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SYMPLECTIC GEOMETRY ON MODULI SPACES OF HOLOMORPHIC BUNDLES OVER COMPLEX SURFACES

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Abstract.

We give a comparative description of the Poisson structures on the moduli spaces of flat connections on *real* surfaces and holomorphic Poisson structures on the moduli spaces of holomorphic bundles on *complex* surfaces. The symplectic leaves of the latter are classified by restrictions of the bundles to certain divisors. This can be regarded as fixing a "complex analogue of the holonomy" of a connection along a "complex analogue of the boundary" in analogy with the real case.

Introduction

In this note we discuss the geometry of the momentum map for gauge groups in the following two cases with the aim of emphasizing the analogy between them. We start by recalling a description of the Poisson structure on the moduli space of flat connections over a two-dimensional real surface. Our main interest is related to a description of analogous structures on moduli of holomorphic bundles over a two-dimensional complex surface. In the first case we deal with smooth objects while in the second one – with complex analytic objects. Our interest in this subject comes mainly from a desire to understand the origin of a symplectic structure (refs. [Mu, Ko]) and of a Poisson structure (refs. [Bon, Bot]) on moduli of holomorphic bundles over complex surfaces in a way which would be parallel to the consideration of flat connections over real surfaces. It is worth mentioning that the both cases above are also of interest from the mathematical physics point of view. Other related important results on the geometry of moduli spaces of holomorphic bundles on certain complex surfaces deal with the study of the symplectic structure related to a Kähler form, or a hyperkähler structure (cf., e.g., refs. [Do, KN, LMNS]).

We adopt the viewpoint of considering the case of holomorphic bundles on complex surfaces as a certain complexification of the case of flat connections on real surfaces. This approach is parallel to the geometric complexification method suggested by V. Arnold in ref. [Ar]. To be more precise, rather than formally complexifying, we replace locally constant sheaves (corresponding to flat connections) by sheaves of holomorphic sections (or, one could say, d/dx is replaced by $\partial/\partial\bar{z}$). Being rather simple by itself this leads however to certain curious ideas some of which we will sketch below (cf., also refs.[Wi, EF, FK, Kh, DT, Th, FKT, KR]).

The consideration of flat connections rests on the notions of holonomy and curvature. One needs to equate the curvature to zero, which leads to the "flatness" condition, while the holonomy is crucial as the "only remaining" part of what can characterize a flat connection modulo gauge transformations. Thus, to pass to "complexified" objects, i.e., to holomorphic bundles over complex surfaces, we need to know what the complex analogues of holonomy and curvature are. It is somewhat easier with curvature. For a (0,1)-connection (i.e., $\bar{\partial}$-connection) one can define its curvature (0,2)-form. However, as we shall see below (cf., §3), a better analogue of the curvature (in the context of symplectic geometry on the space of $\bar{\partial}$-connections) will be a certain (2,2)-form. This form is the wedge product of the above (0,2)-form with a meromorphic (or holomorphic) (2,0)-form on the surface. The meromorphic (2,0)-form will play the role of orientation of the surface in the complex analytic situation. We specifically concentrate on the case of a meromorphic (2,0)-form with logarithmic singularities. This gives rise to a complex analogue of the notion of a real surface with boundary, namely, a complex surface with the polar set of a (2,0)-form.

The problem of finding a proper complex analogue of the notion of holonomy of a flat connection (or, better to say, monodromy) is more intricate. Let us first consider the case of a real surface with boundary. Then, for a flat connection on it, we have certain group elements, the monodromies, associated with every closed loop in the surface. In the case when the loop is homotopic to a boundary component we say that we deal with a monodromy around a hole in the surface. The monodromies around the holes play a distinctive role in the study of Poisson geometry of the moduli space of flat connections. The latter space is, in fact, a Poisson manifold whose Poisson structure is degenerate in the case when the surface has holes (i.e., non-empty boundary). The symplectic leaves of that Poisson manifold can be defined by fixing the conjugacy classes of the monodromies around the holes. Thus one can find out a proper "complexified" notion of the monodromy around a hole provided one is able to single out the symplectic leaves in the Poisson manifold of moduli of holomorphic bundles on a complex surface with "boundary" (in the sense mentioned above). This will be discussed in §3. In the case of a loop which is not homotopic to the boundary we do not know yet a proper complex analogue of the monodromy. It would be interesting to see what should replace "a loop," a concept from homotopy theory, in the complex analytic setting.

It turns out, however, that at least the corresponding homology theory can be constructed. The above approach leads one to certain complex analytic analogues of the notions of chains, boundary, and cycles. This is mentioned in §2.

§1. Real case: Poisson structures on moduli spaces of flat connections.

First we recall several results on Poisson structures on the space of flat connections on *real* surfaces with boundary. Our consideration of holomorphic bundles on *complex* surfaces below, in §3, will be parallel to the case of real surfaces. We follow the papers [AB, FR] in the exposition of the real case.

In the real case G stands for a simple simply connected *compact* Lie group, and $\mathfrak{g} = \operatorname{Lie} G$ is its Lie algebra with a chosen nondegenerate invariant quadratic form, which we denote by tr. Let Σ be an oriented compact surface which may have boundary $\Gamma = \partial\Sigma$ consisting of several components, $\Gamma = \cup_1^k \Gamma_j$. Denote by E a (trivial) principle G-bundle over Σ. Let $\mathcal{A}^\Sigma$ be the affine space of all smooth connections in E. It is convenient to fix any trivialization of E and identify $\mathcal{A}^\Sigma$ with the vector space $\Omega^1(\Sigma, \mathfrak{g})$ of smooth $\mathfrak{g}$-valued 1-forms on the surface:

$$\mathcal{A}^\Sigma = \{d + A \mid A \in \Omega^1(\Sigma, \mathfrak{g})\} \ .$$

Definition 1.1. The space $\mathcal{A}^\Sigma$ is in a natural way a symplectic manifold with the symplectic structure

$$W := \int_\Sigma \operatorname{tr}(\delta A \wedge \delta A) \ , \tag{1}$$

where δ is the exterior differential on $\mathcal{A}^\Sigma$, and $\wedge$ stands to denote the wedge product both on $\mathcal{A}^\Sigma$ and Σ.

Proposition 1.2. *The symplectic structure W is invariant with respect to the gauge transformations*

$$A \mapsto g^{-1}Ag + g^{-1}dg \ ,$$

where g is an element of the group of gauge transformations, G^Σ, i.e., it is a smooth G-valued function on the surface Σ.

The infinitesimal gauge transformations forming a Lie algebra $\mathfrak{g}^\Sigma$ are generated on the symplectic manifold $\mathcal{A}^\Sigma$ by certain Hamiltonian functions.

Proposition 1.3. *An infinitesimal gauge transformation ϵ is generated by the Hamiltonian function*

$$H_\epsilon = \int_\Sigma \operatorname{tr}(\epsilon(dA + A \wedge A)) - \int_{\partial\Sigma} \operatorname{tr}(\epsilon A) \ .$$

Proof. Hamiltonian vector field X corresponding to any Hamiltonian H is defined by its action on functions $f(A)$:

$$L_X f = \{H, f\} = \int_\Sigma \operatorname{tr}\left(\frac{\delta H}{\delta A} \wedge \frac{\delta f}{\delta A}\right),$$

where the latter expression is the Poisson bracket corresponding to eq.(1). It suffices to consider the coordinate function $f(A) = A$:

$$L_X A = \{H, A\} = \frac{\delta H}{\delta A}.$$

Then for the above Hamiltonian H_ϵ we have

$$L_{X_\epsilon} A = \frac{\delta H_\epsilon}{\delta A} = \nabla_A \epsilon,$$

where $\nabla_A \epsilon = d\epsilon + [A, \epsilon]$ is an infinitesimal gauge transformation. Indeed, for $F(A) := dA + A \wedge A$ we have $\delta F = \nabla_A \delta A$, and then

$$\begin{aligned} \delta H_\epsilon &= \int_\Sigma \mathrm{tr}\,(\epsilon\,\delta F) - \int_{\partial\Sigma} \mathrm{tr}(\epsilon\,\delta A) \\ &= \int_\Sigma \mathrm{tr}\,(\epsilon\,\nabla_A \delta A) - \int_{\partial\Sigma} \mathrm{tr}(\epsilon\,\delta A) = \int_\Sigma \mathrm{tr}\,(\delta A \wedge \nabla_A \epsilon)\,. \end{aligned}$$

In the last equality we used the Stokes formula. □

The Hamiltonian function generating a given gauge transformation is defined only up to an additive constant. Hence, generally speaking, the Poisson bracket between two such Hamiltonians reproduces the commutation relation in the gauge algebra $\mathfrak{g}^\Sigma$ only up to a cocycle:

$$\{H_{\epsilon_1}, H_{\epsilon_2}\} = H_{[\epsilon_1, \epsilon_2]} + c(\epsilon_1, \epsilon_2)\ .$$

Proposition 1.4. *For the above choice of Hamiltonians the cocycle is*

$$c(\epsilon_1, \epsilon_2) = \int_{\partial\Sigma} \mathrm{tr}(\epsilon_1 d\epsilon_2)\ . \tag{2}$$

One can show that this well-known cocycle is nontrivial. Therefore one can define the momentum mapping not for the algebra of gauge transformations, but only for its central extension by the 2-cocycle (2).

To define this mapping we need some notations. Let $\hat{\mathfrak{g}}^\Sigma$ denote the Lie algebra of gauge transformations centrally extended by the cocycle (2), and $\hat{G}^\Sigma$ be the corresponding group. The infinite-dimensional space $\hat{\mathfrak{g}}^\Sigma$ is the space of pairs (ϵ, z), where ϵ is a $\mathfrak{g}$-valued function on the surface Σ and z is a real number.

We define the space $(\hat{\mathfrak{g}}^\Sigma)^*$, dual to $\hat{\mathfrak{g}}^\Sigma$, as consisting of triples (F, C, x), where F is a $\mathfrak{g}$-valued 2-form on Σ, C is a $\mathfrak{g}$-valued 1-form on the boundary of Σ, and x is a real number. The nondegenerate pairing $<,>$ between the spaces $\hat{\mathfrak{g}}^\Sigma$ and $(\hat{\mathfrak{g}}^\Sigma)^*$ is the following:

$$< (F, C, x), (\epsilon, z) >= \int_\Sigma \mathrm{tr}(\epsilon F) - \int_{\partial\Sigma} \mathrm{tr}(\epsilon C) + zx\ .$$

Let us consider the action of $\hat{G}^\Sigma$ on $\mathcal{A}^\Sigma$ generated by the action of G^Σ. That is to say, the center of $\hat{G}^\Sigma$ acts trivially.

Proposition 1.5. *The centrally extended group $\hat{G}^\Sigma$ of gauge transformations acts on $\mathcal{A}^\Sigma$ in a Hamiltonian way. The momentum map for the action of the corresponding gauge algebra $\hat{\mathfrak{g}}^\Sigma$ is the mapping $\mathcal{A}^\Sigma \to (\hat{\mathfrak{g}}^\Sigma)^*$ given by the curvature and by the restriction of the connection form to the boundary:*

$$A \mapsto (dA + A \wedge A,\ A|_{\partial\Sigma},\ 1)\ .$$

Let us introduce the following notation. For a manifold Σ and its submanifold $\Gamma \subset \Sigma$ denote by G^Σ_Γ the group of gauge transformations on Σ "based on Γ": $G^\Sigma_\Gamma = \{g \in C^\infty(\Sigma, G) \mid g|_\Gamma = \mathrm{id}\}$, and by $\mathfrak{g}^\Sigma_\Gamma$ the corresponding Lie algebra.

Modifying slightly the last proposition one gets the following

Corollary 1.6. *The group G^Σ_Γ acts on $\mathcal{A}^\Sigma$ in a Hamiltonian way. The momentum map for the action of the corresponding Lie algebra $\mathfrak{g}^\Sigma_\Gamma$ is the mapping $\mathcal{A}^\Sigma \to (\mathfrak{g}^\Sigma_\Gamma)^*$ given by the curvature:*

$$A \mapsto dA + A \wedge A\ .$$

Remark. Note that the group G^Σ_Γ is not centrally extended, but still $G^\Sigma_\Gamma \subset \hat{G}^\Sigma$.

Now consider the Hamiltonian reduction $\mathcal{A}^\Sigma // G^\Sigma_\Gamma$ of the space of connections $\mathcal{A}^\Sigma$ with respect to the group G^Σ_Γ of gauge transformations equal to the identity on the boundary $\Gamma = \partial\Sigma$. This yields the space of flat connections on Σ modulo gauge transformations from G^Σ_Γ,

$$\mathcal{M}_{\Sigma,\Gamma} = \{d + A \in \mathcal{A}^\Sigma \mid dA + A \wedge A = 0\}/G^\Sigma_\Gamma\ .$$

By definition of Hamiltonian reduction, the space $\mathcal{M}_{\Sigma,\Gamma}$ is symplectic (though, certainly, infinite-dimensional and, generally speaking, with singularities). It can be mapped to certain familiar Poisson manifolds. It is well known that the space of G-connections on a circle can be identified with the space of coadjoint representation of the affine Kac–Moody algebra equipped with the standard Kirillov–Kostant Poisson structure. The relation with the Poisson (in fact, symplectic) structure on $\mathcal{M}_{\Sigma,\Gamma}$ is given by the following proposition.

Proposition 1.7. *The mapping from the space $\mathcal{M}_{\Sigma,\Gamma}$ to the Kac–Moody coadjoint representation space sending a flat connection on the surface Σ to its restriction to a boundary component is a Poisson mapping.*

Proof. This mapping is essentially the momentum mapping for the action of gauge transformations on the boundary. □

Remarks. i) Here and below we always mean the nonsingular parts of the moduli spaces when describing the symplectic (or Poisson) structures on them.

ii) We are going to describe now the quotient of our symplectic manifold by the Hamiltonian action of a group. The result is always a *Poisson* manifold, while the momentum map helps to determine the symplectic leaves in it (see, ref. [We]).

Consider the quotient of the space $\mathcal{M}_{\Sigma,\Gamma}$ by the whole group $\hat{G}^{\Sigma}$ of centrally extended gauge transformations. The latter group acts on $\mathcal{M}_{\Sigma,\Gamma}$ since gauge transformations equal to the identity on the boundary form a normal subgroup G^{Σ}_{Γ} in $\hat{G}^{\Sigma}$. The quotient space

$$\mathcal{M}_{\Sigma} = \{d + A \in \mathcal{A}^{\Sigma} \mid dA + A \wedge A = 0\}/\hat{G}^{\Sigma}$$

is a finite-dimensional Poisson manifold (with singularities).

Proposition 1.8. *The space $\mathcal{M}_{\Sigma}$ of flat G-connections modulo gauge transformations on a surface Σ with holes inherits a Poisson structure from the space of all (smooth) G-connections. The symplectic leaves of this structure are parameterized by the conjugacy classes of holonomies around the holes (that is, a symplectic leaf is singled out by fixing the conjugacy class of the holonomy around each hole).*

Proof. The symplectic leaves of $\mathcal{M}_{\Sigma}$ are in one-to-one correspondence with the coadjoint orbits of the (centrally extended) affine Lie algebra on a circle (or the direct sum of several copies of the affine algebras if the boundary of the surface Σ consists of several components). These coadjoint orbits, in turn, are parameterized by the conjugacy classes of holonomies around the circle. $\square$

Remark. The above proposition should not be understood as that the conjugacy classes of holonomies around the holes can be taken arbitrary, since the holonomies of a flat connection on the surface obey certain relations coming from the fundamental group $\pi_1(\Sigma)$. For example, if Σ is a sphere with n holes then the product of all n holonomies has to be id $\in G$ (provided one has chosen the same base point and a convenient orientation for all n loops encircling the holes).

§2. The Stokes–Leray formula.

In order to develop the symplectic geometry related with holomorphic bundles on complex surfaces in a way analogous to what have been considered in the last section we will need a complex analogue of the Stokes formula. This will be nothing but a simple multidimensional generalization of the Cauchy formula.

Higher-dimensional residue. Let γ be a meromorphic n-form on a compact complex n-dimensional manifold M with poles on a smooth complex hypersurface $N \subset M$. Here and below we consider the forms with logarithmic singularities only (i.e., with the first order poles).[1] Let f be a function defining N in a neighborhood

[1] In this paper we restrict ourselves to the case of top-degree meromorphic forms having singularities on smooth divisors. In this case a logarithmic singularity is the same as a first order pole. It is the formulation "γ with logarithmic singularities" which should be kept if one would like to consider the situation of a non-smooth divisor of poles.

of some point $p \in N$. Then locally, in a certain neighborhood $U(p)$, the n-form γ can be decomposed into the sum

$$\gamma = \frac{df}{f} \wedge \alpha + \beta \ ,$$

where α and β are holomorphic in $U(p)$. One can show, that the restriction $\alpha|_N$ is a well-defined (i.e. independent of f) holomorphic $(n-1)$-form on N.

Definition 2.1. The holomorphic $(n-1)$-form $\alpha|_N$ on N is called the residue of the meromorphic form γ and is denoted by $\operatorname{res}_N \gamma$.

Proposition 2.2. *Let M, N and γ be as above, and let u be a smooth $(n-1)$-form on M. Then the form $\gamma \wedge du$ is integrable on M, and*

$$\int_M \gamma \wedge du = 2\pi i \int_N \operatorname{res}_N \gamma \wedge u \ . \tag{3}$$

Remarks. i) The formula (3) is proved by applying the Stokes formula to reduce the integral to the tubular neighborhood of N, and then by using the standard Cauchy formula in the transversal direction to N (see, e.g., ref. [GS]).

ii) This relation can be, of course, generalized to the case when N is a normal crossing divisor in M by modifying the above definition of a residue. Then, in particular, $\operatorname{res} \gamma$ will define *meromorphic* forms on smooth components of N, the residues of which (i.e. residues of residues) will sum to zero at the intersections of components.

iii) We call this formula the Stokes–Leray formula for it is a part of a much broader, than explained here, Leray theory (cf., ref. [Le]), while, on the other hand, we are going to exploit it as a complex analogue of the usual Stokes formula.

iv) Of course, it is only the $(0, n-1)$-part of the form u which is essential in eq.(3). In the same way, we can write $\bar\partial u$ instead of du:

$$\int_M \gamma \wedge \bar\partial u = 2\pi i \int_N \operatorname{res}_N \gamma \wedge u \ . \tag{3'}$$

Digression on homology and cohomology. The last remark leads one to a simple idea of considering a pairing of Dolbeault cochains with certain "geometric chains", and thus to the construction of the corresponding homology theory, which we will describe here roughly (and in a full detail in ref. [KR]).

Let v be a smooth $(0, k)$-form on a complex manifold M and let (X, α) be a pair consisting of a smooth k-dimensional complex submanifold X in M and a meromorphic k-form α on X possessing logarithmic singularities on a smooth hypersurface Y in X. Now let us consider the pairing between $(0, k)$-forms and the set of such pairs given by the integral

$$\int_X \alpha \wedge v \ . \tag{4}$$

Note that the meromorphic top-degree form α on X is the data which allow us to integrate $(0,k)$-forms over submanifolds of *complex* dimension k, (i.e., to integrate v over X). Therefore, this meromorphic form α can be regarded as a holomorphic analogue of *orientation* of the submanifold X. Furthermore, if $v = \bar{\partial} u$, then, by use of eq.(3′), the integral over X is reduced to the integral over its submanifold Y of one complex dimension less:

$$(3'') \qquad \int_X \alpha \wedge \bar{\partial} u = 2\pi i \int_Y \mathrm{res}_Y\, \alpha \wedge u\ .$$

Thus we can speak of the pair $(Y,\ \mathrm{res}_Y\, \alpha)$ as a holomorphic analogue of the boundary for the pair (X, α). In this sense, eq.(3″) can be viewed on as a complex analogue of the Stokes formula. Suppose now that the pair (X, α) is an analogue of a closed manifold, i.e., that α is holomorphic. Then the integral (4) for a $\bar{\partial}$-closed form v depends in fact only on the Dolbeault cohomology class of v. This line of reasoning can be developed to a homology theory (ref. [KR]) which plays the same role with respect to Dolbeault cohomology as singular homology plays with respect to De Rham cohomology.

Remark. Thus, a pair (M, γ) with a *holomorphic* form γ of degree equal to $\dim_{\mathbb{C}} M$ can be regarded as a holomorphic analogue of an *oriented closed* manifold. Sometimes it is necessary to require that γ *has no zeros* (then M has to be a Calabi–Yau or an abelian manifold) in which case one can speak of a holomorphic analogue of a *smooth oriented closed* manifold. If γ is meromorphic (rather than holomorphic) with only first order poles one can speak of "a manifold with boundary" (and the above remark about the possible zeros of γ applies in this case as well).

In the next section we would like to exploit the above understanding of what are the proper holomorphic analogues of orientation and boundary (a similar point of view was useful in refs. [FK, DT] for other gauge-theoretic constructions).

§3. Complex case: Poisson structures on moduli spaces of holomorphic bundles.

Let S be a compact *complex* surface ($\dim_{\mathbb{C}} S = 2$). We are going to describe a Poisson structure on the moduli space of holomorphic vector bundles[2] with a *complex* reductive group G as the structure group ($G \subset GL(n, \mathbb{C})$) on S. Let us do this in analogy with the consideration of flat connections in §1. First of all, in order to define an analogue of the symplectic structure in eq.(1) we have to fix a holomorphic analogue of the orientation. According to the heuristic argument in §2, we have to choose a meromorphic 2-form on the surface S. Let σ be a meromorphic 2-form on S, such that its divisor of poles P is a smooth curve in S

[2] By the moduli space we shall always understand a local universal family near a smooth point.

and that σ has there a logarithmic singularity. The curve P will play the role of the boundary of the surface in our considerations. Let us assume additionally that σ has no zeros (the situation analogous to a smooth oriented real surface). Then P is an anticanonical divisor on S and has to be an elliptic curve, or, may be, a number of nonintersecting elliptic curves. These are analogous to the circles constituting the boundary of a real smooth surface. In what follows we shall assume that S is endowed with such a 2-form σ. (Example: $S = \mathbb{C}P^2$ with a smooth cubic as an anticanonical divisor. As a matter of fact, many Fano surfaces fall into this class.) If it happens that σ has no zeros and no poles (i.e., S is "oriented, without boundary") it means that we deal with either a K3 or an abelian surface. (Note that the further consideration can be extended with minimal changes to the case of a non-smooth divisor P, in particular, to P consisting of several components intersecting transversally. Example: $S = \mathbb{C}P^2$ with $\sigma = dxdy/xy$.)

Let E be a smooth vector G-bundle over S which can be endowed with a holomorphic structure and $\operatorname{End} E$ be the corresponding bundle of endomorphisms with the fiber $\mathfrak{g} = \operatorname{Lie}(G)$. Let $\mathcal{A}^S$ denote the infinite-dimensional affine space of smooth $\bar\partial$-connections in E. By choosing a reference holomorphic structure $\bar\partial_0$, $\bar\partial_0^{\,2} = 0$, in E, the space $\mathcal{A}^S$ can be identified with the vector space $\Omega^{(0,1)}(S, \operatorname{End} E)$ of $(\operatorname{End} E)$-valued $(0,1)$-forms on S, i.e.

$$\mathcal{A}^S = \{\bar\partial_0 + A \,|\, A \in \Omega^{(0,1)}(S, \operatorname{End} E)\} \ .$$

In what follows, instead of $\bar\partial_0$, we shall write simply $\bar\partial$ keeping in mind that this corresponds to a reference holomorphic structure in E when it applies to sections of E or associated bundles.

Definition 3.1. The space $\mathcal{A}^S$ possesses a natural holomorphic symplectic structure

$$W_{\mathbb{C}} := \int_S \sigma \wedge \operatorname{tr}(\delta A_1 \wedge \delta A_2) \ ,$$

where σ is the holomorphic "orientation" of S, while the other notations are essentially the same as in Definition 1.1 above.

After such a definition we can repeat the contents of §1 more or less word by word.

Proposition 3.2. *The symplectic structure $W_{\mathbb{C}}$ is invariant with respect to the gauge transformations*

$$A \mapsto g^{-1} A g + g^{-1} \bar\partial g \ ,$$

where g is an element of the group of gauge transformations, i.e., the group of automorphisms of the smooth bundle E. Abusing notation we denote this group by G^S.

The infinitesimal gauge transformations forming the Lie algebra $\mathfrak{g}^S = \Gamma(S, \mathrm{End}\, E)$ (where Γ denotes the space of C^∞-sections) are generated on the symplectic manifold $\mathcal{A}^S$ by certain Hamiltonian functions.

Proposition 3.3. *An infinitesimal gauge transformation ϵ is generated by the Hamiltonian function*

$$H_\epsilon = \int_S \sigma \wedge \mathrm{tr}(\epsilon\, (\bar\partial A + A \wedge A)) - 2\pi i \int_P \mathrm{res}_P\ \sigma \wedge \mathrm{tr}(\epsilon A)\ .$$

Proof. It is the proof of this Proposition where the Stokes–Leray formula of §2 is used instead of the usual Stokes formula being the only modification in comparison with the proof of Proposition 1.3 in §1. Now we have

$$\begin{aligned} \delta H_\epsilon &= \int_S \sigma \wedge \mathrm{tr}\,(\epsilon\, \delta F) - \int_P \mathrm{res}_P\ \sigma \wedge \mathrm{tr}(\epsilon\, \delta A) \\ &= \int_S \sigma \wedge \mathrm{tr}\,(\epsilon\, \nabla_A\, \delta A) - \int_P \mathrm{res}_P\ \sigma \wedge \mathrm{tr}(\epsilon\, \delta A) = \int_S \sigma \wedge \mathrm{tr}\,\left(\delta A \wedge \overline{\nabla}_A\, \epsilon\right). \end{aligned}$$

Here $\overline{\nabla}_A\, \epsilon = \bar\partial\epsilon + [A, \epsilon]$ is an infinitesimal gauge transformation of a $\bar\partial$-connection. □

In the same way, the commutation relations of these Hamiltonians get centrally extended:

$$\{H_{\epsilon_1}, H_{\epsilon_2}\} = H_{[\epsilon_1, \epsilon_2]} + c(\epsilon_1, \epsilon_2)\ ,$$

by the following cocycle (cf., eq.(2)):

Proposition 3.4.

$$c(\epsilon_1, \epsilon_2) = 2\pi i \int_P \mathrm{res}_P\ \sigma \wedge \mathrm{tr}(\epsilon_1 \bar\partial \epsilon_2)\ . \tag{5}$$

Let $\hat{\mathfrak{g}}^{(S,\sigma)}$ denote the Lie algebra of gauge transformations on S centrally extended by the cocycle (5), and $\hat G^{(S,\sigma)}$ be the corresponding group (cf., [FK])[3]. The infinite-dimensional space $\hat{\mathfrak{g}}^{(S,\sigma)}$ is the space of pairs (ϵ, z), where ϵ is a $\mathfrak{g}$-valued function on the surface S and z is a complex number.

We will take the following space of triples (F, C, x) as the space $(\hat{\mathfrak{g}}^{(S,\sigma)})^*$ dual to $\hat{\mathfrak{g}}^{(S,\sigma)}$. Here F is a $(\mathrm{End}\, E)$-valued (0,2)-form on S, C is a $(\mathrm{End}\, E)$-valued (0,1)-form on the "boundary" (i.e., polar set) P of S, and x is a complex number. The nondegenerate pairing $<,>$ between the spaces $\hat{\mathfrak{g}}^{(S,\sigma)}$ and $(\hat{\mathfrak{g}}^{(S,\sigma)})^*$ is the following:

$$< (F, C, x), (\epsilon, z) > = \int_S \sigma \wedge \mathrm{tr}(\epsilon F) - 2\pi i \int_P \mathrm{res}_P\ \sigma \wedge \mathrm{tr}(\epsilon C) + zx\ .$$

Let us consider the action of $\hat G^{(S,\sigma)}$ on $\mathcal{A}^S$ generated by the action of G^S. That is to say, the center of $\hat G^{(S,\sigma)}$ acts trivially.

[3] It is not important which of the possible central extensions of the group G^S will be taken here since it is only the Lie algebra that matters. However, if one will be able to consider the quantum counterpart of the Poisson geometry a difference may appear.

Proposition 3.5. *The centrally extended group of gauge transformations, $\hat{G}^{(S,\sigma)}$, acts on $\mathcal{A}^S$ in a Hamiltonian way. The momentum map for the action of the corresponding gauge algebra $\hat{\mathfrak{g}}^{(S,\sigma)}$ is the mapping $\mathcal{A}^S \to (\hat{\mathfrak{g}}^{(S,\sigma)})^*$ given by the $(0,2)$-curvature and by the restriction of the $\bar{\partial}$-connection form to the "boundary":*

$$A \mapsto (\bar{\partial}A + A \wedge A,\ A|_P,\ 1)\ .$$

Let us denote, as before, by G^S_P the group of gauge transformations on S based on P: $G^S_P = \{g \in G^S \mid g|_P = \mathrm{id}\}$, and by $\mathfrak{g}^S_P$ the corresponding Lie algebra.

Modifying slightly the last proposition, one obtains the following

Corollary 3.6. *The group G^S_P acts on $\mathcal{A}^S$ in a Hamiltonian way. The momentum map for the action of the corresponding Lie algebra $\mathfrak{g}^S_P$ is the mapping $\mathcal{A}^S \to (\mathfrak{g}^S_P)^*$ given by the curvature:*

$$A \mapsto \bar{\partial}A + A \wedge A\ .$$

Remark. Note that the group G^Σ_Γ is not centrally extended, but still $G^S_P \subset \hat{G}^{S,\sigma}$.

Consider now the (holomorphic) Hamiltonian reduction $\mathcal{A}^S//G^S_P$ of the space of $\bar{\partial}$-connections $\mathcal{A}^S$ with respect to the group G^S_P. The result will be the space of integrable $\bar{\partial}$-connections in the bundle E on S modulo gauge transformations from G^S_P. Such connections, with vanishing $(0,2)$-form of the curvature tensor, are in one-to-one correspondence with *holomorphic structures* in the complex bundle E. Thus the holomorphic Hamiltonian reduction leads us to the consideration of the *space of all holomorphic structures in the bundle E modulo gauge equivalence trivial on P*. The corresponding quotient space $\mathcal{M}_{S,P}$, which we consider only locally, near some of its smooth points, is, by construction, an (infinite-dimensional) symplectic manifold.

On the other hand, let us consider now the space of holomorphic structures in a smooth bundle on a complex one-dimensional manifold by taking P as such a manifold and $E|_P$ as the bundle on P:

$$\mathcal{C} := \{\bar{\partial}_0 + C \mid C \in \Omega^{(0,1)}(P, \mathrm{End}\, E|_P)\}\ .$$

Here $\bar{\partial}_0$ is also understood as the restriction to P of our reference holomorphic structure. The space $\mathcal{C}$ of holomorphic structures in a bundle on an elliptic curve (or a sum of such spaces if P consists of several disjoint components) is in fact an affine subspace in a vector space dual to the Lie algebra $\hat{\mathfrak{g}}^{P,\beta}$. The latter is defined as the central extension of $\mathfrak{g}^P = \Gamma(P, \mathrm{End}\, E|_P)$ by the cocycle

$$c_\beta(\epsilon_1, \epsilon_2) = 2\pi i \int_P \beta \wedge \mathrm{tr}(\epsilon_1 \bar{\partial} \epsilon_2)\ .$$

This, of course, should be compared with eq.(5). In what follows we set $\beta = \mathrm{res}_P\, \sigma$ in which case $\hat{\mathfrak{g}}^{P,\beta} = \hat{\mathfrak{g}}^{S,\sigma}/\mathfrak{g}^S_P$. The Lie algebra $\hat{\mathfrak{g}}^{P,\beta}$ (the two-dimensional current algebra, in terminology of refs. [EF, FK], or "double loop algebra") plays the role of the loop algebra, which appeared in §1, while $\mathcal{C}$ plays the role of the space of connections on a circle. This point of view was suggested in refs. [EF, FK], where more details can be found. We mention only the pairing which defines $\mathcal{C}$ as an affine subspace of the dual space to $\hat{\mathfrak{g}}^{P,\beta}$:

$$<(\epsilon, z)\,,(C, x)>= 2\pi i \int_P \beta \wedge \mathrm{tr}(\epsilon C) + zx\,, \text{ where}$$
$$(\epsilon, z) \in \hat{\mathfrak{g}}^{P,\beta}\,, (\bar{\partial}_0 + C) \in \mathcal{C}, \text{ and } x, z \in \mathbb{C}\,.$$

Thus, the pairs $(C, 1)$ define an affine subspace in $(\hat{\mathfrak{g}}^{P,\beta})^*$. We shall consider the standard Kirillov–Kostant Poisson structure on $(\hat{\mathfrak{g}}^{P,\beta})^*$. Its symplectic leaves are, as always, the coadjoint orbits, which in our case correspond to isomorphism classes of holomorphic bundles on P.

Proposition 3.7. *The mapping from the space $\mathcal{M}_{S,P}$ to the coadjoint representation space, $(\hat{\mathfrak{g}}^{P,\beta})^*$, sending an integrable $\bar{\partial}$-connection on the surface S to its restriction to P is a Poisson mapping.*

Proof. This mapping is essentially the momentum mapping for the action of gauge transformations on the boundary. □

Now we consider the quotient of the space $\mathcal{M}_{S,P}$ by the whole group $\hat{G}^{S,\sigma}$ of centrally extended gauge transformations. The latter group acts on $\mathcal{M}_{S,P}$ since gauge transformations equal to the identity on P form a normal subgroup G^S_P in $\hat{G}^{S,\sigma}$.

The quotient space

$$\{\bar{\partial} + A \in \mathcal{A}^S \,|\, \bar{\partial}A + A \wedge A = 0\}/\hat{G}^{(S,\sigma)}$$

represents the set of isomorphism classes of holomorphic bundles on S (corresponding to a given underlying topological bundle E). Then, by construction, the local smooth moduli space $\mathcal{M}_S$ of holomorphic bundles on S is a finite-dimensional Poisson manifold. The symplectic leaves in it are described in terms of coadjoint orbits in $\left(\hat{\mathfrak{g}}^{P,\beta}\right)^*$ as follows (ref. [FKR]).

Proposition 3.8. *The local moduli space $\mathcal{M}_S$ of holomorphic bundles possesses a (holomorphic) Poisson structure. The symplectic leaves of this structure are parameterized by the moduli of their restrictions to the anticanonical divisor $P \subset S$ (that is a symplectic leaf is singled out by fixing the isomorphism class of the restriction to the elliptic curve P, or the isomorphism classes of restrictions to each curve if P consists of several such curves).*

Remarks. i) With minor modifications the above proposition holds for P consisting of several components intersecting transversally. In the latter case the corresponding gauge group $\hat{G}^{P,\beta}$ is the current group on a punctured Riemann surface, described in ref. [FK].

ii) The above proposition should not be understood as that the isomorphism classes of bundles on P can be taken arbitrary; rather they have to satisfy the condition that they arise as restrictions of bundles defined over S.

Another description of symplectic structure. Here we give an alternative description of the symplectic structure on the symplectic leaves mentioned in Proposition 3.8.

Let F be a holomorphic bundle on S corresponding to a smooth point in $\mathcal{M}_S$, e.g., $H^2(S, \mathrm{End}\, F) = 0$ (we assume here that the structure group is now $G = GL(n, \mathbb{C})$). Let K_S denote the canonical line bundle of S, so that by assumption K_S^{-1} possesses a holomorphic section, say, η and P is the divisor of zeros of η. Denote the sheaves of holomorphic sections of the bundles under consideration by the same symbols as the bundles themselves. We can write down the following exact sequence of sheaves:

$$0 \to \mathrm{End}\, F \otimes K_S \to \mathrm{End}\, F \to \mathrm{End}\, F \otimes \mathcal{O}_P \to 0 \ ,$$

where the second map is given by the multiplication by η, while $\mathcal{O}_P$ is the structure sheaf of the submanifold P. The associated long exact sequence reads as follows:

$$\begin{aligned} 0 \to & H^0(S, \mathrm{End}\, F) \to H^0(P, \mathrm{End}\, F|_P) \to \\ & H^1(S, \mathrm{End}\, F \otimes K_S) \to H^1(S, \mathrm{End}\, F) \to H^1(P, \mathrm{End}\, F|_P) \to \\ & H^2(S, \mathrm{End}\, F \otimes K_S) \to 0 \ , \end{aligned} \tag{6}$$

where we have used Serre duality to set $H^0(S, \mathrm{End}\, F \otimes K_S) = (H^2(S, \mathrm{End}\, F))^* = 0$. [4]

The cohomology group $H^1(S, \mathrm{End}\, F)$ represents the space of infinitesimal deformations of F. It describes the tangent space to $\mathcal{M}_S$ at the point F [5], while the space tangent to the symplectic leaf $\mathcal{S}_F$ at F should correspond (according to Proposition 3.8) to such deformations of F which leave $F|_P$ unchanged. Thus, that tangent space corresponds to the kernel of the restriction map

$$T_F\, \mathcal{S}_F = \ker\left(H^1(S, \mathrm{End}\, F) \to H^1(P, \mathrm{End}\, F|_P) \right) ,$$

or, by the exactness of (6), to the quotient

$$T_F\, \mathcal{S}_F = H^1(S, \mathrm{End}\, F \otimes K_S)/I \ ,$$

[4] The cohomology groups $H^0(S, \mathrm{End}\, F)$ and $H^2(S, \mathrm{End}\, F \otimes K_S)$ appearing also in eq.(6) are of equal dimension by Serre duality and are isomorphic to $\mathbb{C}$ if we further require F to be a *simple* bundle. The groups $H^1(S, \mathrm{End}\, F)$ and $H^1(S, \mathrm{End}\, F \otimes K_S)$ are also related by Serre duality.

[5] A description of the Poisson bivector on $\mathcal{M}_S$ in these terms was given in refs. [Mu, Bon, Bot].

where

$$I = \operatorname{im}\left(H^0(P, \operatorname{End} F|_P) \to H^1(S, \operatorname{End} F \otimes K_S) \right) .$$

Now consider the pairing

$$H^1(S, \operatorname{End} F) \otimes H^1(S, \operatorname{End} F \otimes K_S) \to H^2(S, K_S) \cong \mathbb{C} ,$$

induced by the multiplication in cohomology and taking trace in $\operatorname{End} F$. This map can be restricted in the first factor to the subspace $T_F\,\mathcal{S}_F \subset H^1(S, \operatorname{End} F)$, which gives us

$$T_F\,\mathcal{S}_F \otimes H^1(S, \operatorname{End} F \otimes K_S) \to \mathbb{C} .$$

By considering the second factor, one observes that this pairing vanishes on the subspace $T_F\,\mathcal{S}_F \otimes I$, thus, descending to a map of $T_F\,\mathcal{S}_F \otimes \big(H^1(S, \operatorname{End} F \otimes K_S)/I\big)$ or, finally,

$$T_F\,\mathcal{S}_F \otimes T_F\,\mathcal{S}_F \to \mathbb{C} . \tag{7}$$

One can check now that this pairing is skew-symmetric and defines a 2-form on $\mathcal{S}_F$ which coincides with the symplectic structure discussed in Proposition 3.8.

It is more difficult to prove the closedness of the 2-form (7) in the latter description than by means of the Hamiltonian reduction discussed above. Note also that the alternative construction also has an analogue for the moduli space of flat connections, which is recovered by substituting locally constant sheaves instead of sheaves of holomorphic sections.

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