HOMOLOGY OF i-CONNECTED GRAPHS AND INVARIANTS OF KNOTS, PLANE ARRANGEMENTS, ETC.

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Abstract. We describe several objects and problems of homological combinatorics, arising naturally in the theory of discriminants and plane arrangements, in particular the homology groups of complexes of connected and two-connected graphs and hypergraphs. Knot-theoretical motivations and applications are indicated, as well as first results of related calculations of homology groups of spaces of knots and generic plane curves. Unsolved problems are discussed.

Introduction

The topological study of discriminants, i.e. of spaces of singular geometrical objects, was started by V. Arnold about 1968, see [2], [3]. It is closely related to the investigation of complementary spaces of nonsingular objects.

Later ([40], [38]) it became clear that a main tool in calculating homology groups of such objects are the simplicial (or, more generally, conical, see [39]) resolutions of discriminant spaces.

Below we discuss combinatorial structures, arising naturally in this study, in particular complexes of connected and two-connected graphs (which appear in the calculation of homology groups of spaces of knots and links, [41]) and more general complexes of connected k-hypergraphs, which appear in the study of spaces of k-fold points free generic maps $M \to \mathbb{R}^n$, see [44].

Another issue of these objects is the homological theory of plane arrangements (also founded by Arnold, see [1]), in which simplicial resolutions also are extremely effective.

In § 3.5 we present the first nontrivial positive-dimensional cohomology classes of spaces of knots in $\mathbb{R}^n$, obtained by these techniques. In § 5 we formulate main nonsolved problems of the theory: 1) the homotopy

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splitting conjecture (which states a far extension of the Kontsevich realization theorem for knot invariants), 2) the calculation of equivariant homology groups of our complexes, and 3) the realization of new cohomology classes of knot spaces (which are by now calculated in quite implicit terms).

1. Combinatorial theory

1.1. **Complexes of connected graphs.** Let $\langle k\rangle$ be a set of cardinality k, say the set of natural numbers $1, 2, \dots, k$. Consider the simplex $\Delta(k)$ with $\binom{k}{2}$ vertices, which are in one-to-one correspondence with all two-element subsets of $\langle k\rangle$. Any face of this simplex can be depicted by a graph with k nodes corresponding to elements of $\langle k\rangle$: this graph contains the segment $\overline{ab}$, connecting two elements $a, b \in \langle k\rangle$, if and only if the vertex, corresponding to the set $\{a, b\}$, belongs to our face. Below we consider only the graphs which can be obtained in this way, i.e. having no loops and no multiple edges, but maybe with isolated nodes.

The set of all faces of the simplex $\Delta(k)$ forms an acyclic simplicial complex, which also will be denoted by $\Delta(k)$. Canonical generators of this complex are graphs with ordered edges, while permuting the edges we send such a generator to $\pm$ itself depending on the parity of the permutation. We will always choose the generator, corresponding to the lexicographic order of edges induced by some fixed order of initial k nodes. The boundary of a graph is the formal sum of all graphs obtained from it by removing one of its edges, taken with coefficients 1 or -1.

A graph is called *connected* if any two points of $\langle k\rangle$ can be joined by a chain of its segments.

Denote by $M(k)$ the set of all faces in $\Delta(k)$ corresponding to all nonconnected graphs. Obviously, it spans a subcomplex of the acyclic complex $\Delta(k)$.

Definition. The *complex of connected graphs* associated with the set $\langle k\rangle$ is the quotient complex $\Delta(k)/M(k)$; this quotient complex is denoted by $\Delta^1(k)$.

Example 1. Suppose that $k = 3$ so that the possible nodes of graphs are numbered by 1, 2 and 3. The simplex $\Delta(3)$ is a triangle, whose vertices are called $(1, 2)$, $(1, 3)$ and $(2, 3)$. Among its 7 faces only four correspond to connected graphs, namely, all faces of dimension 1 or 2. In particular, the homology group $H_i(\Delta^1(3))$ is trivial if $i \neq 1$ and is isomorphic to $\mathbb{Z}^2$ if $i = 1$.

THEOREM 1. *a) For any k, the group $H_i(\Delta^1(k))$ is trivial for all $i \neq k-2$, and $H_{k-2}(\Delta^1(k)) \simeq \mathbb{Z}^{(k-1)!}$.*

b) A basis in the group $H_{k-2}(\Delta^1(k))$ consists of all snake-wise (homeomorphic to a segment) connected trees, one of whose endpoints is fixed.

Statement a) of this theorem is a corollary of a general theorem of Folkman [22] on homology of geometric lattices, see e.g. [13]. A proof of b) and another proof of a), based on the Goresky–MacPherson formula for the homology of plane arrangements, is given in [43], see also § 2.1 below.

The number $(k-1)!$ from this theorem is well-known to specialists in the theory of hyperplane arrangements. Consider the space $\mathbb{C}^k$ with complex coordinates $x_1, \dots, x_k$, and the subset $\Sigma \subset \mathbb{C}^k$ which is the union of all complex hyperplanes distinguished by equations $x_i = x_j$, $1 \leq i < j \leq k$. The cohomology ring of its complement $\mathbb{C}^k \backslash \Sigma$ was calculated by Arnold and Fuchs about 1968 (see the work [1], in which the topological study of plane arrangements was essentially started).

THEOREM 2 (see [1]). *The cohomology ring of the space $\mathbb{C}^k \backslash \Sigma$ is isomorphic to that of the product space*

$$S^1 \times \vee_2 S^1 \times \cdots \times \vee_{k-1} S^1, \tag{1}$$

where $\vee_m S^1$ is the wedge of m circles. In particular the Poincaré polynomial of this space is equal to $(1+t)(1+2t)\cdots(1+(k-1)t)$, and the upper non-trivial homology group of our space, $H^{k-1}(\mathbb{C}^k \backslash \Sigma)$, is equal to $\mathbb{Z}^{(k-1)!}$.

We will see a little later that $(k-1)!$ from this statement and $(k-1)!$ from Theorem 1 are one and the same $(k-1)!$.

1.2. **Homology of 2-connected graphs.** DEFINITION. A graph with k nodes is l*-connected* if it is connected, and removing from it any j nodes, $j < l$, together with all incident edges, we obtain also a connected graph (with $k-j$ nodes). Again, the set of all not l-connected graphs is a subcomplex of $\Delta(k)$.

The complex $\Delta^l(k)$ of l*-connected graphs* is the quotient complex of $\Delta(k)$, generated by all faces, corresponding to l-connected graphs. Its homology group is obviously isomorphic to the Borel–Moore homology

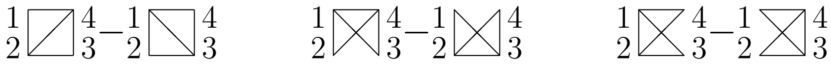


Figure 1. Basic chains for two-connected graphs with 4 vertices

group[1] of the simplex $\Delta(k)$ (considered as a topological space) with all not l-connected faces removed.

In what follows we will consider only the complexes of connected (=1-connected) and 2-connected graphs. Many interesting results concerning the cases of greater l, and also their relations with classical problems of combinatorics, are given in the work [8].

Theorem 3. For any k, the group $H_i(\Delta^2(k))$ is trivial if $i \neq 2k-4$ and is isomorphic to $\mathbb{Z}^{(k-2)!}$ if $i = 2k-4$.

This theorem was proved independently and in different ways by E. Babson, A. Björner, S. Linusson, J. Shareshian, and V. Welker, on one hand, and almost simultaneously (only a day later) by V. Turchin on the other, see [8], [36].

Example 2. The unique face of $\Delta(3)$, corresponding to a two-connected graph, is the triangle itself. Thus $H_i(\Delta^2(3))$ is trivial if $i \neq 2$ and is isomorphic to $\mathbb{Z}$ if $i = 2$.

The simplex $\Delta(4)$ has $\binom{4}{2} = 6$ vertices. The corresponding complex of two-connected graphs consists of the simplex itself, all 6 its faces of dimension 4, and 3 faces of dimension 3, corresponding to all cyclic graphs of length 4. It is easy to calculate that $H_i(\Delta^2(4)) = 0$ for $i \neq 4$ and $H_4(\Delta^2(4)) \simeq \mathbb{Z}^2$. Namely, this homology group is generated by three 4-chains, any of which is the difference of two graphs with 5 edges, obtained from the complete graph by removing edges, connecting complementary pairs of points, see fig. 1. Such basic chains are numbered by unordered partitions of four points into two pairs and satisfy one relation: the sum of all three chains is equal to the boundary of the complete graph.

1.3. **Complexes of connected hypergraphs.** Consider the same k-element set $\langle k \rangle$ as in § 1.1. For any $m < k$ denote by $\Delta_m(k)$ the simplex with $\binom{k}{m}$ vertices, corresponding to all m-point subsets of $\langle k \rangle$. Faces of this simplex, i.e. collections of m-element subsets, are called the

[1] i.e. the homology group of the complex of *locally finite* singular chains, or, isomorphically, the homology group of the one-point compactification reduced modulo the added point. (Recall that the one-point compactification of a *compact* space is this space with a separated additional point.)

m-*hypergraphs* with given k vertices; in particular the 2-hypergraphs are just the usual graphs.

A face of $\Delta_m(k)$ is *connected* if any two points of $\langle k\rangle$ can be joined by a chain of points, any two neighbors in which belong to a m-element subset corresponding to some vertex of this face.

Again, the union of all non-connected faces forms a subcomplex of the (acyclic) simplicial complex formed by all faces of the simplex $\Delta_m(k)$. The *complex* $\Delta^1_m(k)$ *of connected* m-*hypergraphs* is defined as the corresponding quotient complex.

EXAMPLE 3. $\Delta^1_2(k) \equiv \Delta^1(k)$, see § 1.1.

The simplex $\Delta_3(4)$ is a tetrahedron; all its nonconnected faces are just its vertices, therefore the group $H_*(\Delta^1_3(4))$ is isomorphic to $\mathbb{Z}^3$ in dimension 1 and trivial in all other dimensions.

The simplex $\Delta_3(5)$ has $\binom{5}{3} = 10$ vertices. All its non-connected faces are: 0) all 10 vertices; 1) 15 segments; 2) 20 triangles; 3) 5 tetrahedra.

These complexes were studied [17] in connection with the topological study of spaces of generic plane curves and complements of plane arrangements, see § 2 below; another motivation comes from the complexity theory, see [16], [14].

In particular, the following facts were proved.

THEOREM 4 (see [17]). *The simplex $\Delta_m(k)$ reduced modulo the union of non-connected faces is homotopy equivalent to a wedge of spheres, in particular all homology groups $H_d(\Delta^1_m(k))$ are torsion-free. Moreover, these groups can be nontrivial only for d equal to $k-(m-2)t-2$, $1 \leq t \leq k/m$. The rank of these groups always is a multiple of $\binom{k-1}{m-1}$, and in the higher possible dimension $d = k-m$ the rank of $H_{k-m}(\Delta^1_m(k))$ is equal to $\binom{k-1}{m-1}$.*

A general formula for these ranks also is given in [17] (see Theorem 4.5 there), but it is much more complicated.

A. Merkov [27] calculated some similar homology groups for complexes of (hyper)graphs with colored nodes, which appear in problems like the homotopy classification of links or *ornaments*, where only (multiple) intersections of *different* components are forbidden, cf. [20], [21], [44].

Certainly, similar problems on the homology groups of complexes $\Delta^l_m(k)$ of l-connected m-hypergraphs with arbitrary l also can be considered.

2. Topology of plane arrangements and simplicial resolutions

2.1. **Goresky-MacPherson formula and a proof of Theorem 1.** Theorem 2 is the first result of the topological theory of (complements of) plane arrangements; for some further results see [18] and [30]. One of final results here is the general formula by Goresky and MacPherson [23], expressing the cohomology group of the complement of an arbitrary collection of planes in $\mathbb{R}^m$.

Let $\{V_j\}$, $j \in J$, be any finite set of affine planes (of arbitrary dimensions) in $\mathbb{R}^m$, $V = \cup_{j\in J} V_j$, and we are interested in the cohomology group of $\mathbb{R}^m \setminus V$. First of all, by the Alexander duality this group is isomorphic to the Borel–Moore homology group of the set V:

$$\tilde{H}^i(\mathbb{R}^m \setminus V) \simeq \bar{H}_{m-i-1}(V), \tag{2}$$

where $\tilde{H}^*$ is the cohomology group reduced modulo a point. For any subset $I \subset J$ denote by V_I the plane $\cap_{j\in I} V_j$.

Theorem 5. *There is splitting formula*

$$\bar{H}_r(V) \simeq \bigoplus_{L=V_I} \bar{H}_{r-\dim L}(K(L)), \tag{3}$$

where summation is taken over all geometrically distinct *planes L of the form V_I, $I \subset J$, and the space $K(L)$ is defined as follows.*

For any plane $L = V_I$, consider the simplex $\Delta(L)$, whose vertices correspond formally to all indices $j \in J$ such that $V_j \supset L$. To any face of this simplex there corresponds a plane, also containing L, namely the intersection of all planes V_j corresponding to all vertices of this face. This face is called *marginal* if this intersection plane is strictly greater than L. Certainly, all marginal faces constitute a subcomplex of $\Delta(L)$. The space $K(L)$ is defined as the simplex $\Delta(L)$ from which all the marginal faces are removed.

Remark. The Euler characteristic of this complex, $\chi(K(L))$, is nothing other than the Möbius function of the plane L, well-known to the combinatorialists.

Theorem 5 (with the complex $K(L)$ replaced by a different but homologous complex $\tilde{K}(L)$, defined in terms of the order complex of the plane arrangement V) is a version of Theorem A from § III.1.3 of [23].

Corollary 1. *The cohomology group of the complement of an arbitrary affine plane arrangement V depends only on its combinatorial* *data, i.e. on dimensions of all spaces V_I (and the information which of them are empty).*

EXAMPLE 4. Suppose that $m = 2k$, so that $\mathbb{R}^m \sim \mathbb{C}^k$, and V is the arrangement of complex hyperplanes studied in Theorem 2. Consider the smallest stratum $L = V_I$ of this arrangement, i.e. the (complex one-dimensional) intersection of all planes $\{x_i = x_j\}$. The corresponding simplex $\Delta(L)$ can be naturally identified with the simplex $\Delta(k)$ from Theorem 1: its vertex corresponding to the plane $\{x_i = x_j\}$ corresponds also to the edge $\overline{ij}$ of the complete graph. It is easy to see that the marginal faces of this simplex are exactly the ones corresponding to nonconnected graphs.

Now we have everything to prove part a) of Theorem 1. Indeed, if $i < k-2$, then $H_i(\Delta^1(k)) = 0$, because there are no connected graphs with k nodes and $< (k-1)$ edges. To prove the same for $i > k-2$ consider the space $\mathbb{R}^k$ and the real hyperplane arrangement V in it given by the same (real) equations $x_i = x_j$. By the formula (3), applied to the smallest stratum $V_I \sim \mathbb{R}^1$ of this arrangement, the i-dimensional homology group of the complex $\Delta^1(k)$ is the direct summand in the $(i+1)$-dimensional homology group of the $(k-1)$-dimensional variety $\bar{V}$, hence it is trivial for $i > k-2$. So, we have proved that the unique non-trivial homology group of the complex $\Delta^1(k)$ is H_{k-2}. Moreover, this is true for any coefficient group, thus by the formula of universal coefficients there is no torsion in this group. To calculate its rank, it remains to count the Euler characteristic of the complex of connected (or, equivalently, nonconnected) graphs, which is easy and terminates the proof of Theorem 1a).

Also Theorem 4 concerning complexes of connected hypergraphs has an arrangement-theoretical application. Consider the "m-equal" variety in $\mathbb{C}^k$ or $\mathbb{R}^k$, i.e. the union of all $\binom{k}{m}$ planes given by equations

$$x_{i_1} = x_{i_2} = \cdots = x_{i_m}$$

for some $1 \leq i_1 < \cdots < i_m \leq k$, see [17]. Then the complex $K(L)$ from the formula (3), corresponding to the smallest stratum $L = \{x_1 = x_2 = \cdots = x_k\}$, coincides with the complex $\Delta^1_m(k)$.

2.2. **Homotopical version of the Goresky-MacPherson formula.** In fact, the formula (3) has a homotopical version, expressing the homotopy type of the one-point compactification $\bar{V}$ of V in the same terms.

Theorem 6 (see [43], [49]). *There is a homotopy equivalence*

$$\bar{V} \sim \bigvee_{L=V_I} \Sigma^{\dim L} \left(\overline{K(L)}\right), \tag{4}$$

where Σ^i is the i-fold suspension, and $\overline{K(L)}$ is the one-point compactification of $K(L)$ or, which is the same, the quotient space of $\Delta(L)$ by the union of the marginal faces.

This theorem implies in particular, that in the statement of Corollary 1 the words *"the cohomology group"* can be replaced by *"the stable homotopy type"*, in particular by *"any extraordinary cohomology group"*.

2.3. **Geometrical resolutions of plane arrangements.** Theorem 6 (especially its proof from [43]) is a model application of a general method of computing homology groups of *complements of discriminant spaces*. It is based on the techniques of *simplicial resolutions*, which is just a continuous analog of the combinatorial formula of inclusions and exclusions. All the above combinatorial and graph-theoretical notions, like the complexes of connected and 2-connected (hyper)graphs and marginal faces appear naturally in this method.

For two arrangements of 3-lines, shown in the left-hand part of fig. 2, these resolutions are given in the right-hand part of the same picture.

The general construction is as follows. First, we embed all our planes V_j separately and generically into the space $\mathbb{R}^N$ of a very large dimension. For any point $x \in V$ we take all its images under all these embeddings (the number of them is equal to the number of planes V_j containing x) and denote by $\Delta(x)$ the convex hull in $\mathbb{R}^N$ of all these images, see fig. 2. If N is sufficiently large and the planes V_j are embedded generically, then for any x this convex hull is a simplex, whose vertices correspond to all these images, and such simplices corresponding to different points x do not meet in $\mathbb{R}^N$. The resolution space $\tilde{V}$ is defined as the union of all such simplices. The topological type of this space does not depend on the choice of generic embeddings $V_j \to \mathbb{R}^N$. There is natural map $\tilde{V} \to V$: any simplex $\Delta(x)$ is mapped into the point x. This map is proper and defines a homotopy equivalence $\tilde{V} \sim V$; moreover it can be extended by continuity to the map of one-point compactifications $\overline{\tilde{V}} \to \bar{V}$, which *also is a homotopy equivalence*. The space $\tilde{V}$ is called the *simplicial resolution* of V. It turns out that often it is much easier to study the topology of $\tilde{V}$ than that of V.

2.4. **Proper pre-image and filtration in the resolution space.** The next useful notion here is the *proper pre-image* of a plane $V_I \subset V$.

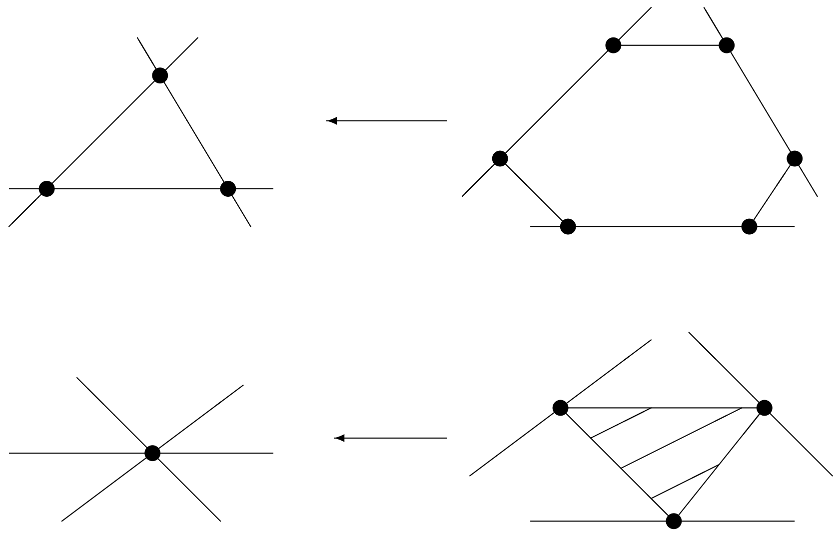


Figure 2. Simplicial resolutions

Consider all *generic* points of V_I, i.e. the points which do not belong to strictly smaller planes $V_J \subset V_I$, and take the union of complete pre-images in $\tilde{V}$ of these generic points. The *proper pre-image* $\tilde{V}_I$ of the plane V_I is the closure of this union. It is easy to see that this proper pre-image is naturally homeomorphic to the direct product $V_I \times \Delta(V_I)$ (where the simplex $\Delta(V_I)$ is defined after Theorem 5). E.g. in the upper (respectively, lower) row of fig. 2 the *complete* pre-image of any line of V is the corresponding line in $\tilde{V}$ together with two segments joining it to other lines (respectively, with the central triangle). However the corresponding proper pre-images in both cases consist only on the corresponding lines in $\tilde{V}$.

The space $\tilde{V}$ (and hence also its compactification $\overline{\tilde{V}}$) admits a useful filtration. Namely, its term $F_i(\tilde{V})$, $i = 1, \ldots, m$, is the union of all proper pre-images of planes $V_I \subset V$, whose *codimensions* $m - \dim V_I$ do not exceed i. In the case of the compactified space $\overline{\tilde{V}}$ we add to all these terms the added point (which is the term $\overline{F}_0$ of the corresponding filtration $\{\overline{F}_j\}$).

Now we can explain the notion of the marginal face. Indeed, the term $F_i \backslash F_{i-1}$ of our filtration is the union of proper pre-images

$$\tilde{V}_I \sim V_I \times \Delta(V_I) \sim \mathbb{R}^{m-i} \times \Delta(V_I) \tag{5}$$

of all planes V_I of codimension exactly i, from which their intersections with all smaller terms of the filtration are removed. It is easy to understand that for any V_I this intersection corresponds in the right-hand part of (5) to the direct product $\mathbb{R}^{m-i}\times$ (the union of marginal faces of $\Delta(V_I)$).

The splitting formula (3) can be now interpreted in the following way. Consider the spectral sequence $E^r_{p,q} \to \bar{H}_{p+q}(\tilde{V}) \simeq \bar{H}_{p+q}(V)$. By definition, its first term $E^1_{p,q}$ is isomorphic to $\bar{H}_{p+q}(F_p(\tilde{V})\setminus F_{p-1}(\tilde{V}))$. Then this spectral sequence degenerates in the first term: $E^\infty_{p,q} \equiv E^1_{p,q}$.

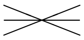
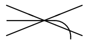
The homotopical extension (4) of this formula means that our spectral sequence (and the filtered space $\overline{\tilde{V}}$) splits in much stronger homotopical sense:

$$\overline{\tilde{V}} \sim \overline{F}_1 \vee (\overline{F}_2/\overline{F}_1) \vee \ldots \vee (\overline{F}_m/\overline{F}_{m-1}). \tag{6}$$

Remark. The method of simplicial resolutions is a continuous version of the combinatorial inclusion-exclusion formula. Indeed, suppose we have a finite union of finite sets, $A = \cup A_i$. To calculate the cardinality of this union, we construct the simplicial resolution of A, i.e. first take all sets A_i separately; further, if some point $x \in A$ belongs to two different sets A_i, A_j, then we join the corresponding points of the separated sets A_i, A_j by a segment; if $x \in A_i \cap A_j \cap A_k$ then after this step we get three segments, which should be filled in by a triangle, etc. The resulting complex $\tilde{A}$ is obviously homotopy equivalent to the initial space A : from any point $x \in A$ we obtain a simplex of a certain dimension. In particular the Euler characteristics of both sets coincide. But the Euler characteristic of the discrete set A is just its cardinality, and that of $\tilde{A}$ is the total number of points in all separated sets A_i, minus the number of segments, plus the number of triangles etc. The similar construction, generalizing both this discrete situation and the above-described case of plane arrangements, can be applied to the calculation of homology groups of an arbitrary "tame" union of topological sets: say, it is sufficient to demand that this union is a CW-complex, and all (multiple) intersections of subsets V_i are its subcomplexes.

Remark. However, our choice of the filtration in the resolved plane arrangement is not so standard. Indeed, there is a canonical filtration in the simplicial resolution of an union $V = \bigcup V_i$: its term F_i consists of all inserted simplices of dimensions $\leq i-1$. The corresponding spectral sequence is called the Mayer-Vietoris spectral sequence. It turns out that in the case of plane arrangements this filtration is not perfect: the Mayer-Vietoris spectral sequence degenerates very slowly, and its intermediate terms are horrible, unlike the case of the filtration described earlier.

Remark. Another simplicial resolution (which leads to the genuine Goresky-MacPherson formula and the version of Theorem 6 obtained in [49]) is based on the notion of the order complex $\kappa(V)$ of our plane arrangement V and all its planes V_I ordered by inclusions. (For the notion of the order complex of a partially ordered set, see e.g. [23].) Namely, this resolution can be constructed as a subset of the direct product $\kappa(V) \times \mathbb{R}^m$: to any plane $V_I \subset V$ we associate the subcomplex $\kappa(V_I) \subset \kappa(V)$ subordinate to this plane, and define the simplicial resolution $\check{V}$ as the union of direct products $\kappa(V_I) \times V_I$ over all such planes.

I often prefer to use the former order complex, because its construction is more local. Indeed, the following two "arrangements"   coincide close to the most complicated point, but their resolutions based on the notion of the order complex do not. Certainly, the second "arrangement" does not exist. However, the theory of plane arrangements is just a nice tool for debugging general topological methods of algebraic geometry. Therefore we need to think always, which of our constructions can be generalized immediately, and which use too much the features of the linear algebra.

3. Applications to the knot theory

3.1. **"Alexander duality".** Let us recall the "discriminant-theoretical" approach to the construction of knot invariants (and, more generally, to the calculation of cohomology groups of spaces of knots in $\mathbb{R}^n$, $n \geq 3$.)

Consider the functional space $\mathcal{K}$ of all C^∞-smooth maps $S^1 \to \mathbb{R}^n$. Let $\Sigma \subset \mathcal{K}$ be the *discriminant set* of all maps which are not smooth embeddings, i.e. have either self-intersections or singular points. The numerical invariants of knots in $\mathbb{R}^3$ are nothing but the elements of the group $H^0(\mathcal{K}\setminus\Sigma)$.

To investigate this group (as well as the similar higher-dimensional cohomology groups, in particular in the case $n > 3$) we use a sort of the Alexander duality[2]

$$\tilde{H}^*(\mathcal{K}\setminus\Sigma) \sim \bar{H}_{\dim \mathcal{K} - * - 1}(\Sigma). \tag{7}$$

[2]In the similar finite-dimensional case (when the functional space $\mathcal{K}$ is the space of polynomials of a given degree in one variable, and Σ is the set of polynomials with multiple roots) this reduction was used first by Arnold in his work [2], in which the topological study of complements of generalized discriminants was initiated, see also [3].

Of course, in our infinite-dimensional situation the formula (7) is senseless, however the method of simplicial resolutions allows us to supply it with some sense in the following way. We consider sufficiently large finite-dimensional approximating subspaces $\mathcal{K}^d \subset \mathcal{K}$ (say, given by Fourier polynomials of restricted degree), so that the groups $H^*(\mathcal{K}^d \backslash \Sigma)$ weakly converge to $H^*(\mathcal{K} \backslash \Sigma)$, and make the similar reduction

$$\tilde{H}^*(\mathcal{K}^d \backslash \Sigma) \sim \bar{H}_{\dim \mathcal{K}^d - * - 1}(\Sigma \cap \mathcal{K}^d) \tag{8}$$

in any of them. Then we consider the simplicial resolution of the corresponding set $\Sigma \cap \mathcal{K}^d$ (see the next subsection), and calculate the groups $\bar{H}_i(\Sigma \cap \mathcal{K}^d)$ of small *codimension* $\dim \mathcal{K} - i$ by means of the corresponding spectral sequence. It turns out that these spectral sequences (more precisely, the Alexander dual cohomological spectral sequences converging to groups $H^*(\mathcal{K}^d \backslash \Sigma)$ and given by the formal inversion $E_r^{p,q}(\mathcal{K}^d) \equiv E^r_{-p, \dim \mathcal{K}^d - q - 1}(\mathcal{K}^d)$) stabilize when the approximations get better and better. The stable spectral sequence converges to some subgroup of the left-hand side of (7) (moreover, if $n > 3$ then to entire this group). Therefore we can work with the space $\mathcal{K}$ and its subset Σ as with spaces of very large but finite dimensions.

3.2. **Simplicial resolution.** Keeping all this in mind, let us describe the simplicial resolution of the discriminant set Σ.

First, we consider the *tautological normalization* of Σ (which is the direct analog of the first step "to embed all spaces V_i separately" in the construction from § 2.3). It is defined by eliminating the quantifiers. Indeed, the space Σ is defined by the formula

(9)
$$\Sigma = \left\{ f \in \mathcal{K} | \exists x, y \in S^1 : ((x \neq y) \& (f(x) = f(y)) \;\; or \;\; ((x = y) \& (f'(x) = 0))) \right\}.$$

Its normalization σ_1 is defined as the set of pairs $(f; (x, y))$ satisfying the same conditions, namely by the formula

(10)
$$\sigma_1 = \{(f; (x, y)) | \, ((x \neq y) \& (f(x) = f(y)) \;\; or \;\; ((x = y) \& (f'(x) = 0))\}.$$

By definition it is a subset of the direct product $\mathcal{K} \times \overline{B(S^1, 2)}$, where $\overline{B(S^1, 2)}$ is the space of all unordered pairs of points in X (may be coinciding); this space is diffeomorphic to the closed Möbius band. More precisely, σ_1 is the space of a vector bundle with base $\overline{B(S^1, 2)}$, whose fibers are vector subspaces of codimension n in $\mathcal{K}$. This bundle (respectively, entire manifold σ_1) is orientable if and only if n is even (respectively, odd). There is natural epimorphic map $\sigma_1 \to \Sigma$ induced by the obvious projection of the product $\mathcal{K} \times \overline{B(S^1, 2)}$ onto the first factor. Its pre-image over any point $f \in \Sigma$ consists of all pairs $(x, y) \subset S^1$ such that $f(x) = f(y)$ $(x \neq y)$ or $x = y$ and $f'(x) = 0$.

Exactly as in the case of plane arrangements, for any such point f we consider the simplex $\delta(f)$ whose vertices correspond to all such pairs (x, y) *respected by* f. (Using only generic finite-dimensional approximations $\mathcal{K}^d \subset \mathcal{K}$ we can avoid the points f for which the number of such pairs is infinite.) The simplicial resolution σ is defined as the union of all such simplices:

$$\sigma = \bigcup_{f \in \Sigma} f \times \delta(f). \tag{11}$$

The obvious projection $\pi : \sigma \to \Sigma$ induces an isomorphism of Borel–Moore homology groups,

$$\bar{H}_*(\sigma \cap \pi^{-1}(\mathcal{K}^d)) \simeq \bar{H}_*(\Sigma \cap \mathcal{K}),$$

for any approximating space $\mathcal{K}^d$.

3.3. **Filtration.** The space σ admits a useful filtration (by the complexities of underlying maps $f \in \Sigma$) which allows us to calculate these homology groups. Namely, given a map $f \in \Sigma$, its *complexity* $c(f)$ is the number of *independent* conditions as in the right-hand part of (9) or (10), satisfied by f.

EXAMPLE 5. If f has p different double points and no other singularities, then $c(f) = p$. If f has exactly one self-intersection point of multiplicity k then $c(f) = k - 1$. In these two cases the corresponding simplices $\delta(f)$ have respectively p and $\binom{k}{2}$ vertices.

Now, we define the term F_c of our filtration in σ as the closure of the union of all simplices $f \times \delta(f)$ over all $f \in \Sigma$ of complexity $\leq c$.

In particular the manifold σ_1 coincides with the first term F_1 of this filtration.

This filtration is motivated by the fact that for any c the difference $F_c \backslash F_{c-1}$ is the space of a $(\dim \mathcal{K} - nc)$-dimensional vector bundle over a tame finite-dimensional base, consisting of finitely many smooth strata corresponding to all combinatorial types of singularities of complexity c. So we kill the infinite-dimensionality of our problem.

EXAMPLE 6. If f has exactly p different double points, and no other singularities, then for any c $\delta(f) \cap F_c$ is the union of all $(< c)$-dimensional faces of the $(p-1)$-dimensional simplex $\delta(f)$.

If f has exactly one self-intersection point of multiplicity k, then the corresponding simplex $f \times \delta(f)$ lies in F_{k-1} and is naturally identified with the simplex $\Delta(k)$ of graphs, whose k nodes correspond to all k

points of S^1 glued together by the map f. Its intersection with the smaller term of filtration, $\delta(f) \cap F_{k-2}$, is exactly the union of all faces corresponding to nonconnected graphs. The stratum of σ (respectively, of $\sigma \setminus F_{k-2}$), corresponding to such singular points, is the fiber bundle, whose base is the configuration space $B(S^1, k)$ of all k-subsets in S^1, and the fiber over such a configuration is the product of the simplex $\Delta(k)$ (respectively, the complex $\Delta^1(k)$) and the subspace of codimension $n(k-1)$ in $\mathcal{K}$ (consisting of all maps $S^1 \to \mathbb{R}^n$ gluing together all points of this k-subset).

The study of more complex strata (corresponding to finitely many multiple points) can be reduced to this one: the corresponding simplices $\delta(f)$ are the joins of smaller simplices corresponding to all multiple points of f, and their subsets lying in the highest possible term of the filtration are the joins of similar subsets for these smaller simplices.

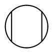
E.g. the part of $F_p \setminus F_{p-1}$, corresponding to the singularity type "p double points" is the space of a fiber bundle, whose base is the space of all *chord diagrams* (i.e. of collections of $2p$ points in S^1 partitioned into pairs), and the fiber is the product of an *open* $(p-1)$-dimensional simplex and a vector space of codimension np in $\mathcal{K}$. Almost all of these $(p-1)$-dimensional simplices are the interior parts of simplices $f \times \delta(f)$ for maps f having exactly p different double points, but not all: say, if f has $p' > p$ double points, then all $(p-1)$-dimensional open faces of the corresponding $(p'-1)$-dimensional simplex also belong to this part of $F_p \setminus F_{p-1}$.

Theorem 1 implies in particular the following corollary: calculating the knot invariants we may ignore the singular maps with intersections of multiplicity > 3 (and maps with 3-fold intersections can provide only relations, and not generators in the space of invariants.) Similarly, if we calculate the 1-dimensional cohomology group of the space of knots in $\mathbb{R}^3$, then only maps with intersections of multiplicity 2, 3 and at most one point of multiplicity 4 should be taken into account.

3.4. **Auxiliary filtrations and two-connected graphs.** Kontsevich proved that in the case of coefficients in $\mathbb{C}$ our stable spectral sequence degenerates at the first term: $E_\infty \equiv E_1$. In the case $n = 3$ for the diagonal terms $E^{-i,i}$, responsible for the knot invariants, this follows from his integral representation, see [25], [10]. However, the complexity of calculation of this first term (i.e., of Borel–Moore homology groups of terms $F_p \setminus F_{p-1}$) grows exponentially with p. Even the modern computers was able to calculate the terms $E^{-i,i}$ only for $i \leq 9$, see [9], and the calculation of terms $E^{p,q}$ with $p + q > 0$, responsible for the

$(p+q)$-dimensional cohomology of the space of knots is even much more complicated.

To accomplish these calculations, we use *auxiliary filtrations* in the terms $F_p \backslash F_{p-1}$ of the *main filtration* generating this spectral sequence (or, equivalently, in the bases of corresponding $(\dim \mathcal{K} - np)$-dimensional vector bundles, see § 3.3). There are two competing auxiliary filtrations, both defined in terms of the numbers of points in S^1 involved in the definition of corresponding strata; sometimes one of them is more convenient and sometimes the other.

Example 7. Let $p = 2$. The term $F_2 \backslash F_1$ of the main filtration consists of strata of 4 main types, corresponding to maps with the following singularities: S4) two double points (there are two connected strata of this type depending on the two possible dispositions of two pairs of points in S^1:  and ); S3$_1$) one triple point; S3$_2$) one double point and one singular point (at which $f' = 0$) not coinciding with points meeting at this double point; S2$_1$) two singular points; S2$_2$) a pair of points x, y such that $f(x) = f(y)$ and $f'(x) = 0$. The corresponding *direct auxiliary filtration* $\phi_2 \subset \phi_3 \subset \phi_4 = F_2 \backslash F_1$ is defined as follows: ϕ_2 is the union of strata $S2_1$ and $S2_2$; adding strata S3$_1$ and S3$_2$ we get the term ϕ_3, and adding strata of type $S4$ we get the entire space $F_2 \backslash F_1$.

For arbitrary p, the similar filtration in $F_p \backslash F_{p-1}$ consists of terms

$$\phi_{[(p-1)/2]+1} \subset \cdots \subset \phi_{2p}.$$

The standard calculations of finite-order knot invariants of any order p are based on this filtration: the components of its top term $\phi_{2p} \backslash \phi_{2p-1}$ are in a obvious one-to-one correspondence with "p-chord diagrams", and the components of $\phi_{2p-1} \backslash \phi_{2p-2}$ generate the "4-term" and "trivial" relations among them.

The *reversed auxiliary filtration* in $F_p \backslash F_{p-1}$ is defined by decreasing the number of involved points, see [47], [39]. E.g. in the case $p = 2$ there are exactly two terms: $\Phi_0 =$ (the closure of the union of strata of type $S4$) (this closure covers also sets of types $S3_2$, $S2_1$ and $S2_2$) and $\Phi_1 = F_2 \backslash F_1$ obtained from it by adding the remaining part of the stratum of type $S3_1$.

Consider especially the strata of types $S4$ and $S3_1$. They both are spaces of $\dim \mathcal{K} - 2n$-dimensional vector bundles over the 5-dimensional bases $s4$, $s3_1$. Namely, $s4$ is a fiber bundle, whose 4-dimensional base consists of all distinct pairs of distinct points of S^1, and the fiber over such a configuration is the segment $\delta(f)$ for any generic map f respecting this configuration, see § 3.2. Endpoints of these segments belong to the smaller term F_1 of the main filtration. Similarly, $s3_1$ is the fiber bundle, whose 3-dimensional base is the configuration space $B(S^1, 3)$ of all triples of different points, and the fiber over such a triple (a, b, c) is the triangle $\Delta(a, b, c) \sim \Delta(3)$, whose vertices are called ab, ac and bc. These vertices also belong to the term F_1 of the main filtration: indeed, they correspond to *non-connected* faces of the triangle $\Delta(3)$. Thus, the calculation of the homology group of $F_2 \setminus F_1$ by means of the direct auxiliary filtration involves the calculation of the Borel–Moore homology group of the fiber bundle over $B(S^1, 3)$, whose fiber is the complex $\Delta^1(3)$ of connected graphs with 3 nodes.

Further, the sides of such a triangle belong to the closure of the stratum $s4$, and hence to the term Φ_0 of the reversed auxiliary filtration. Therefore, calculating the same homology group by means of this filtration we need to consider the homology of a similar fiber bundle with the same base and the open triangle $\Delta(3) \setminus \partial\Delta(3) \equiv \Delta^2(3)$ for a fiber. The entire this calculation is more economical by the following reasons. Previously, the sides of these triangles divided the piece S4 into two strata, which we needed thus to count separately, as well as the strata $\mathrm{S}3_2$, $\mathrm{S}2_1$ and $\mathrm{S}2_2$. In the new approach, these sides are added to the union of these strata, which constitutes a single connected manifold with a simple topology: it is a fiber bundle over the configuration space $B(\bar{M}, 2)$, $\bar{M}$=the closed Möbius band. The remaining part $\Phi_1 \setminus \Phi_0$ is even simpler. The result of the corresponding calculation is presented in Theorem 8 below.

In a similar way, for any p the term $F_p \setminus F_{p-1}$ of the main filtration consists of p terms of the reversed filtration, $\Phi_0 \subset \Phi_1 \subset \cdots \subset \Phi_{p-1}$. Its term Φ_i is the closure of the union of strata $\mathrm{S}j_\alpha$, in whose definition only configurations of j points, $2p - i \leq j \leq 2p$, are involved.

The highest term $\Phi_{p-1} \setminus \Phi_{p-2}$ of this filtration is smooth and is covered by simplices $\Delta(p + 1)$ of graphs, whose vertices correspond to some configurations of $p + 1$ points of S^1, see § 1. Its faces, corresponding to nonconnected graphs, belong to the smaller term F_{p-1} of the main filtration, and faces corresponding to not 2-connected graphs belong to the smaller term Φ_{p-2} of the reversed auxiliary filtration, so that our term $\Phi_{p-1} \setminus \Phi_{p-2}$ is fibered into complexes $\Delta^2(p + 1)$.

The following two properties of this reversed filtration probably prove that it is natural and essential.

3.4.1. *Multiplicativity.* Our (double) filtration is multiplicative: if two knot invariants of finite orders a (respectively, b) have in $F_a \backslash F_{a-1}$ (respectively, in $F_b \backslash F_{b-1}$) auxiliary filtrations α and β (i.e. they can be realized as linking numbers with cycles, whose intersections with $F_a \backslash F_{a-1}$ and $F_b \backslash F_{b-1}$ lie in terms Φ_α and Φ_β), then their product is of order $\leq a+b$ (this was proved in [25]) and, moreover, its filtration in $F_{a+b} \backslash F_{a+b-1}$ is $\leq \alpha + \beta$.

3.4.2. *Higher indices of knot invariants.* In the standard theory of finite-order knot invariants (see e.g. [11], [5], [10]), one considers the indices of *singular knots* with p transverse selfintersection points: they are defined as alternated sums of values of the invariant at 2^p neighboring nonsingular knots. In fact, we can define similar indices for arbitrary singular knots with selfintersection points of arbitrary finite multiplicities.

E.g. if a map $f : S^1 \to \mathbb{R}^3$ has exactly one generic selfintersection point of multiplicity k, then this index takes value in the group $\bar{H}_{2k-4}(\Delta^2(k))$ (which is, by Theorem 3, isomorphic to $\mathbb{Z}^{(k-2)!}$). Moreover, if our knot invariant is of order $\leq k-1$, then similar index is well-defined also for f with nongeneric k-fold points, see [47], [39].

3.5. **First results of calculations.** Here we present the first calculations of higher-dimensional cohomology classes of spaces of knots in $\mathbb{R}^n$, obtained by means of the above-described techniques.

3.5.1. *Compact knots.* First we consider the space of *compact knots*, i.e. of all smooth embeddings $S^1 \to \mathbb{R}^n$, $n \geq 3$. It is well-known that there are no first order invariants of knots in $\mathbb{R}^3$, i.e. 0-dimensional cohomology classes of the space $\mathcal{K} \setminus \Sigma$ defined as linking numbers with $(\dim \mathcal{K} - 1)$-dimensional cycles lying in the first term of the canonical filtration of the resolved discriminant. However the positive-dimensional cohomology classes of order 1 (defined as linking numbers with cycles of greater codimensions) exist; the simplest of them proves that the space of *unknots* in $\mathbb{R}^3$ is not simply-connected.

Theorem 7 (see [47]). *A. For any $n \geq 3$, the subgroup $F^*_{1,\mathbb{Z}_2} \subset H^*(\mathcal{K} \backslash \Sigma, \mathbb{Z}_2)$ of first-order cohomology classes of the space of knots in $\mathbb{R}^n$ consists of exactly two non-trivial homogeneous components $F^{n-2}_{1,\mathbb{Z}_2} \sim F^{n-1}_{1,\mathbb{Z}_2} \sim \mathbb{Z}_2$.*

B. If n is even, then both these cohomology classes give rise to integer cohomology classes, i.e. $F^{n-2}_{1,\mathbb{Z}} \sim F^{n-1}_{1,\mathbb{Z}} \sim \mathbb{Z}$, and there are no other non-trivial integer cohomology groups $F^d_{1,\mathbb{Z}}$, $d \neq n-2, n-1$.

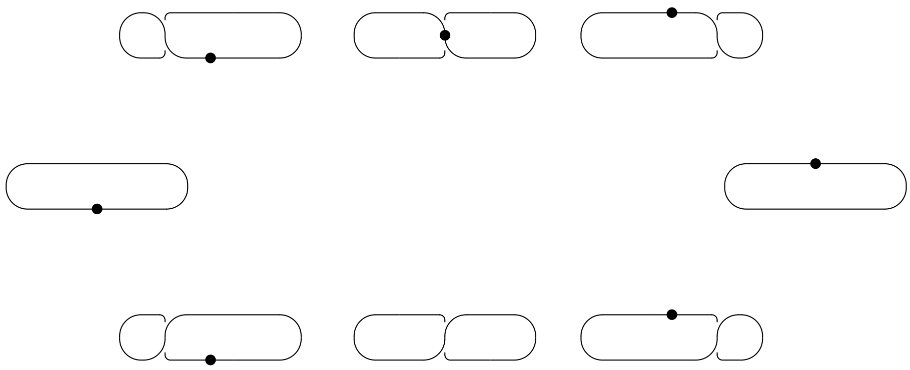


FIGURE 3. Non-trivial 1-cycle in the space of unknots

C. If n is odd, then the generator of the group $F^{n-1}_{1,\mathbb{Z}_2}$ is equal to the first Steenrod operation applied to the generator of $F^{n-2}_{1,\mathbb{Z}_2}$.

D. The generator of the group $F^{n-2}_{1,\mathbb{Z}_2}$ can be defined as the linking number with the $\mathbb{Z}_2$-fundamental cycle of the variety $\Gamma \subset \Sigma$, formed by all maps $\phi : S^1 \to \mathbb{R}^n$, gluing together some two opposite *points of S^1. The generator of the group $F^{n-1}_{1,\mathbb{Z}_2}$ is the linking number with the $\mathbb{Z}_2$-fundamental cycle of the subvariety in Γ, formed by all maps, gluing together some two* fixed *opposite points, say, the points 0 and π. Moreover, if n is even, then these two varieties are orientable, and the groups $F^{n-2}_{1,\mathbb{Z}}$, $F^{n-1}_{1,\mathbb{Z}}$ are generated by the linking numbers with the corresponding $\mathbb{Z}$-fundamental cycles.*

E. If $n = 3$, then the cycles, generating the groups $F^1_{1,\mathbb{Z}_2}$ and $F^2_{1,\mathbb{Z}_2}$, are non-trivial already in the restriction to the component of the unknot in $\mathbb{R}^3$.

Let us demonstrate the nontrivial 1-cycle in the space of unknots in $\mathbb{R}^3$. Set $\mathcal{K} = C^\infty(S^1, \mathbb{R}^3)$. Consider the loop $\Lambda : S^1 \to \mathcal{K}\backslash\Sigma$, some whose eight points are shown in fig. 3. Note that any two (un)knots of this family, placed in this picture one over the other, have the same projection to the "blackboard" $\mathbb{R}^2$. Let us connect any such two unknots by a segment in $\mathcal{K}$, along which the projection to $\mathbb{R}^2$ also is preserved. The union of these segments is a disc in $\mathcal{K}$, spanning the loop Λ; it is obvious that the $\mathbb{Z}_2$-intersection number of this disc with the variety Γ is equal to 1, in particular the class $\{\Gamma\} \in H^1(\mathcal{K}\backslash\Sigma, \mathbb{Z}_2)$ is non-trivial.

On the other hand, it is easy to see that this loop Λ is homotopic to the loop Λ', consisting of knots, obtained from the standard embedding $\phi : S^1 \to \mathbb{R}^2 \subset \mathbb{R}^3$ by rotations by all angles $\alpha \in [0, 2\pi]$ around any diagonal of $\phi(S^1)$, and also to the loop Λ'', consisting of all knots ϕ_τ,

$\tau \in [0, 2\pi]$, having the same image as ϕ and given by the formula $\phi_\tau(\alpha) = \phi(\alpha + \tau)$.

THEOREM 8 (see [47]). *For any $n \geq 3$ all additional integer-valued cohomology groups $F^i_{2,\mathbb{Z}}/F^i_{1,\mathbb{Z}}$ of order 2 of the space of compact knots in $\mathbb{R}^n$ are trivial in all dimensions i other than $2n-6$ and $2n-3$. If $n > 3$, then both these groups in these two dimensions are isomorphic to $\mathbb{Z}$. If $n = 3$, then the first of them (the 0-dimensional one) also is equal to $\mathbb{Z}$ (and is generated by the well-known knot invariant), and the second (3-dimensional) is cyclic (maybe infinite and maybe trivial).*

3.5.2. *Calculations for long knots.* Almost all cohomology classes described in the previous subsubsection exist because the circle itself has nontrivial topology. To avoid this influence, let us consider the space of *long* (or *noncompact*) knots in $\mathbb{R}^n$, i.e. of embeddings $\mathbb{R}^1 \to \mathbb{R}^n$ coinciding with a fixed linear embedding outside a compact subset in $\mathbb{R}^1$.

Then there are no cohomology classes of order 1, and exactly one $(2n-6)$-dimensional class of order 2.

Further, there is exactly two more classes of order 3: they have dimensions $3n-9$ and $3n-8$.

If $n = 3$, then the first of them also is a well-known knot invariant, and the second was discovered in a computer calculation by D. M. Teiblum and V. E. Turchin (based on the cellular decomposition of spaces $F_p \setminus F_{p-1}$ introduced in [41]). In fact, all these calculations are 2-periodic on n, thus they discovered also all these $(3n-8)$-dimensional classes for all odd n. Using the technology of two-connected graphs, I was able to repeat this result "by hands", and to do the similar calculation for all even n (which is the "super" analog of the odd-dimensional situation).

4. SPACES OF m-FOLD POINTS FREE MAPS AND COMPLEXES OF CONNECTED m-HYPERGRAPHS

Suppose that M is a l-dimensional manifold, $l < n$, and we consider the space of all smooth maps $M \to \mathbb{R}^n$ having no points of multiplicity m (i.e. such that there are no m points $x_1, \ldots, x_m \in M$ with $f(x_1) = \cdots = f(x_m)$). Suppose that $(n-l)m > n$, so that the discriminant set, consisting of maps with forbidden m-fold points, has a positive codimension in the space of all maps $M \to \mathbb{R}^n$.

EXAMPLE 8. Besides the spaces of knot and links (where $m = 2$) the most famous examples of such spaces are the spaces of triple-points free plane curves studied in [6], [7], [35], [32], [39] etc.: in this case $M = S^1, n = 2, m = 3$.

Again, the corresponding discriminant set can be resolved in almost the same way as in § 3.2; the first step of this construction (and simultaneously the first term of the natural filtration) is the space σ_1 consisting of all pairs of the form (a map $f : M \to \mathbb{R}^n$; an unordered subset $(x_1, \ldots, x_m) \subset M$) such that $f(x_1) = \cdots = f(x_m)$. This space is a fiber bundle over the configuration space $B(M, m)$ and its fibers are vector subspaces of codimension $(m-1)n$ in the space of all maps $M \to \mathbb{R}^n$.

Now, consider the stratum of the discriminant, consisting of all maps f having one k-fold point in $\mathbb{R}^n$, $k \geq m$. Then over generic points of this stratum we get in the resolution space a simplex with $\binom{k}{m}$ vertices. These simplices belong to the $(k-m+1)$-st term of the natural filtration, and some of their faces belong even to smaller terms of the filtration. It is easy to see that they are exactly the faces corresponding to not connected m-hypergraphs, see § 1.3.

Remark. Already the first step of the construction of the resolution, i.e. the manifold σ_1, can be useful. Indeed, the *first-order* cohomology classes of the space of nonsingular maps are exactly those given by linking numbers with direct images (under the obvious map $\bar{H}_*(\sigma_1) \to \bar{H}_*(\Sigma)$) of cycles lying in σ_1.

Consider the case $M = S^1$. It is easy to calculate that the manifold σ_1 is orientable if and only if the number $(n-1)(m-1)$ is even. If additionally we are working with the plane curves, $n = 2$, then it means that m should be odd. E.g., the *strangeness* (i.e. the invariant of triple points free immersed curves, defined in [6], [7] as the linking number with entire Σ) is well defined; on the other hand the similar one-dimensional cohomology class of the space of plane curves without 4-fold selfintersections (and more complicated singularities) is not defined over the integers.

5. Problems

5.1. **Homotopy degeneration of the main spectral sequence.** The Kontsevich's splitting theorem states that the main spectral sequence calculating the *complex* cohomology of spaces of knots in $\mathbb{R}^n$, $n \geq 3$, defined by the filtration from § 3.3, degenerates at the first term: $E_\infty^{p,q} \equiv E_1^{p,q}$. (For an explicit description of this spectral sequence see e.g. [39], [47], and in the case $n = 3$ also [41], [38].)

I think that in fact a much more strong *homotopy splitting* holds, which is similar to the splitting (4) for plane arrangements. Of course, in this case we have two new problems: a) all terms of the filtration

$F_p(\sigma)$ are infinite-dimensional spaces, and b) there is infinitely many of them. To formulate our conjecture correctly, we do the following.

Note that for any filtering degree p the corresponding term F_p of the resolved discriminant σ stabilizes in the following sense. For any approximating subspace $\mathcal{K}^d \subset \mathcal{K}$ denote by $F_p^d(\sigma)$ the set $F_p(\sigma) \cap \pi^{-1}(\mathcal{K}^d)$, i.e. the corresponding term of the filtration of the resolution of the space $\Sigma \cap \mathcal{K}^d$. Then for any p there exist such a good approximating subspace $\mathcal{K}^d$ that for any better approximation $\mathcal{K}^s \supset \mathcal{K}^d$ the corresponding term $F_p^s(\sigma)$ is homeomorphic (as a filtered space) to the direct product $F_p^d(\sigma) \times \mathbb{R}^{(\dim \mathcal{K}^s - \dim \mathcal{K}^d)}$.

Now the HOMOTOPY SPLITTING CONJECTURE is formulated as follows: *For any p and any sufficiently large approximating space $\mathcal{K}^d \subset \mathcal{K}$ (such that the above stabilization property for the space F_p^d holds), its one-point compactification $\overline{F_p^d} \equiv \overline{F_p^d(\sigma)}$ is homeomorphic to the wedge*

$$\overline{F_1^d} \vee (\overline{F_2^d}/\overline{F_1^d}) \vee \ldots \vee (\overline{F_p^d}/\overline{F_{p-1}^d}). \tag{12}$$

There is a large series of similar situations when analogous discriminants admit similar splittings. Among them, besides the above-described case of plane arrangements, there are discriminants in spaces of

- real and complex monic polynomials in one variable (discriminants consisting of polynomials with roots of multiplicity $\geq k$), see [2], [3], [38], [39];
- systems of complex or real polynomials (discriminants defined as sets of systems with common roots), see [19], [38];
- maps $S^m \to \mathbb{R}^n$, $m < n+1$ (discriminant defined as the space of maps intersecting the origin in $\mathbb{R}^n$, so that its complement is the space $\Omega^m(\mathbb{R}^n \backslash 0) \sim \Omega^m S^{n-1}$);
- linear endomorphisms of K^n, $K = \mathbb{R}, \mathbb{C}$ or $\mathbf{H}$ (discriminant defined as the determinant variety of non-isomorphic endomorphisms).

All these spaces have a common feature: they consist of maps into a vector space (as well as the knot spaces discussed now). However there are examples of similar situations when the target space is not so easy, and the spectral sequence does not degenerates; see [46], [39], [24].

5.2. **Relation with the graph-complex of trees.** It seems likely that on the level of calculating the knot invariants the study of two-connected graphs is more or less equivalent to that of the Konsevich's graph-complex of trees, see [10].

PROBLEM: to establish a direct isomorphism between these two theories[3].

5.3. **Equivariant homology of complexes $\Delta^1(k), \Delta^2(k), \Delta^1_m(k)$ etc. with respect to the natural action of the permutation group $S(k)$.** The topological study of knots, generic plane curves etc. is just a model example of a large class of problems, one of which concerns the topology of the space of maps $M \to \mathbb{R}^n$ having no m-fold points of the image (and maybe no some other similar singularities). In the same way as above, this problem leads to the investigation of the (Borel–Moore) homology of the following fibered space. Its base is the configuration space $B(M, k)$ of all k-point subsets of the manifold M, and the fiber over such a configuration $X = (x_1, \dots, x_k)$ is the complex $\Delta^1_m(k)$, considered as the union of all *connected* open faces of the simplex $\Delta_m(k)$, whose $\binom{k}{m}$ vertices are in a fixed one-to-one correspondence with all m-point subsets of the configuration X.

Certainly, it is impossible to solve these problems for all manifolds M simultaneously. However, the space of all k-configuration spaces has a universal object, $B(\mathbb{R}^\infty, k)$, from which all others are induced by an arbitrary embedding $M \to \mathbb{R}^\infty$. It seems reasonable first to calculate the cohomology of our fiber bundle over this space, and then, for any particular M, to calculate the cohomology of the similar bundle over $B(M, k)$ as the result of some interaction of the "coefficient subring" induced from the cohomology of the universal bundle, and ingredients coming from the homological features of M. But $B(\mathbb{R}^\infty, k)$ is a realization of the $K(S(k), 1)$-space, and the homology of our universal $\Delta^1_m(k)$-bundle on it is nothing but the equivariant cohomology of the complex $\Delta^1_m(k)$ with respect to the $S(k)$-action induced by permuting the k points.

PROBLEM: to calculate these homology groups. The similar problem holds for all complexes $\Delta^j_m(k)$ (and for $j = 2$ this problem even has a discriminant-theoretical sense); moreover, for some parities of n, m and $\dim M$, we need to consider the similar equivariant cohomology with coefficients in the representation $S(k) \to Aut(\mathbb{Z})$ sending the odd permutations to the multiplication by -1.

5.4. **Realization of the Teiblum–Turchin cocycle.** The $(3n-8)$-dimensional order 3 cohomology class of the space of knots in $\mathbb{R}^n$, mentioned in § 3.5.2, is defined only as the linking number with a certain cycle in the discriminant. Moreover, in the case $n = 3$ it is not

[3]This problem was solved in Fall 1997 by V. Turchin, see [37]

proved that this cycle is not homologous to zero, and hence that this 1-dimensional class is non-trivial.

PROBLEM: to realize this cocycle in intrinsic terms of the space of knots, and to construct cycles in this space, on which it takes non-zero values. In the case $n = 3$, what is the simplest component of the space of knots containing such a cycle?

The same problems for the $(2n-3)$-dimensional cohomology class of order 2 of the space of compact knots, see Theorem 8.

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