of (mono)configurations. Namely, consider a multi-index $A = (a_1 \geq \dots \geq a_k)$, where all a_i are natural numbers greater than 1. Given a topological space N (say, $N = S^1$), a *multi-configuration of type A* in N is a collection of $a_1 + \dots + a_k$ distinct points in N divided into groups of cardinalities a_1, \dots, a_k . Denote by $V(N, A)$ the set of all A -configurations in N . It is convenient to consider any such configuration as a subspace (even a subring) in the space of continuous (or smooth if N is a manifold) functions $N \rightarrow \mathbb{R}^1$: namely, as the space of all functions taking equal values at the points of any group. The codimension of this subspace is equal to $\sum_{i=1}^k (a_i - 1)$, therefore this number is called the *complexity* of the multi-index A and of any multi-configuration of type A . Let $\overline{V(N, A)}$ be the closure of $V(N, A)$ in the corresponding Grassmannian topology. For example, if $N = S^1$, $k = 1$ and $a_1 = 2$, then the space $V(N, A)$ is the configuration space $B(S^1, 2)$, i.e. an open Möbius band, and $\overline{V(N, A)}$ is the space $\overline{B(S^1, 2)}$, i.e. a closed Möbius band.

For any natural number s , we consider the union $\rho(N, s)$ of spaces $\overline{V(N, A)}$ over all possible multi-indices A of complexity s . The disjoint union of such unions

with different s is a poset under the natural subordination of multi-configurations (this subordination can be interpreted as the inverse inclusion of the corresponding functional subspaces).

E. ORDER COMPLEXES. For any natural d consider the topological order complex $\Omega(S^1, d)$ of all spaces $\rho(N, s)$ with $s \leq d$. The homological study of these order complexes $\Omega(S^1, d)$ is known as the theory of finite-type knot invariants (and other cohomology classes of the space of knots) and is very complicated. Indeed, the homology group of the quotient space $\Omega(S^1, d)/\Omega(S^1, d-1)$ is the first (and, accordingly to M. Kontsevich, in the case of rational coefficients also the last) step in the calculation of all such invariants and classes of order d modulo similar classes of order $d-1$.

F. THE SUPPORT. The wedge $\{(p, q) : p \leq 0, q + (n-2)p \geq 0\}$. If $M^n = \mathbb{R}^n$ then we can indicate also the upper boundary of this support: $q \leq np + \lfloor -p/2 \rfloor + 1$, see [60], [55], [56].

G. CONVERGENCE. If $n > 3$, then the spectral sequence converges to the entire cohomology group of the space of knots in M^n , and there are only finitely many nonzero terms $E_1^{p,q}$ on any line $\{p+q = \text{const}\}$. If additionally $M^n = \mathbb{R}^n$ then all terms $E_1^{p,q}$ are finitely generated (and explicitly described in [55], [56], [69]). On the other hand, for $n = 3$ already the problem of the convergence on the line $\{p+q = 0\}$ (responsible for the 0-dimensional cohomology classes, i.e. the knot invariants) is unsolved (and is now one of the main problems of knot theory).

H. DEGENERATION. The well-known Kontsevich integral [37] proves that for $M^n = \mathbb{R}^3$ the spectral sequence (with complex coefficients) degenerates on the main diagonal $\{p+q = 0\}$. The same construction proves degeneration for any \mathbb{R}^n on the main diagonal $\{q + (n-2)p = 0\}$. Moreover, Kontsevich knows (at least since 1994) a similar proof of degeneracy of the entire spectral sequence for any \mathbb{R}^n : $E_\infty/\mathbb{C} \equiv E_1/\mathbb{C}$ (still unpublished). My guess is that a) for *long knots* $\mathbb{R}^1 \rightarrow \mathbb{R}^n$ even a homotopy splitting of the discriminant like (2) holds (in some exact ‘stable’ sense, although the discriminant and all terms of its filtration are infinite dimensional, see Problem 5.1 in [72]), but b) for standard knots $S^1 \rightarrow \mathbb{R}^n$ such a splitting (and moreover even the integer homology splitting) does not hold because of torsion terms arising from the topological nontriviality of the source manifold S^1 . On the other hand, there are easy counterexamples to the degeneracy property in the case of manifolds $M^n \neq \mathbb{R}^n$, see e.g. [66].

I. STABILIZATION. Our spectral sequences calculating the cohomology groups of spaces of knots in vector spaces $\mathbb{R}^n, \mathbb{R}^m$ are very similar if n and m are of the same parity. Namely, in this case their initial terms E_1 coincide up to shifts: $E_1^{p,q-pn}(n) \simeq E_1^{p,q-pm}(m)$. In the case of spectral sequences with coefficients in \mathbb{Z}_2 such isomorphisms hold independently of the parity of $m-n$. All this follows immediately from the cellular structure of the resolved discriminant, see [60], [55].

J. COMPARISON AND SMALE–HIRSCH PRINCIPLE. Any smooth function $f : S^1 \rightarrow \mathbb{R}^1$ defines a curve in \mathbb{R}^n given by its $(n-1)$ -jet extension $(f, f', \dots, f^{n-1}) : S^1 \rightarrow \mathbb{R}^n$. Such a curve is called a *holonomic knot* if this map is a smooth embedding. If $n > 3$ then the space of holonomic knots (or links) in \mathbb{R}^n is homotopy equivalent to the space of standard knots (respectively, links with the same number of components). If $n = 3$ then any isotopy class of links can be represented by a holonomic link

(see [65]); conversely, any two holonomic links isotopic in the space of all links are isotopic also in the space of holonomic links (see [16]).

K. EXPLICIT FORMULAS for all cohomology classes can be found (as well as in all other problems discussed here) by the direct calculation of the spectral sequence. In the case of knots in \mathbb{R}^n , any element γ of the term $E_1^{p,q}$ can be encoded as a linear combination of certain graphs (known as *chord diagrams* for $q + p(n - 2) = 0$ and slightly more general for arbitrary p and q) satisfying certain homological condition (ensuring that the corresponding chain in the term $F_i \setminus F_{i-1}$ of the filtration, $i = -p$, consisting of cells encoded by these graphs, actually is a cycle modulo F_{i-1}). The conscientious calculation of the spectral sequence is the following process: we find the boundary of the chain γ in the next term $F_{i-1} \setminus F_{i-2}$ and span it by some chain in this term (it does no matter that this chain will be of infinite dimension). Then find the boundary of this spanning chain in $F_{i-2} \setminus F_{i-3}$, etc. By Kontsevich's theorem, all these steps can actually be performed, i.e. such spanning chains always exist (at least in the case of complex coefficients). At the last step we already have a cycle in all of Σ and span it by a relative chain in the space of all maps $N^1 \rightarrow \mathbb{R}^n \text{ mod } \Sigma$. The corresponding explicit formula works as follows: to any generic cycle in the space of knots $\mathcal{F} \setminus \Sigma$ it associates the number of its intersection points with this relative chain (counted with appropriate signs).

For invariants (i.e. 0-dimensional cohomology classes) of knots in \mathbb{R}^3 , some combinatorial expressions were obtained by J. Lannes, M. Polyak and O. Viro, P. Cartier, S. Piunikhin, S. Tyurina, a.o., see [38], [45], [53], [54]. It was then proved by M. Goussarov [31] that expressions of Polyak–Viro type exist for any invariants of finite filtration for long knots $\mathbb{R}^1 \rightarrow \mathbb{R}^3$.

These expressions (and their extensions to the case of $n > 3$) arise naturally in the above-described algorithm if we choose the spanning chains in some natural way (a semialgebraic chain in \mathbb{R}^N , distinguished by several equations, the last of which is $f(x) = g(x)$, should be spanned by the chain given by all the same conditions with the last one replaced by $f(x) \geq g(x)$, etc.).

Several examples of cohomology classes of spaces of knots other than the knot invariants (and not related to them by the stabilization mentioned in item I above) are known, see [66], [69], [56]. Namely, for ‘compact’ knots $S^1 \rightarrow \mathbb{R}^n$ there are two linearly independent cohomology classes of filtration 1 (of dimensions $n - 2$ and $n - 1$) and two cohomology classes of filtration 2 (one of which is the well-known knot invariant or its stabilization mentioned in I and has dimension $2(n - 3)$, and the second is of dimension $2n - 3$). For ‘long’ knots $\mathbb{R}^1 \rightarrow \mathbb{R}^n$ there are no cohomology classes of filtration 1 or 2 other than the knot invariant or its stabilization, and in filtration 3 for any n there is exactly one more independent cohomology class having dimension $3n - 8$: it was found by D. Teiblum and V. Turchin in the case of odd n and in [69], [56] for even n . Combinatorial formulas for all these classes will be given in [74].

L. MULTIPLICATION. The multiplication formula for chord diagrams expressing the multiplication of corresponding knot invariants was found by Kontsevich. For a similar formula for higher cohomology classes, see [52].

There exists a huge theory of finite type knot invariants (i.e. zero-dimensional cohomology classes), see [14]. In this case the natural filtration on the space of such classes has an elementary characterization in terms of finite differences (see e.g. §0.2 in [60]). However the direct translation of this elementary definition turns out to be very misleading if one tries to apply it to different problems such as the study of generic plane curves or the calculation of higher-dimensional cohomology classes of spaces of knots. The families of classes and invariants arising from such a direct translation are usually unnatural and only weakly related to more classical ones, and the algebraic structures describing them are non aesthetic¹. The reason for this is that in these cases *the singularities of discriminant spaces essential for the calculation of these classes and invariants are more complicated than just normal crossings*.

4.9. Generic plane curves. There are dozens of problems of this kind. We shall consider four of them and denote them by (i), (si), (d) and (o) (for ‘immersions’, ‘immersions/strangeness’, ‘doodles’ and ‘ornaments’, respectively).

A. THE SPACE consists for (i) and (si) of all smooth *immersions* of a circle (or a collection of circles) to \mathbb{R}^2 , for (d) of all smooth maps of a circle to \mathbb{R}^2 , and for (o) of all smooth maps of the union N^1 of $s \geq 3$ circles to \mathbb{R}^2 .

B. THE DISCRIMINANT consists for (i) of non-generic immersions (i.e. having self-tangencies or triple points of the image); for (si) of immersions with triple points; for (d) of maps with triple points or their degenerations (i.e. double points at one of which $f' = 0$ or single points at which $f' = f'' = 0$); and for (o) of maps such that images of some 3 *different* components meet at some point in \mathbb{R}^2 .

The study of the complement of the discriminant in case (i) (and its subproblem (si)) was initiated in [8], [9]; for (d) a similar problem was formulated in [62], [63] and studied in various versions in [36], [41], [42], [70] under the name of the *theory of doodles*. The problem (o) in the general multidimensional situation goes back at least to Kronecker, see also [26], [27], [35]. The homological problems related to the study of the corresponding resolved discriminants were formulated in [62], [63] and studied, in particular, in [18], [63], [40].

C. THE TAUTOLOGICAL RESOLUTION for (si) is fibered over the configuration space $B(S^1, 3)$. The fiber over a triple $(x, y, z) \subset S^1$ consists of all *immersions* f such that $f(x) = f(y) = f(z)$. For (i) the resolution space also includes the set fibered over $B(S^1, 2)$ whose fiber over $(x, y) \subset S^1$ is the space of immersions f such that $f(x) = f(y)$ and $f'(x)$ is collinear with $f'(y)$. For (d) the resolution space is fibered over the completed configuration space $\overline{B(S^1, 3)} = S^3(S^1)$ (or $\overline{B(N^1, 3)}$ if we consider many-component curves). The fiber consists of maps gluing together the points of the configuration, in particular it is an affine subspace of codimension 4 in the space of all maps. For (o) all is the same, but only the configurations whose points belong to different components of the curve are considered.

We mention below only the problems (si), (d) and (o). The remaining theory (i)\(si) includes, among others, Arnold’s basic invariants J^+ and J^- (dual to the

¹A nice counterexample is provided by the theory of finite type invariants of 3-manifolds started by T. Ohtsuki and extended by S. Garoufalidis, M. Goussarov and others. Unfortunately I cannot include this wonderful theory in the general framework of discriminant theory

sets of immersed curves with codirected and counter-directed self-tangencies), and the theory of Legendrian knots, see [9], [11].

D. THE POSET. See the description of the poset of multi-singularities in §4.8, with the sole difference that all indices a_i of a multi-index A should be greater than 2, and not only greater than 1. Additionally, in the case (si) we do not consider completions of spaces of multi-configurations, but only their regular points, and in the case (o) we consider only multi-configurations such that in any of the groups of cardinalities a_1, \dots, a_k there are present points of at least three different components of the manifold N^1 .

E. TOPOLOGY OF THE ORDER COMPLEX is a good open problem.

F. THE SUPPORT. For problems (d), (o) the wedge $\{(p, q) : p \leq 0, p + q \geq 0\}$. For the analogous problem on plane curves without self-intersections of multiplicity k , the wedge $\{(p, q) : p \leq 0, p + q \geq k - 3\}$.

G. CONVERGENCE. In the cases (is) and (d), the terms $E^{p,q}$ with $p + q = 0$ converge (weakly) to the group of invariants of the corresponding objects: all nonequivalent doodles or generic immersions can be distinguished by finite-type invariants, see [42].

H. DEGENERATION. For cases (o) and (d), the spectral sequences conjecturally degenerate at the first term on the diagonal $\{p + q = 0\}$, i.e. $E_\infty^{p,q} = E_1^{p,q}$ for such p, q . This is an experimental fact proved (by A. B. Merkov) up to filtration 6, as well as for many other important cases, but in general it is a conjecture.

K. EXPLICIT FORMULAS. The first explicit formulas for non-obvious finite-type invariants of generic curves in problem (o) were found in [63]: we can count all intersection points of some two components of a curve with their signs (i.e. mutual orientations of these components at these points) and weights (which are polynomial functions of the indices of these points with respect to different components of the curve), see [63]. Strong generalizations of these *index-type invariants* were found in [40], see also [46]. Similar expressions were found in [50], [51] for the simplest *strangeness* invariant in problem (si) of [8], [9] and some of its generalizations; combinatorial expressions for the invariants J^+ , J^- were constructed in [76].

4.10. **Nonsingular hypersurfaces of degree d in $\mathbb{C}\mathbb{P}^n$.** A. THE SPACE $\mathcal{H}(d, n)$ of all homogeneous polynomials $\mathbb{C}^{n+1} \rightarrow \mathbb{C}$ of degree d .

B. THE DISCRIMINANT consists of polynomials whose zeros form a non-smooth hypersurface in $\mathbb{C}\mathbb{P}^n$.

C. THE TAUTOLOGICAL RESOLUTION is fibered over $\mathbb{C}\mathbb{P}^n$ with fiber over x equal to the space of polynomials with zero differential at the corresponding line in \mathbb{C}^{n+1} .

D. THE POSET. We take the space of all possible singular sets defined by such polynomials in $\mathbb{C}\mathbb{P}^n$ and supply it with the topology induced from the Grassmann manifolds $G_i(\mathcal{H}(d, n))$ by the map sending any such set into the space of all polynomials whose singular set contains this one. Finally, we take the closures of these spaces in this topology.

E. THE ORDER COMPLEX. For $d = 2$, this order complex coincides with that considered in §4.7 for the space $End(\mathbb{C}^{n+1})$, and hence its link is homeomorphic to the sphere of dimension $(n + 1)^2 - 2$. For $d = 3$ and $n = 2, 3$ the *rational* homology

groups of these links vanish in all positive dimensions. For $(d, n) = (4, 2)$ its rational Poincaré polynomial is equal to $t^{14}(1+t^3)(1+t^5)$, see [68].

G. CONVERGENCE. Yes, as the dimension is finite.

H. DEGENERATION. For all cases calculated with $d > 2$, i.e. for $(d, n) = (3, 2), (3, 3)$ or $(4, 2)$ the *rational* spectral sequence converges at the first term: $E_\infty/\mathbb{Q} \equiv E_1/\mathbb{Q}$. However, this is just an experimental fact, and not a part of a general theorem. Also, in the most ‘rigid’ case $d = 2$ the spectral sequence does not degenerate at E_1 , see [68].

I. STABILIZATION. For any fixed n and $d \rightarrow \infty$ the corresponding spectral sequences stabilize to one calculating the cohomology of the space of non-vanishing sections of an $(n+1)$ -dimensional vector bundle over $\mathbb{C}P^n$.

J. COMPARISON AND SMALE–HIRSCH PRINCIPLE. The gradient mapping $\mathcal{H}(3, 2) \rightarrow \mathcal{H}(2, 2)^3$, sending any polynomial of degree 3 in \mathbb{C}^3 into the triple of its partial derivatives, induces an isomorphism between the *rational* cohomology groups of the space $\mathcal{H}(3, 2) \setminus \Sigma$ of nonsingular polynomials and the space of quadratic vector fields in \mathbb{C}^3 with unique singular point at 0. This isomorphism follows by comparison of the corresponding spectral sequences, see [68].

K. EXPLICIT FORMULAS. For $d = 2$, the spaces $\mathcal{H}(d, n) \setminus \Sigma$ are homotopy equivalent to the corresponding Lagrange Grassmannian manifolds $U(n+1)/O(n+1)$, whose homology groups are well known together with their various realizations. For any d, n the projective linear group $PGL(n+1, \mathbb{C})$ acts on the space of nonsingular hypersurfaces in \mathbb{C}^{n+1} . In particular any orbit is the image of a map of this group into this space. If $(d, n) = (3, 2)$ or $(3, 3)$ then any such map defines an isomorphism of rational cohomology groups. Thus the cohomology classes of our spaces can be expressed in terms of generators of the cohomology of the group. On the other hand, the Poincaré polynomial of the rational cohomology group of the space of nonsingular *quartics* in $\mathbb{C}P^2$ is equal to $(1+t^3)(1+t^5)(1+t^6)$ (see [68]), i.e., we have a new generator of dimension 6. As J. Steenbrink explained to me, this generator is induced from a cohomology class of the moduli space of curves of genus 3: the cohomology groups of this space were calculated by E. Looijenga in [34].

The problem of calculating topological invariants of spaces of nonsingular plane algebraic curves was posed by V. Arnold, see problems 1970-13 and 1981-13 in [11]. V. Kharlamov [33] used the topology of the *real* discriminant in the parallel theory of rigid isotopy classification of real algebraic plane curves.

4.11. Hermitian matrices with simple spectra. A. THE SPACE of all Hermitian operators in \mathbb{C}^n .

B. THE DISCRIMINANT is the set of operators having at least two equal eigenvalues. This set (as well as its complement) was studied in Arnold’s papers [4], [10]. For some related physical motivations see also [44]. The cohomology ring of the complementary space is well known at least since [19] (as it coincides with the cohomology ring of the space of complete flags in \mathbb{C}^n), nevertheless the induced ‘stable’ structures in it arising from the resolutions seem to be interesting.

C. THE TAUTOLOGICAL RESOLUTION is fibered over the Grassmannian manifold $G_2(\mathbb{C}^n)$; the fiber over any point is the space of all operators whose restrictions to the corresponding 2-plane are scalar.

- D. **THE POSET** is indexed by the same multi-indices $A = (a_1, \dots, a_k)$, $a_i \geq 2$, as in §4.8 (with the additional condition $\sum a_i \leq n$).
- E. **THE LINK**. The explicit formula for the ranks of its homology groups in the case of general n is unknown to me. It was calculated in [71] for $n = 3, 4$ and 5: the Poincaré polynomials of such rational groups (reduced modulo a point) are equal in these cases to $t^2(1+t^2)$, $t^3(1+t^4)(1+t^2+t^4)$ and $t^4(1+t^2+t^4+t^6)(1+t^2+t^4+t^6+t^8+t^{10})$ respectively. For any n , such rational homology groups are trivial in all dimensions of the same parity as n .
- G. **CONVERGENCE**. Yes, as the problem is finite dimensional.
- H. **DEGENERATION**. $E_\infty \equiv E_1$ in the case of rational coefficients. However, homotopy splitting surely does not hold since the groups $E_1^{p,q}$ with integer coefficients can have torsion (due to coinciding indices a_i). See [71].
- I. **STABILIZATION**. There is a natural stabilization of our spectral sequences as n increases. The stabilized spectral sequence converges to the cohomology group of the space of infinite Hermitian matrices with simple spectra (with the topology of the direct limit) and provides a natural filtration on this group. All cells $E_1^{p,q}$ of this stable spectral sequence are finitely generated, although the limit cohomology group certainly is not; thus we get the notion of *finite type cohomology classes* of the space of infinite Hermitian matrices with simple spectra, see [71].
- K. **EXPLICIT FORMULAS**. A few are given in [71]. Finding the others (i.e. the expression of our filtration in the terms of the Chern classes of tautological bundles) seems to be an interesting problem.
- L. **MULTIPLICATION**. A conjectural multiplication formula was given in [71] but is not yet proved.

4.12. (Stabilized) cohomology groups of complements of bifurcation diagrams of zeros of complex function singularities.

- A. **THE SPACE** is the parameter space of a deformation (say, a versal deformation) of a complex function germ $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ with isolated singularity. E.g., the space of complex polynomials as in §4.2 considered as a deformation of the function x^d .
- B. **THE DISCRIMINANT** is (the germ at the origin of) the set of parameter values, for which the corresponding perturbation of f has a critical point close to the origin in \mathbb{C}^n with critical value 0.
- C. **THE TAUTOLOGICAL RESOLUTION** is fibered over (a neighborhood of the origin in) \mathbb{C}^n , the fiber over x consists of all parameter values for which the corresponding perturbed function has x as a critical point.
- D. **THE POSET**. If the singularity is sufficiently complicated with respect to the number d , and the deformation is versal, then its elements corresponding to planes of complex codimension $\leq d(n+1)$ are the (completed) configuration spaces $B(\mathbb{C}^n, i)$, $i = 1, 2, \dots, d$ (see [57]).
- F. **THE SUPPORT** is in the wedge $\{(p, q) : p \leq 0, q + 2p \geq 0\}$.
- G. **CONVERGENCE**. Yes, as the problem is finite dimensional (for any particular f and its deformation).
- H. **DEGENERATION**. If the singularity is sufficiently complicated with respect to d (as in D) then all its differentials d^r , $r \geq 1$, act trivially on the groups $E_r^{p,q}$ with $p \geq -d$. Since by F all nontrivial groups $E_r^{p,q}$ with smaller d satisfy the

inequality $p + q > d$, we get the degeneration $E_\infty^{p,q} \equiv E_1^{p,q}$ for pairs (p, q) in the domain $\{p + q < d\}$, see [57]. Moreover, for such d we have a homotopy splitting $\bar{F}_d \sim \bar{F}_1 \vee (\bar{F}_2/\bar{F}_1) \vee \dots \vee (\bar{F}_d/\bar{F}_{d-1})$, see [55], [56].

I. STABILIZATION. If a singularity f is ‘more complicated’ than g (i.e. in any versal deformation of f we have singularities equivalent to g), then the parameter space of a versal deformation of g can be embedded into that of f in such a way that the discriminant goes to the discriminant, see problem 17 in [5] and also problems 1975-19, 1976-28 and 1980-15 in [11]. These embeddings induce morphisms of the corresponding spectral sequences, which stabilize to a limit spectral sequence calculating the limit cohomology group (which can be thought of as that of the complement of the discriminant of an immensely complicated isolated singularity).

J. COMPARISON AND SMALE–HIRSCH PRINCIPLE. The stable spectral sequence coincides with that (described in §4.6) calculating the cohomology ring of the space $\Omega^{2n}(\mathbb{R}^{2n+2} \setminus 0)$ of all continuous maps $\mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n+2}$ with fixed behavior at infinity and avoiding 0. For any isolated complex function singularity and its deformation, the 1-jet extensions of functions $[\varphi \mapsto (\varphi, \partial\varphi/\partial x_1, \dots, \partial\varphi/\partial x_n)]$ define an embedding of the complement of the corresponding discriminant into this iterated loop space. Our comparison theorem proves that for sufficiently complicated functions this map induces a homology (and even stable homotopy) equivalence up to some high dimension.

K. EXPLICIT FORMULAS. Follow from those for iterated loop spaces.

The above stabilization map defined by adjacency of functions was proposed by Arnold about 1975 (see [5]) together with the problem of computing the corresponding stable cohomology rings (and of proving that they are well defined). Stating this problem was one of the main steps in all the theory described in this section, since it forced one to find ‘stable’ structures of discriminants and ‘stable’ methods of computing their homology groups. The corresponding resolution and stable spectral sequence were constructed in April 1985, see [57]. This was the first of the series of calculations listed in this paper.

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