COMBINATORIAL FORMULAS FOR COHOMOLOGY OF SPACES OF KNOTS

V.A. VASSILIEV

Abstract. An algorithmic method of finding combinatorial formulas for knot invariants and other cohomology classes of spaces of knots in $\mathbb{R}^n$, $n \geq 3$, is described. In the case of invariants of knots in $\mathbb{R}^3$, we have a formal algorithm whose input is an arbitrary *weight system*, i.e. a possible principal part of a finite type invariant, and the output is either a proof of the fact that this weight system actually does not correspond to any knot invariant or an effective description of some invariant with this principal part, namely a finite collection of easy subvarieties of full dimension in the space of spatial curves such that the value of this invariant on any generic knot is equal to the sum of multiplicities of these varieties in a neighborhood of the knot. (In examples, the former possibility never occurs.) This algorithm is formally realized over $\mathbb{Z}_2$, but its generalization to the case of arbitrary coefficients is just a technical task.

A similar method of realizing higher dimensional cohomology classes of spaces of knots is not completely formalized yet, however it always works successively in particular examples of comparatively low dimensional cohomology classes; it has proved the existence of several positive-dimensional classes predicted by algebraic computations.

This method is based on a deep analogy between the knot theory and the theory of plane arrangements,

Key words: knot invariant, combinatorial formula, knot space, plane arrangement, simplicial resolution, discriminant

1. Introduction

The study of knot invariants is only a subproblem of the more natural problem on the cohomology ring of the space of knots. Indeed, the numerical invariants of knots in M^3 are just the 0-dimensional cohomology classes of this space. Recent works by V. Turchin show that the finite type cohomology ring of the space of knots in $\mathbb{R}^3$ has a beautiful algebraic structure; probably the easiest description of the group of finite type invariants can be derived from it by an obvious factorization, see [25], [26], and § 7 below.

It is very convenient to consider simultaneously the spaces of knots in all spaces $\mathbb{R}^n$, $n \geq 3$. If $n > 3$ then all their cohomology classes are of finite type (filtration) in the sense of [29], [32] and can be calculated by the spectral sequence introduced in these works.

We shall mainly consider the spaces of *long knots* in $\mathbb{R}^n$, i.e. of embeddings $\mathbb{R}^1 \to \mathbb{R}^n$ coinciding with a fixed linear embedding outside some segment in $\mathbb{R}^1$, see Fig. 1. The cohomology rings of spaces of long knots and usual "compact" knots (i.e. smooth embeddings $S^1 \to \mathbb{R}^n$) are closely related. The invariants of knots of both types in $\mathbb{R}^3$ are in a natural one to one correspondence, but in higher dimensions the cohomology ring of the space of compact knots is more complicated than that for long knots.

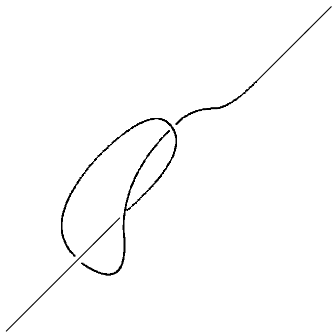


FIGURE 1. A long knot

Any cohomology class of the space of knots (both long or compact) can be realized by the linking number with some cycle (of infinite dimension but finite codimension) in the *discriminant space* of all maps $\mathbb{R}^1 \to \mathbb{R}^n$ or $S^1 \to \mathbb{R}^n$ which are not the knots, i.e. have singular or intersection points.

Definition 1 (see §2 below)**.** Given a cohomology class of the space of knots in $\mathbb{R}^n$ (e.g. a knot invariant if $n = 3$), a *combinatorial formula* for it is an arbitrary *easy subalgebraic relative cycle* in the space of curves modulo the discriminant space Σ, such that our class is equal to the linking number with the boundary of this cycle.

There is a deep analogy between the knot theory and theory of *affine plane arrangements*, i.e. of finite families of affine planes of arbitrary (maybe different) dimensions in $\mathbb{R}^N$. This analogy is determined by the fact that the discriminant Σ in the space $\mathcal{K}$ of curves also is swept out by a family of planes in $\mathcal{K}$. This family is not discrete: it is parametrized by all unordered pairs of points $a, b \in \mathbb{R}^1$. Namely, to any such pair the plane $L(a, b)$ corresponds consisting of maps $\mathbb{R}^1 \to \mathbb{R}^n$ such that $f(a) = f(b)$ if $a \neq b$ or $f'(a) = 0$ if $a = b$.

Short lists of parallel notions of both theories are summarized in [34], [35].

In both theories, it is extremely useful to consider the *simplicial resolutions* of discriminants (respectively, arrangements), and in particular the related spectral sequences. Our method of constructing the combinatorial formulas for cohomology of the knot space $\mathcal{K} \setminus \Sigma$ is nothing else than the direct calculation of such a spectral sequence.[1]

All other methods known to the author of finding combinatorial formulas of all knot invariants deal in an essential way with spatial pictures, drawing the knot diagrams and watching their homotopies. By making a priori geometric choices, our method is purely combinatorial and deals only with easily encodable events such as chord diagrams and their natural generalizations.

[1]By the calculation of a homological spectral sequence one usually means the calculation of its isomorphism class, in particular the existence theorem claiming that any element of the calculated group $E^\infty_{p,q}$ can be extended to a cycle with this principal part; the proof of this theorem usually is implicit and follows from vanishing of all homological obstructions to such an extension. By the *direct* calculation I mean an explicit step by step construction of such cycles.

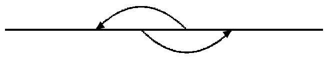


FIGURE 2. Polyak-Viro formula for the Casson invariant

I shall show here how does this method calculate the simplest non-trivial knot invariant (of order 2). Also we present the results of its work for the next invariant (of order 3), for the Teiblum–Turchin $(3n-8)$-dimensional cocycle (also of order 3) of the space of long knots in $\mathbb{R}^n$, and for all cohomology classes of order ≤ 2 of the spaces of compact knots $S^1 \hookrightarrow \mathbb{R}^n$.

A spectral sequence providing cohomology classes of the space of long knots in $\mathbb{R}^3$ was introduced in [29]. It defines some natural filtration (degree) of these classes. In the simplest case of 0-dimensional classes this filtration has an easy characterization in terms of induced indices of knots with finitely many transverse intersection points, see e.g. §0.2 in [29] and also [2]. The simplest positive-dimensional cohomology class following from this spectral sequence was discovered by D.M. Teiblum and V.E. Turchin in 1995. Again, this was rather an existence theorem: the calculation of the term $E_1^{-3,4}$ of the spectral sequence together with the fact that all the groups into which the differentials of this term can act are trivial. Moreover, it was not known whether this class actually is nontrivial.

Immediate generalizations $E_r^{p,q}(n)$ of this spectral sequence calculate all cohomology groups of spaces of knots in $\mathbb{R}^n$, $n > 3$. These sequences have a natural periodicity property: if m and n are of the same parity or we calculate $\mathbb{Z}_2$-cohomology, then $E_1^{p,q} \sim E_1^{p,q+p(m-n)}$; for m and n of different parities these groups are "super"analogues of one another. Some first calculations, including the extension of the Teiblum–Turchin class to the case of even n and the calculation of all cohomology classes of degree ≤ 2 of the space of compact knots, were given in [33], [31].

A special part of the classes arising from these spectral sequences was then studied in [7]: it are exactly the classes occurring as stabilizations of knot invariants. Also, in [7] a different filtration on the cohomology ring of the space of knots was introduced under the name "Vassiliev-order". Its value on a cohomology class always is no greater than our filtration, but often is strictly smaller: for instance, the "Vassiliev-order" of the Teiblum–Turchin class is equal to zero.

The talk follows my papers [34] and [36].

I thank A.B. Merkov and referee very much for many remarks. This work was supported in part by grants RFBR–01-01-00660, INTAS–00-0259, and NWO–047-008-005.

2. What is a combinatorial formula for a cohomology class of the space of knots

This is a comment to the above Definition 1. First, let us consider an example.

Probably the first non-trivial combinatorial formulas for certain finite type knot invariants were given by J. Lannes in [17]. For some other approaches to combinatorial formulas see e.g. [6], [20], [4], [24], [5]. The best-known and convenient combinatorial expressions for such invariants are the *Polyak–Viro formulas* [21] represented by pictures like Fig. 2 or linear combinations of similar pictures. Fig. 2 should be read as follows. Consider a generic long knot $f : \mathbb{R}^1 \to \mathbb{R}^3$. A *representation* of the picture of Fig. 2 in this knot is any collection of points $a < b < c < d \subset \mathbb{R}^1$ such that $f(a)$ lies below $f(c)$ and $f(d)$ lies below $f(b)$ (with respect to a chosen direction in $\mathbb{R}^3$). The value of this picture on our knot is equal to the number of its representations counted with appropriate signs. An immediate calculation shows that this number is a knot invariant of order 2. Moreover, for any $n > 3$ the same diagram describes a $2(n-3)$-dimensional cohomology class of the space of long knots in $\mathbb{R}^n$, the unique basic class of filtration 2. General Polyak–Viro diagrams consist of several oriented arcs connecting different points of $\mathbb{R}^1$. M. Goussarov has proved that any finite type invariant of long knots can be represented by a linear combination of such diagrams, see [13].

Let us understand in which sense the above diagram defines a subalgebraic chain.

Consider the Cartesian product $\mathcal{K} \times \mathbb{R}^4$ of the space of curves $f : \mathbb{R}^1 \to \mathbb{R}^n$ and the space of quadruples of points $a, b, c, d \in \mathbb{R}^1$. Then all the above conditions obviously define a collection of linear conditions in this enlarged space: 5 conditions of inequality type and $2(n-1)$ conditions of equality type. The relative cycle expressed by the Polyak–Viro formula is just the direct image of the fundamental cycle of the variety distinguished by these conditions under the projection to $\mathcal{K}$.

The proof of the fact that this direct image actually is a relative cycle, all of whose boundary lies in the discriminant, is slightly more complicated, cf. section 5 below.

In the finite dimensional algebraic geometry, the projections of semialgebraic sets or chains remain in the same class of objects by the Tarski-Seidenberg lemma. In our functional space, we do not have a similar fact, therefore we use the word "subalgebraic" for objects obtained as projections of semialgebraic objects from slightly greater spaces. The word "easy" in Definition 1 means that our cycle is a finite linear combination of subalgebraic chains, any of which is the projection of a semialgebraic one defined by no more inequalities than the filtration of the cohomology class; for infinite type cohomology classes easy formulas probably do not exist.

2.1. **Example: a realization of the Teiblum–Turchin cocycle mod 2.** Combinatorial formulas for cohomology classes of positive dimension can be more complicated: I do not see a general expression for them as easy as the Polyak–Viro formulas. Now we give a formula for the simplest such cohomology class, the Teiblum–Turchin class, found by D.M. Teiblum and V.E. Turchin in 1995 and described in [31], [33]. This class is of dimension $3n - 8$ (in particular of dimension 1 if $n = 3$) and of degree 3 in the sense of the natural filtration in the resolved discriminant.

Our combinatorial formula for this class is shown in Fig. 3. The sense of its three pictures is explained in three items of the following theorem.

Denote by $\mathbb{R}^{n-1}$ the quotient space of $\mathbb{R}^n$ by the chosen direction (defining the notions "above" and "below"). Choose a direction "to the east" in this space $\mathbb{R}^{n-1}$.

Theorem 1 (see [34])**.** *For any $n \geq 3$, the value of the reduced mod 2 Teiblum–Turchin class on any generic $(3n-8)$-dimensional singular cycle in the space of long knots in*

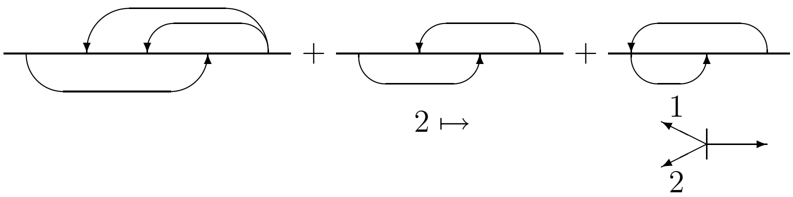


FIGURE 3. Combinatorial formula for the Teiblum–Turchin cocycle

$\mathbb{R}^n$ is equal to the parity of the number of points of this cycle corresponding to knots $f : \mathbb{R}^1 \to \mathbb{R}^n$ such that one of three holds:

1) there are five points $a < b < c < d < e$ in $\mathbb{R}^1$ such that $f(a)$ is above $f(d)$, and $f(e)$ is above both $f(c)$ and $f(b)$;

2) there are four points $a < b < c < d$ in $\mathbb{R}^1$ such that $f(a)$ is above $f(c)$, $f(b)$ is below $f(d)$, and the projection of the derivative $f'(b)$ to $\mathbb{R}^{n-1}$ is directed "to the east";

3) there are three points $a < b < c$ in $\mathbb{R}^1$such that $f(a)$ is above $f(b)$ but below $f(c)$, and the "exterior" angle in $\mathbb{R}^{n-1}$ formed by projections of $f'(a)$ and $f'(b)$ contains the direction "to the east" (i.e. this direction is equal to a linear combination of these projections, and at least one of coefficients in this combination is non-positive).

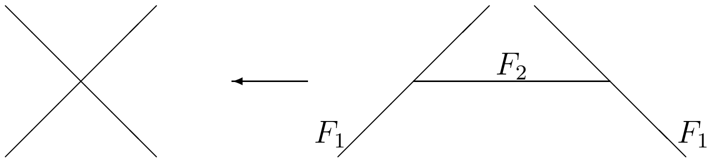
If for some knot f there exist several point configurations $a < b < \cdots$ satisfying these conditions, then the point f of the cycle should be counted for with the corresponding multiplicity.

Corollary 1. *The Teiblum–Turchin cocycle defines a non-trivial cohomology class in the most interesting case $n = 3$.*

(For $n > 3$ the same follows from dimensional reasons.)

Indeed, let us consider the connected sum of two equal (long) trefoil knots in $\mathbb{R}^3$ and a path in the space of knots connecting this knot with itself as in the proof of the commutativity of the knot semigroup: we shrink the first summand, move it "through" the second, and then blow up again. It was proved in [34] that this closed path in the space of long knots has an odd number of intersection points (counted with multiplicities) with the union of three varieties indicated in items 1, 2 and 3 of Theorem 1. But the Teiblum–Turchin cocycle is a well-defined integral cohomology class, see e.g. [31], [33]. We obtain that its reduction mod 2 takes the non-zero value on the reduction mod 2 of a well-defined integral homology class; hence also the pairing of these integral classes is not equal to zero.

Remark 1. There is an unpublished conjecture by R. Budney, F. Cohen and A. Hatcher that the space of long knots in $\mathbb{R}^3$ is a two-fold loop space which is, up to homotopy equivalence, freely generated over the little squares operad by the spaces of prime knots (whose homotopy types are discussed in [14]). The above proof of Corollary 1 shows that this little square action is non-trivial. I thank the Referee for communicating this remark.



a b

Figure 4. Resolution of the cross

3. Simplicial resolutions of plane arrangements and discriminants

The most convenient way of calculating homology groups of plane arrangements (and also of discriminants in the spaces of curves) is provided by the method of *simplicial resolutions* that is a continuous analog of the combinatorial formula of inclusions and exclusions.

Our main example will be the line arrangement $\mathbf{X} \subset \mathbb{R}^2$ consisting of two crossing lines, see Fig. 4a. Suppose that we need to calculate its *Borel–Moore homology group* $\bar{H}_*(\mathbf{X})$, i.e. the homology group of its one-point compactification $\bar{\mathbf{X}}$ reduced modulo the added point. This group is related via Alexander duality with the usual (reduced modulo a point) cohomology group of the complementary space:

$$\bar{H}_i(\mathbf{X}) \sim \tilde{H}^{N-i-1}(\mathbb{R}^N \setminus \mathbf{X})$$

(in our case $N = 2$). The simplicial resolution of $\mathbf{X}$ is shown in Fig. 4b. Namely, we first take two lines forming $\mathbf{X}$ separately, and then join their common point by a segment. The resulting space $\mathbf{X}!$ admits a natural proper projection to $\mathbf{X}$ defining an isomorphism of Borel–Moore homology groups of these spaces (and moreover extending to a homotopy equivalence of their one-point compactifications). It admits a natural increasing filtration: its term F_1 consists of two divorced lines, and F_2 coincides with entire $\mathbf{X}!$.

In the case of a general arrangement Ψ in $\mathbb{R}^N$ one also takes first all planes forming it separately, and then inserts simplices spanning their common points in such a way that the resulting space $\Psi!$ admits a proper projection to Ψ with contractible (although maybe different) fibers. There are several constructions of the simplicial resolution; we use the one defined in terms of the *order complex* of our arrangement. For its definition, see e.g. [12], [38], or [35]. This resolved space $\Psi!$ always admits a natural increasing filtration of length $\leq n-1$: its ith term is the union of all *proper preimages* under the projection $\Psi! \to \Psi$ of all intersection planes of our arrangement having codimension $\leq i$ in $\mathbb{R}^N$.

In the case of the arrangement $\mathbf{X}$ of Fig. 4, the corresponding spectral sequence calculating the Borel–Moore homology group is as follows: its unique two non-zero terms

$$E^1_{p,q} \equiv \bar{H}_{p+q}(F_p \setminus F_{p-1})$$

are $E^1_{1,0} = \bar{H}_1(F_1) \cong \mathbb{Z}^2$ and $E^1_{2,-1} = \bar{H}_1(F_2 \setminus F_1) \cong \mathbb{Z}$. The reduced cohomology classes of $\mathbb{R}^2 \setminus \mathbf{X}$ of degree (= filtration) one are exactly the linear combinations of linking numbers with either of two lines forming $\mathbf{X}$.

In particular we see that the homological spectral sequence calculating the group $\bar{H}_*(\mathbf{X})$ stabilizes at the first term, and we have

$$\bar{H}_i(\mathbf{X}) \equiv \bar{H}_i(\mathbf{X}!) \cong E^1_{1,i-1} \oplus E^1_{2,i-2}. \tag{1}$$

This is the general fact taking place for arbitrary affine plane arrangements. Moreover, there is the *homotopy splitting* formula [38], [30]: the one-point compactification of any affine plane arrangement is homotopy equivalent to the wedge of one-point compactifications of spaces $F_k \setminus F_{k-1}$ of the natural filtration of the simplicial resolution of this arrangement. The homological version of this splitting is the Goresky-MacPherson formula [12] for the cohomology of the complement of an arrangement.

The simplicial resolution of the discriminant set Σ in the space $\mathcal{K}$ of parametric curves $f : \mathbb{R}^1 \to \mathbb{R}^n$ (with fixed behavior at the infinity) can be constructed in precisely the same way. First we take the *tautological normalization* of Σ, i.e. the total space of the affine bundle, whose base is the configuration space $\overline{B(\mathbb{R}^1, 2)}$ of all unordered pairs of points $a, b \in \mathbb{R}^1$, and the fiber over such a point is the affine subspace $L(a, b) \subset \mathcal{K}$ consisting of all maps $f : \mathbb{R}^1 \to \mathbb{R}^n$ such that $f(a) = f(b)$ if $a \neq b$ or $f'(a) = 0$ if $a = b$. This normalized space is supplied with the obvious projection onto Σ and is the natural analog of the "union of lines taken separately", i.e. the set F_1 in Fig. 4b. However such spaces $L(a, b)$ with different pairs (a, b) intersect in $\mathcal{K}$; therefore we need to span their corresponding points by segments, triangles, etc. in such a way that the resulting space σ admits a natural projection onto Σ, all whose fibers are contractible.

The exact construction of this space σ can be formulated in the terms of the (naturally topologized) order complex of the space of all affine subspaces in $\mathcal{K}$ equal to intersections of several spaces of type $L(a, b)$. It also admits a natural increasing filtration $\sigma_1 \subset \sigma_2 \subset \dots$, whose first term coincides with the tautological normalization and the common term σ_i is the union of proper preimages of all planes $L(a_1, b_1) \cap L(a_2, b_2) \cap \dots$ of codimensions $\leq i$.

The resulting space σ is very similar to Σ. If $n > 3$ then their Borel–Moore homology groups of finite codimension are well defined and isomorphic to each other (and are Alexander dual to the cohomology group of the space of knots). If $n = 3$ then the situation is more complicated. A priori only a part of Borel–Moore homology classes of Σ (= cohomology classes of the space of knots in $\mathbb{R}^3$) can be represented by images of cycles from σ. These are exactly the *finite type* cohomology classes; their *degrees* (or *orders*) are defined by our filtration in σ. However this subgroup is quite ample: at this time, no non-trivial homology class of the space of knots in $\mathbb{R}^3$ is known on which all the finite type cohomology classes vanish.

The first term of the filtration, σ_1, is homologically trivial: it is the total space of an affine bundle over the half-plane $\mathbb{R}^2/\{(a, b) = (b, a)\}$, whose fiber is an affine space of codimension n in $\mathcal{K}$. Therefore $\bar{H}_*(\sigma_1) \equiv 0$, and the first column $E^r_{1,q}$ of the related spectral sequence identically vanishes. It is convenient to split the space σ_1 into two cells: one is the affine bundle over the open half-plane $\{(a < b)\}$, and the second is equal to its boundary and is fibered over the line $\{(a, a)\}$. Similar natural decompositions into open cells exist for all terms $\sigma_i \setminus \sigma_{i-1}$ of our spectral sequence.

Further, easy calculations show that the second column $E^1_{2,q} \equiv \bar{H}_{2+q}(\sigma_2 \setminus \sigma_1)$ is generated by the space of a fiber bundle over a 4-dimensional open cell with fiber equal to the product of an open interval and an affine subspace of codimension $2n$ in $\mathcal{K}$. By dimensional reasons, this column survives up to E^∞; moreover, the corresponding cycles survive the projection $\sigma \to \Sigma$ and form a subgroup isomorphic to $\mathbb{Z}$ in the Borel–Moore homology group of codimension $2n-5$ of Σ. By the Alexander duality this means that the unique non-trivial group of cohomology classes of degree 2 of the space of long knots lies in dimension $n-3$ and is isomorphic to $\mathbb{Z}$; for $n=3$ it is generated by the Casson knot invariant v_2.

We shall discuss this basic class v_2 in parallel with the class A generating the group of degree 2 Borel–Moore homology classes of the line arrangement from Fig. 4 reduced modulo the group of classes of degree 1.

The *principal part* of A is the homology class generating the group $\bar{H}_1(F_2(\mathbf{X}!)\setminus F_1(\mathbf{X}!))$, i.e. simply the fundamental class of the horizontal interval in Fig. 4b.

In a similar way, the principal part of the class v_2 is the cycle generating the group $\bar{H}_*(\sigma_2 \setminus \sigma_1)$. This cycle in $\sigma_2 \setminus \sigma_1$ is swept out by the triples of the form

$$\big((a_1 < a_2 < b_1 < b_2) \subset \mathbb{R}^1, f, t\big)\,, \tag{2}$$

where f is a map $\mathbb{R}^1 \to \mathbb{R}^n$ such that $f(a_1) = f(b_1)$, $f(a_2) = f(b_2)$, and $t \in (-1, 1)$ is the parameter along an inserted interval arising in the construction of the simplicial resolution and analogous to the horizontal interval in Fig. 4b. The endpoints of any such interval lie in the bigger cell $\breve{\sigma}_1$ of the term σ_1, i.e. in the space of pairs

$$\big((a < b) \subset \mathbb{R}^1, f\big) \tag{3}$$

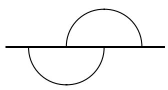
such that $f(a) = f(b)$. Namely, these endpoints sweep out the sets of points (3) in $\breve{\sigma}_1$ such that additionally $f(a') = f(b')$ for some pair of points $a' < b' \in \mathbb{R}^1$ where either

$$a' < a < b' < b \tag{4}$$

or

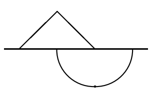
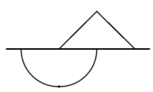
$$a < a' < b < b'. \tag{5}$$

Our basic cycle (2) in $\sigma_2 \setminus \sigma_1$ is naturally depicted by the "chord diagram"

(6)  ,

which indicates the mutual disposition of possible pairs of points (a_i, b_i) glued together by the maps f participating in its definition.

Two summands of its boundary in σ_1 corresponding to two possible dispositions (4) and (5) will be denoted respectively by two parts of the expression

(7) $(-1)^n$  $-$ 

(the signs in all such expressions depend on the orientation conventions; here we use the ones from [34]).

These two summands are exact analogues of the two endpoints of the horizontal segment in Fig. 4b.

4. "Combinatorial formulas" for cohomology of plane arrangements

Consider again the relative cycle A in the resolved cross $\mathbf{X}!$ generating the group $E^1_{2,-1} \equiv \bar{H}_1(F_2 \setminus F_1)$. Although we have the splitting (1) for the group

$$\bar{H}_1(\mathbf{X}) \cong \tilde{H}^0(\mathbb{R}^2 \setminus \mathbf{X}),$$

this cycle A itself does not define any cohomology class and cannot take values on the zero-homologous 0-cycles in $\mathbb{R}^2 \setminus \mathbf{X}$. The formula (1) says only that this relative cycle *can be* extended to a Borel–Moore cycle in all of $\mathbf{X}$, and hence to define such a homology class. However, in order to define such a class correctly we need an *explicit construction* of such an extension.

Namely, we need first to consider the boundary of this relative cycle in F_1. Formula (1) says us that it can be *spanned* there, i.e. represented as a boundary of a locally finite chain in F_1. We need to choose such a chain A_1, then the difference $A - A_1$ will be a cycle in all of $\mathbf{X}!$. We take the direct image of this cycle in $\mathbf{X}$ and choose a relative cycle in the pair $(\mathbb{R}^2, \mathbf{X})$ spanning this cycle in $\mathbb{R}^2$. This cycle can already take values on particular points of the space $\mathbb{R}^2 \setminus \mathbf{X}$, defining thus a "combinatorial formula". In the case of a general arrangement in $\mathbb{R}^N$ we, by the very definition of the spectral sequence of a filtered space, need to do the same, but with more steps. Given an element of the group $E^1_{p,q}$, we start from a locally finite cycle γ in F_p mod F_{p-1} realizing it, then take its boundary $d^1(\gamma)$ in $F_{p-1} \setminus F_{p-2}$, span it there by a chain γ_1, consider the boundary $d^2(\gamma)$ of the cycle $\gamma - \gamma_1$ in $F_{p-2} \setminus F_{p-3}$, etc.

In the case of plane arrangements in $\mathbb{R}^N$, there is an obvious way to make all of these choices. Indeed, let us choose an arbitrary constant vector field in $\mathbb{R}^N$ generic with respect to our arrangement. It is convenient to imagine it as the gradient of a generic linear function $\mathcal{L}$. All terms $F_i \setminus F_{i-1}$ of the resolution are smooth manifolds, and we always can lift the function $\mathcal{L}$ to these manifolds and span the cycles $d^k(\gamma)$ by trajectories of gradients of these lifted functions issuing from them; the genericity condition implies that these trajectories are always transversal to these cycles.

In particular, for the arrangement $\mathbf{X}$ this procedure is shown in Fig. 5. We assume that the vector field is directed down. In the left-hand top picture we mark by a thick line only the initial relative cycle $A \subset F_2 \setminus F_1$; in the right-hand top picture we add the segments in F_1 spanning its boundary; in the right-hand bottom picture we see the projection of the resulting cycle to $\mathbf{X}$, and in the left-hand bottom picture we shadow the chain spanning this cycle in $\mathbb{R}^2$ and swept out by the trajectories of our vector field issuing from its points.

In fact, in the case of plane arrangements it is not necessary to accomplish all step of this calculation, because its result can be predicted from the very beginning: it is the realization of the Goresky–MacPherson formula given in [38].

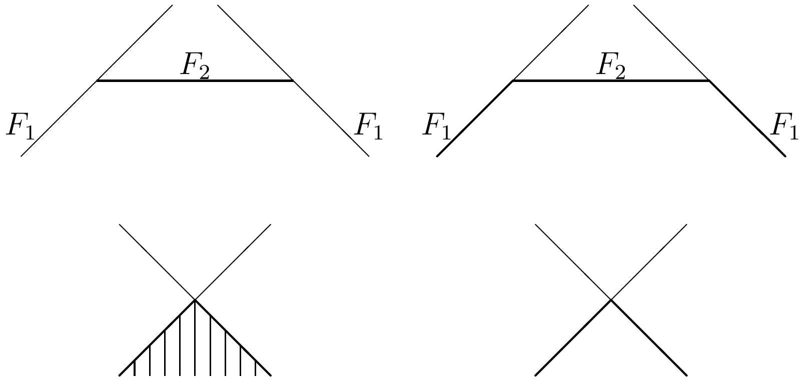


FIGURE 5. "Combinatorial formula" for the degree 2 homology class

Of course, another choice of the direction in $\mathbb{R}^N$ can give a different combinatorial formula, however the difference of these formulas will be a relative cycle of strictly lower filtration.

5. How to calculate the combinatorial formulas for knot spaces

All the same can be done with the homology classes of the discriminant in the space of knots. A formal algorithm performing this calculation of $\mathbb{Z}_2$-valued invariants of knots in $\mathbb{R}^3$ is described and justified in [36]. Its generalization to the case of integral coefficients is straightforward and also will be written soon. Similar formal algorithms for higher-dimensional cohomology classes of spaces of knots in all $\mathbb{R}^n$, $n \geq 3$, also have no chance not to exist, however a precise elaboration of such an algorithm is a more complicated problem: it involves the study of more refined degenerations of knots. Nevertheless, in the first examples (including the Turchin–Teiblum class and all cohomology classes of degree 2 of the space of compact knots) this non-formalized method proved to be efficient: for its results see Theorems 1 and 4 of this paper.

In this section we give the first illustration of our algorithm, calculating a combinatorial formula for the unique degree 2 knot invariant v_2 reduced mod 2.

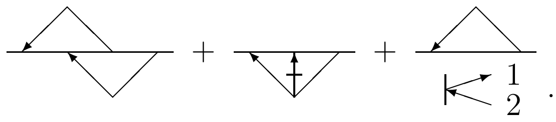
Theorem 2 (see [36])**.** *The value of v_2 on a generic long knot $f : \mathbb{R}^1 \to \mathbb{R}^3$ is equal (mod 2) to the sum of three numbers:*

a) the number of configurations $\{a < b < c < b\} \subset \mathbb{R}^1$ such that $f(c)$ is above $f(a)$ and $f(d)$ is above $f(b)$;

b) the number of configurations $\{a < b < c\}$ such that $f(c)$ is above $f(a)$ and the projection of $f(b)$ to $\mathbb{R}^2$ lies to the east of the (common) projection of $f(a)$ and $f(c)$;

c) the number of configurations $\{a < b\}$ such that $f(b)$ is above $f(a)$ and the direction "to the east" in $\mathbb{R}^2$ is a linear combination of projections of derivatives $f'(a)$ and $f'(b)$, such that the first of these projections participates in this linear combination with a positive coefficient, and the second with a negative one.

These three numbers can be depicted by three summands in the next formula:

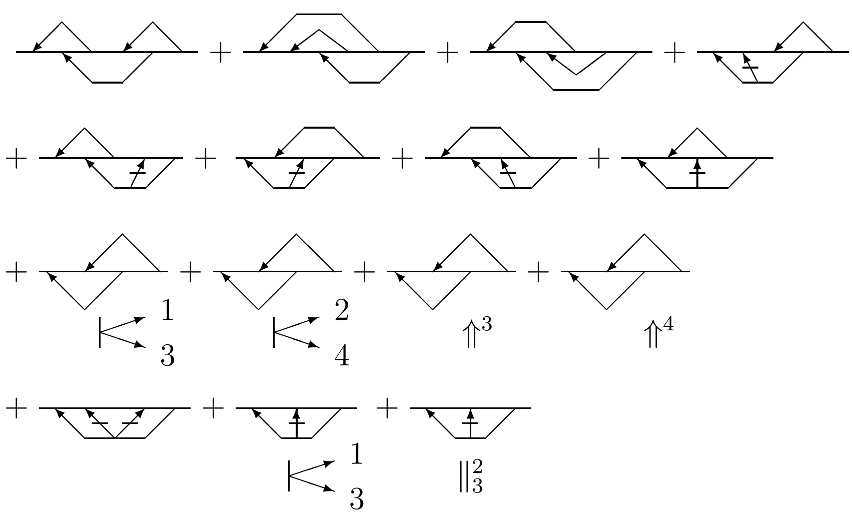


(8)

Remark 2. This formula is obviously more complicated than the Polyak–Viro formula of Fig. 2 for the same invariant. It is easy to improve the forthcoming calculation to obtain exactly the Polyak–Viro formula, see Remark 4 below. I cannot yet formalize this improvement in such a way that my algorithm itself will select the most economical computation.

The combinatorial formula for the third degree invariant, provided by our algorithm, consist of fifteen terms.

Theorem 3 (see [36])**.** *A combinatorial formula for the third degree invariant* (*mod 2*) *is given by the sum of fifteen subvarieties indicated in* (*9*).



(9)

Here the notation $\Uparrow^3$ means that the projection to the blackboard plane $\mathbb{R}^2$ of the derivative of f at the third active point of the picture should be directed into the upper half-plane. The notation $\|_3^2$ means that the similar projections of derivatives at the second and the third active points should be directed into one and the same half-plane of $\mathbb{R}^2$. Other elements of this formula can be decoded exactly as (2). For instance, the first summand in the last row of (9) denotes the number of configurations ($a < b < c < d$) such that $f(d)$ is above $f(a)$, and the projections of $f(b)$ and $f(c)$ to $\mathbb{R}^2$ are to the east of the projection of $f(a)$ or $f(d)$.

The rest of this section is occupied by a proof of Theorem 2.

5.1. **Principal part, first differential, and its homology to zero.** The principal part of v_2 in $\sigma_2 \setminus \sigma_1$ is expressed by the chord diagram shown in formula (6). The boundary of this principal part in σ_1 is shown in (7) (where we can now forget about the signs because the passage to the $\mathbb{Z}_2$-homology).

Let us span this boundary by a chain in the cell $\breve{\sigma}_1$. Recall that the first summand in (7) is the union of all points $((a, b), f) \in \sigma_1$ such that additionally there exist some two points a', b' with $a' < a < b' < b$ such that $f(a') = f(b')$.

It is natural to try to span this chain in $\breve{\sigma}_1$ by (i.e. to represent it as a piece of the boundary of) the set of points $((a, b), f)$ satisfying all the same conditions, but with the equality $f(a') = f(b')$ replaced by the condition that the projections of $f(a')$ and $f(b')$ to the "blackboard plane" $\mathbb{R}^2$ coincide, and the projection of $f(a')$ to its orthogonal line lies below the projection of $f(b')$.

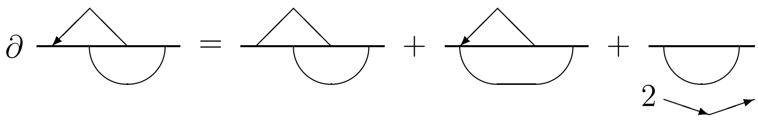
The latter condition is depicted by the "broken arrow" as in the left-hand part of (10).

In a similar way, we try to span the second summand in (7) by the variety depicted in the left-hand side of (11).

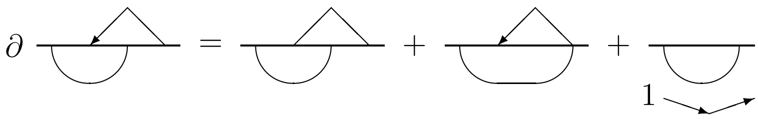
Unfortunately, these two varieties contain additional pieces of boundary, so that their sum does not span the entire chain (7). These pieces correspond to possible degenerations of the configurations of four points a, b, a' and b' participating in the definition of these varieties.

Namely, the full boundaries of these two varieties are described by the right-hand sides of (10) and (11). Let us analyze for instance the first of them.

(10)



(11)



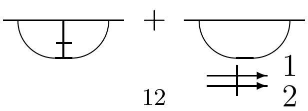
The second summand in its right-hand side appears when the first point a' and the second point a participating in the definition of our variety coincide.

In fact, this formula should have two summands more, arising when the third point b' tends to either a or b. However these two summands coincide since $f(a) = f(b)$ and cancel each other. (This happens also in the similar integral homology calculation: these two summands appear in the integral boundary of our variety with opposite orientations.)

Finally, the last summand arises when a' tends to a and simultaneously b' tends to b. The spatial picture of the corresponding degeneration is shown in Fig. 6a. The labeled arrows in the notation of this summand express the following condition: the projections to the blackboard plane $\mathbb{R}^2$ of derivatives of the knot at the points $a < b$ are co-directed there, but the direction of $f'(a)$ in $\mathbb{R}^3$ goes "above" that of $f'(b)$. The second formula (11) can be analyzed in exactly the same way.

So, we obtain that the cycle (7) is homologous in $\breve{\sigma}_1$ to the sum of second and third summands of right-hand sides of formulas (10) and (11). The sum of their second (respectively, third) summands can be depicted by the first (respectively, the second) summand in the next formula (12).

(12)



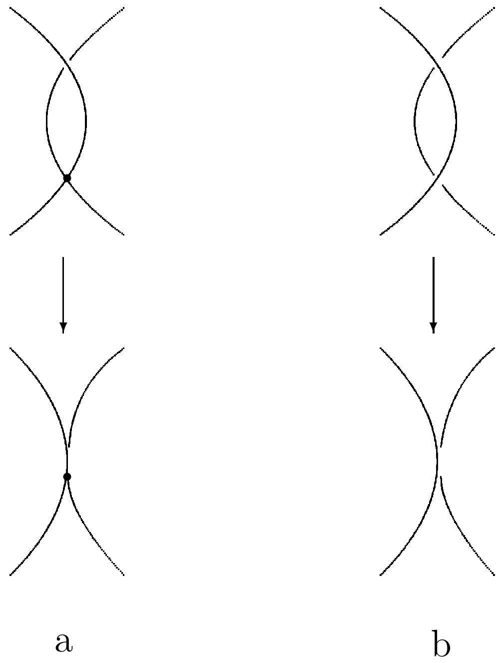


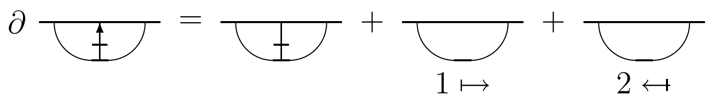
FIGURE 6. Some standard degenerations of singular knot diagrams

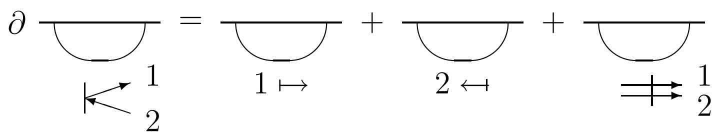
By definition, the first summand in (12) consists of points $((a,b),f) \in \breve{\sigma}_1$ which, in addition to the usual condition $f(a) = f(b)$, satisfy the following one: there is a point $c \in (a,b)$ such that the projection of $f(c)$ to the blackboard plane coincides with that of the point $f(a) \equiv f(b)$. Similarly, the additional condition expressed by two crossed arrows under the second summand in (12) says that projections to the blackboard plane of the derivatives $f'(a)$ and $f'(b)$ should be co-directed.

So, the sum of varieties in left-hand sides of formulas (10), (11) provides the homology between the cycles (7) and (12), and we need to find a zero-homology of the latter cycle. This cycle is simpler than the initial one: it consists of two varieties, each of which is defined by certain conditions on the behavior of the function f at 3 or 2 points of $\mathbb{R}^1$, while for both summands of the initial cycle the number of such "active" points was equal to 4. This situation occurs systematically. On any step of the algorithm we need to span a cycle consisting of several varieties, any of which is defined by certain conditions on the behavior of f at several points. At least one of these conditions should be an equality (e.g. the images in $\mathbb{R}^3$ of some two points should coincide, or some two vectors should be co-directed, etc.) Then we try to span such a variety by a similar variety in whose definition this equality is replaced by the inequality relating the same quantities. The latter variety usually has additional pieces of boundary, but these pieces appear at degenerations of the configurations of active points, hence are simpler than the initial cycle. By induction on the number of active points, the algorithm necessarily terminates. The algorithm inventory thus includes a list of varieties (subalgebraic chains) of codimension 1 (i.e. of dimension equal to that of the discriminant) in the space of curves and in the resolution of its discriminant, of which all our spanning chains can consist, plus the list of all irreducible varieties of codimension 2 of which the boundaries of the previous chains consist. These varieties are listed in §2 of [36]. All the same holds for the similar (not written yet) algorithm over the integers, which will differ from the existing mod 2 one by defining coorientations of all these varieties and calculating their incidence coefficients. The similar inventory for the algorithmic calculation of combinatorial formulas for r-dimensional cohomology classes should include similar lists of varieties of codimensions $r+1$ and $r+2$, thus these algorithms will be more complicated.

Let us apply this process to the cycle (12). We replace the equality-type condition distinguishing its first summand, i.e. the condition "there is a point $c \in (a, b)$ such that the projections of $f(c)$ and $f(a) \equiv f(b)$ to $\mathbb{R}^2$ coincide" by the inequality-type condition: "there is a point $c \in (a, b)$ such that the projection of $f(c)$ to $\mathbb{R}^2$ lies *to the east* of the projection of $f(a) = f(b)$". This condition is depicted by the once crossed arrow as in the left-hand side of (13).

Also we replace the equality-type condition distinguishing the second summand of (12), i.e. the condition "projections of vectors $f'(a)$ and $f'(b)$ to $\mathbb{R}^2$ are co-directed" by the inequality type condition: "the chosen direction "to the east" in $\mathbb{R}^2$ lies in the angle between the projections of two vectors $f'(a)$ and $-f'(b)$". The latter condition is expressed by the subscript under the left-hand side of (14).

(13) 

(14) 

Again, the chains shown in the left-hand sides of equalities (13), (14) contain some additional pieces of boundary described in right-hand sides of these equalities. For instance, the condition $1 \mapsto$ means that the projection to $\mathbb{R}^2$ of the derivative f' at the first active point is directed to the east. The complexities of these additional terms are lower than these of terms which we try to span. For instance, the homology given by the formula (14) does not reduce the number of all active points of its very right-hand picture (i.e. the second picture in (12)), but it replaces a condition involving two such points (expressed by the subscript under this picture) by conditions involving only one point each, thus the complexity of the chain again decreases.

These additional summands in left-hand sides of equalities (13), (14) cancel each other; therefore the sum of all their summands is equal to the cycle (12). The same will necessarily hold also in the similar calculation over the integers (with proper signs before all participating terms) because of the homological conditions: all right-hand parts of our equations and also the cycle which we are going to kill have no boundaries.

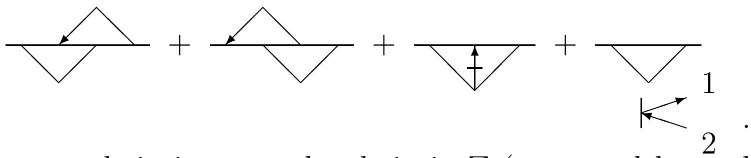
Finally, we obtain that the desired chain spanning the cycle (7) in $\breve{\sigma}_1$ is equal to the sum of four varieties indicated in left-hand sides of equations (10), (11), (13), and (14)). This sum establishes a homology between the cycle (7) and some cycle in the smaller cell of σ_1. By dimensional reasons, the latter cycle can be equal only to the fundamental cycle of the latter cell taken with some coefficient. Also, it is easy to see that the boundary positions of our four varieties form subvarieties of positive codimension in this cell. Therefore this coefficient is equal to zero, and the sum of our four varieties forms a zero-homology of the cycle (7) in all of σ_1; this sum together with the initial relative cycle (6) forms the cycle in σ_2 generating its group $\bar{H}_*(\sigma_2)$.

Remark 3. This is the unique dangerous instant in the integration of weight systems, i.e. the instant when this integration can fail and prove that there is no knot invariant corresponding to our initial weight system γ. Namely, if the degree of γ is equal to k then the (successful) algorithm consists of k steps. Its starting point is the Borel–Moore cycle of maximal dimension in $\sigma_k \setminus \sigma_{k-1}$ encoded by γ (the maximal cells of $\sigma_k \setminus \sigma_{k-1}$ are in the canonical one-to-one correspondence with the k-chord diagrams, while the vice-maximal cells correspond to one-term and four-term relations). On the rth step, the algorithm considers (a geometric realization of) the rth differential $d^r(\gamma)$, which is a Borel–Moore cycle of codimension 1 in $\sigma_{k-r} \setminus \sigma_{k-r-1}$, and tries to span it there, i.e. to represent as the boundary of a subalgebraic chain of the maximal dimension. If this works then the boundary of this spanning chain in $\sigma_{k-r-1} \setminus \sigma_{k-r-2}$ is the initial data $d^{r+1}(\gamma)$ for the next step.

On any such step, our algorithm no problem establishes a homology between $d^r(\gamma)$ and a cycle in the union of vice-maximal cells, the latter cycle can be only a linear combination of these cells. If the homology class of this cycle is equal to zero, i.e. the cycle is equal to the boundary of a linear combination of maximal cells, then we subtract this linear combination from the above homology and obtain the desired zero-homology of $d^r(\gamma)$. However, if this homology class is not equal to zero then we obtain an obstruction to the integration. (By the Kontsevich theorem, this is impossible over the rational numbers, but the obstruction can be a torsion element and thus prevent the integration over the integers.) In principle, it can happen that this obstruction is not fatal: if exactly the same nontrivial homology class arose previously as an obstruction to the integration of a weight system γ' of a lower degree, then the difference of chains obtained in these calculations for γ and γ' is a relative cycle in $(\sigma_k, \sigma_{k-r-1})$ which can be integrated one step further; however the system γ' will remain an example of a non-integrable system.

Fortunately, in all known examples all these obstructions are trivial, thus it remains to formulate the corresponding general conjecture. In fact, in all these examples the triviality of the obstruction follows from the fact that the boundary positions of our homologies (spanning the cycles $d^r(\gamma)$ in maximal cells of $\sigma_{k-r} \setminus \sigma_{k-r-1}$) form subvarieties of positive codimensions in the vice-maximal cells, thus the homological boundaries of these homologies in the union of the latter cells are not only homologous but even equal to zero. A stronger conjecture says that this also is a general situation.

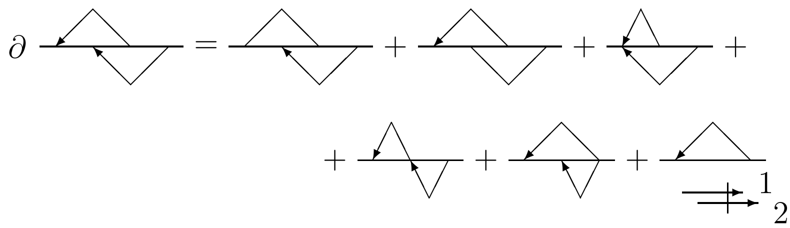
5.2. **Second differential and its homology to zero.** Now we consider the projection of the obtained cycle from σ to Σ. The projection of its part (6) is of codimension 2 and does not participate in cycles of codimension 1 responsible for knot invariants. On the other hand, the projections to Σ of four chains in σ_1 found in the previous subsection (i.e. the left-hand sides of (10), (11), (13) and (14)) are depicted by four summands of the following formula:



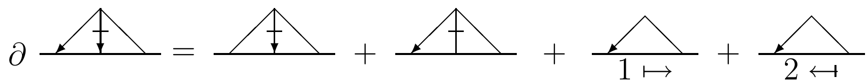
(15)

The passage from a chain in σ_1 to the chain in Σ (expressed by replacing round arcs by broken arcs without arrows) consist in posing the quantifier $\exists$: we replace the set of points $((a,b), f)$ satisfying the condition $f(a) = f(b)$ plus some other conditions by the set of maps f such that there exist points a, b such that $f(a) = f(b)$ plus all the same conditions; if f satisfies these conditions several times then we take it with the corresponding multiplicity.

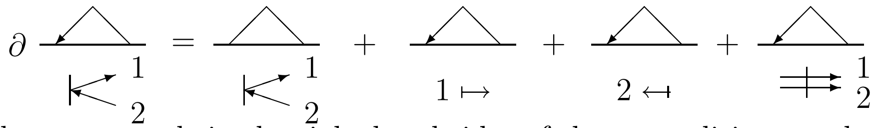
Again, it is natural to try to span the obtained chain by another one, in whose definition the condition $f(a) = f(b)$ is replaced by "$f(a)$ is below $f(b)$ in $\mathbb{R}^3$". In the language of pictures, this variety is obtained from the initial one by putting an arrow at an endpoint of the broken arc. Then we obtain three pictures indicated in left-hand sides of equalities (16)–(18); note that both the first and the second summands in (15) appear as the pieces of the boundary of the left-hand variety of (16).



(16)



(17)



(18)

All the other summands in the right-hand sides of these equalities are obvious, except maybe for the last summand in (16) which is analogous to the last summands in (10), (11), and reflects the second Reidemeister degeneration shown in Fig. 6b.

The sum of the third, fourth, and fifth terms in the right-hand side of (16) equals the second summand in (17). Therefore the sum of right-hand sides of (16)–(18) is equal to the cycle (15), and the sum of chains indicated in left-hand sides of (16)–(18) is the desired combinatorial formula, i.e. the relative cycle of the space of curves modulo Σ, whose boundary coincides with the cycle generating the degree two homology group of Σ. This sum coincides with formula (8), and Theorem 2 is proved.

Remark 4. We could try to kill two summands of (7) not by the left parts of the equalities (10) and (11) but by similar pictures with reversed orientations of broken arcs. If we make such a switch for exactly one of these summands, then the additional summands in the right-hand parts of resulting versions of (10) and (11) cancel each other, so that the sum of their left-hand parts spans the cycle (7). Continuing our algorithm, we obtain in this case exactly the Polyak–Viro formula (2) consisting of a single term, and not of three terms given in Theorem 2. Unfortunately at this stage we have not found a way to make the process both algorithmic and sensitive to such optimal choices.

6. Results for compact knots

In this section we, following [34], describe combinatorial formulas for all cohomology classes of filtration ≤ 2 of the space of compact knots $S^1 \hookrightarrow \mathbb{R}^n$ for any $n \geq 3$. First we, following [33], list all such classes.

We assume that a cyclic coordinate in S^1, i.e. an identification $S^1 \simeq \mathbb{R}^1/2\pi\mathbb{Z}$, is fixed.

Proposition 1 (see [32], [33])**.** *For any $n \geq 3$ the group of $\mathbb{Z}_2$-cohomology classes of degree 1 of the space of compact knots in $\mathbb{R}^n$ is nontrivial only in dimensions $n-2$ and $n-1$, and is isomorphic to $\mathbb{Z}_2$ in these dimensions. Moreover, for (only) even n similar integral cohomology groups in these dimensions are isomorphic to $\mathbb{Z}$. The generator of the $(n-2)$-dimensional group is Alexander dual to the set of discriminant maps $S^1 \to \mathbb{R}^n$ gluing together some two* opposite *points of S^1, and the $(n-1)$-dimensional one is dual to the set of maps gluing some* chosen *opposite points, say 0 and π.*

Indeed, the first degree cohomology classes ere exactly the ones that can be realized by linking numbers with direct images of some Borel–Moore homology classes of the tautological resolution σ_1 of the discriminant. In the case of compact knots, σ_1 is the space of a vector bundle over the closed Moebius band (i.e. the configuration space of unordered pairs of points $(a, b) \subset S^1$), the fiber over the point (a, b) consisting of all maps $f : S^1 \to \mathbb{R}^n$ such that $f(a) = f(b)$ if $a \neq b$ or $f'(a)$ if $a = b$. This bundle is (co)orientable if and only if n is even. This gives us the column $E_1^{-1,*}$. It is obvious that the dual cohomology classes are well-defined (i.e. all differentials acting from this column are trivial). The fact that these classes are nontrivial follows from their very easy realization: for any $n \geq 3$ they are nontrivial already in the restriction to the space of all naturally parametrized big circles of the unit sphere in $\mathbb{R}^n$ (equal to the Stiefel variety $V(n, 2)$). For $n > 3$ this fact follows also immediately from the shape of the spectral sequence.

Proposition 2 (see [31], [33])**.** *Additional classes of degree 2 exist in exactly two dimensions: $2n-6$ and $2n-3$. In dimension $2n-6$ for any n they form a group isomorphic to $\mathbb{Z}$ (for $n = 3$ it is generated by the simplest knot invariant). The group in dimension $2n-3$ is isomorphic to $\mathbb{Z}$ for $n > 3$ and is cyclic for $n = 3$; its generator is Alexander dual to the cycle in the discriminant, whose* principal part (*i.e. the analog of the cycle (6) in the double selfintersection of Σ) is swept out by maps $f : S^1 \to \mathbb{R}^n$ such that for some $\alpha \in S^1$ we have $f(\alpha) = f(\alpha + \pi)$, $f(\alpha + \pi/2) = f(\alpha + 3\pi/2)$.*

Again, the proof of this proposition consists in the direct calculation of the column $E_1^{-2,q}$ of the spectral sequence (which has exactly two non-trivial groups, both isomorphic to $\mathbb{Z}$), and dimensional considerations assuring that all these classes (except maybe for some elements of the group $E_1^{-2,5}$ in the case $n = 3$) survive and define non-trivial elements of the term E_∞.

Below we show in particular that for $n = 3$ the latter group also survives and defines a free cyclic subgroup in the cohomology of the space of knots, see Corollary 3. Now we give explicit combinatorial formulas for all classes mentioned in Propositions 1 and 2.

Theorem 4 (see [34])**.** *For any $n \geq 3$, the values of any of these four cohomology classes on any generic cycle of corresponding dimension in the space $\mathcal{K}_n \setminus \Sigma$ of compact knots in $\mathbb{R}^n$ is equal to the number of points of this cycle, corresponding to knots satisfying the following conditions (and in the case of integer coefficients taken with appropriate signs).*

A. For the $(n-1)$-dimensional class of degree 1: projections of $f(0)$ and $f(\pi)$ into the plane $\mathbb{R}^{n-1}$ coincide, and $f(0)$ is "higher" than $f(\pi)$.

B. For the $(n-2)$-dimensional class of degree 1, one of the following two conditions:

a) there is a point $\alpha \in [0, \pi)$ such that the projections of $f(\alpha)$ and $f(\alpha + \pi)$ to $\mathbb{R}^{n-1}$ coincide, and moreover $f(\alpha)$ is "higher" than $f(\alpha + \pi)$;

b) the projection of the point $f(0)$ to $\mathbb{R}^{n-1}$ lies "to the east" of the projection of $f(\pi)$.

C. For the $(2n-3)$-dimensional class of degree 2, one of the following two conditions:

a) there is a point $\alpha \in [0, \pi/2)$ such that projections of $f(\alpha)$ and $f(\alpha + \pi)$ to $\mathbb{R}^{n-1}$ coincide, projections of $f(\alpha + \pi/2)$ and $f(\alpha + 3\pi/2)$ to $\mathbb{R}^{n-1}$ coincide, and additionally $f(\alpha + \pi)$ is "higher" than $f(\alpha)$ and $f(\alpha + \pi/2)$ is "higher" than $f(\alpha + 3\pi/2)$;

b) projections of $f(0)$ and $f(\pi)$ to $\mathbb{R}^{n-1}$ coincide, $f(\pi)$ is "higher" than $f(0)$, and the projection of $f(\pi/2)$ to $\mathbb{R}^{n-1}$ is "to the east" of the projection of $f(3\pi/2)$.

D. For the $(2n-6)$-dimensional class of degree 2, one of two conditions:

a) there are four distinct points $\alpha, \beta, \gamma, \delta \in S^1$ (whose cyclic coordinates satisfy the inequalities $0 \leq \alpha < \beta < \gamma < \delta < 2\pi$) such that projections of $f(\alpha)$ and $f(\gamma)$ to $\mathbb{R}^{n-1}$ coincide, projections of $f(\beta)$ and $f(\delta)$ to $\mathbb{R}^{n-1}$ coincide, and additionally $f(\gamma)$ is "higher" than $f(\alpha)$ and $f(\beta)$ is "higher" than $f(\delta)$.

b) If $n = 3$ then the second condition is void (and we have only the first one coinciding with the Polyak–Viro formula), but for $n > 3$ we have an additional condition: there are three distinct points β, γ, δ (whose cyclic coordinates satisfy the inequalities $0 < \beta < \gamma < \delta < 2\pi$) such that projections of $f(\gamma)$ and $f(0)$ to $\mathbb{R}^{n-1}$ coincide, $f(\gamma)$ is "higher" than $f(0)$, and the projection of $f(\delta)$ to $\mathbb{R}^{n-1}$ is "to the east" of the projection of $f(\beta)$.

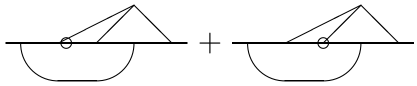
Corollary 2. *For any $n \geq 3$, the basic class of degree 2 and dimension $2n - 3$ takes value ± 1 on the fundamental cycle of the submanifold of the space of knots, consisting of all naturally parametrized great circles of the unit sphere in $\mathbb{R}^n$.*

Indeed, the variety a) of statement C does not intersect this submanifold, and variety b) has with it exactly one intersection point.

In the case of even n the fact that the fundamental class of this submanifold is not homologous to zero in the space of knots was proved in [7] by very different methods.

Corollary 3. *The group of $(2n-3)$-dimensional cohomology classes of degree 2 is free cyclic for $n = 3$ as well.*

Our proofs of Theorems 1 and 4 are very similar to that of Theorem 2. In particular, in the notation of [36] the computation of the combinatorial formula for the Teiblum–Turchin cocycle starts from the generalized weight system



describing some cellular cycle of codimension one in $\sigma_3 \setminus \sigma_2$ and contained in the resolution of the space of curves having one double and one triple point.

Remark 5. In §8 of [7] the map of cohomology rings induced by the inclusion of the space of compact knots into that of immersions $S^1 \to \mathbb{R}^n$ was considered. Our spectral sequence is situated very well for the study of this map (and similar maps for knots in other manifolds), because the "small" discriminant space of maps that are not immersions is a subspace of our discriminant of non-inclusions, and the simplicial resolution of the former discriminant is a subspace of that of the latter. Therefore we arrive at an easy comparison of spectral sequences; in particular these maps in cohomology should respect the filtrations. All multiplicative generators of the rational cohomology ring of the space of immersions are of filtration 1 (if n is even) or 1 and 2 (if n is odd), and the above two propositions say us everything on the cohomology classes of these degrees for the knot space.

Remark 6. In any term $\sigma_k \setminus \sigma_{k-1}$ of the resolution of the discriminant in the space of curves $S^1 \to \mathbb{R}^3$ there is the important open subset $\Box_k$ related with k-fold selfintersections of these curves: it is the space of all triples

$$(C, f, t),$$

where C is a k-chord configuration (i.e. a collection of $2k$ points in S^1 matched into pairs), f is a map $S^1 \to \mathbb{R}^n$ gluing together the points of any pair, and t a point of some $(k-1)$-dimensional simplex arising in the construction of the resolution. For any k the connected components of this set are in the obvious one-to-one correspondence with k-chord diagrams.

The "Vassiliev order" defined in [7] for any cohomology class of the space of knots (or, equivalently, for its Alexander dual Borel–Moore homology class γ of Σ) is the greatest k such that the restriction of γ defines a non-zero class in $\bar{H}_*(\Box_k)$. This index obviously does not exceed the filtration from [29], [33] (defined in exactly the same way but with entire $\sigma_k \setminus \sigma_{k-1}$ instead of $\Box_k$) and sometimes is strictly below it.

7. Appendix: V. Turchin's calculation

The theory of finite type invariants of knots has led to many beautiful algebraic objects, such as the Hopf algebra of chord diagrams and graph-complex of trees, see e.g. [16], [3].

It was shown recently by V. Turchin [25], [26] that these structures are non-separable parts of more general theories, related with entire cohomology rings of spaces of knots and formulated in terms of generalized chord diagrams listed in [29]. The corresponding multiplicative structures resemble the shuffle multiplication in the cohomology of complements of plane arrangements, see [37], [9], [10], but are, of course, much more complicated. It was proved in [25] and [26] that the first term of the main spectral sequence calculating the rational homology of the space of long knots in $\mathbb{R}^n$, $n \geq 3$, can be described in terms of the Hochschild homology of the Poisson algebras operad if n is odd (respectively, of the Gerstenhaber algebras operad if n is even). Namely, the Hochschild homology of these operads is in both cases a polynomial algebra in infinitely many even and odd variables. To obtain the first term of the spectral sequence in the case of even n we need to factor the corresponding polynomial algebra by one odd generator $[x_1, x_2]$. In the case of odd n we need to factor by two generators: one even (equal to $[x_1, x_2]$) and one odd (equal to $[[x_1, x_3], x_2]$).

In particular, the standard bialgebra of chord diagrams factored through the 4-term relations (see [15], [3]) is some subspace in the Hochschild homology of the Poisson algebras operad. To obtain the algebra of finite degree invariants (i.e. cohomology of degree zero in the case $n = 3$) we should factor this bialgebra by one generator $[x_1, x_2]$.

The Poisson operad occurs naturally also in the Goodwillie–Sinha approach to the topology of knot spaces, see [23].

References

[1] V.I. Arnold, On some topological invariants of algebraic functions. *Transact.* (*Trudy*) *Mosc. Mat. Soc.* **21** (1970), 27–46.

[2] Bar-Natan, D. (1994–) *Bibliography of Vassiliev Invariants.* Web publication **http://www.ma.huji.ac.il/~drorbn/VasBib/VasBib.html**

[3] D. Bar-Natan, On the Vassiliev knot invariants, *Topology,* **34** (1995), 423–472.

[4] R.Bott and C.H.Taubes, On the self-linking of knots, *J. of Mathematical Physics,* 35(10), 1994, 5247–5287.

[5] R. Budney, J. Conant, K. Scannell, and D. Sinha. *New perspectives of self-linking.* **math.GT/0303034**, 2003.

[6] P. Cartier, Construction combinatoire des invariants de Vassiliev, *C.R.Acad. Sci. Paris, Série I,* **316** (1993), 1205–1210.

[7] A.S. Cattaneo, P. Cotta-Ramusino, and R. Longoni, Configuration spaces and Vassiliev classes in any dimension, *Algebr. Geom. Topol.,* **2** (2002), 949–1000.

[8] F. Cohen and S. Gitler, Loop spaces of configuration spaces, *Cohomological methods in homotopy theory* (Bellaterra 1998), 59–78, *Progr. Math.,* Birkhauser, Basel, 2001.

[9] P. Deligne, M. Goresky, and R. MacPherson, L'algebre de cohomologie du complément, dans un espace affine, d'une famille finie de sous-espaces affines, *Michigan J. Math.* **48** (2000), 121–136.

[10] M. de Longueville and C. Schultz, The cohomology rings of complements of subspace arrangements, Math. Annalen. 319 (2000), 625–646.

[11] T. Goodwillie, J. Klein, and M. Weiss, Spaces of smooth embeddings, disjunction and surgery. Surveys on surgery theory, Vol. 2, 221–284, *Ann. of Math. Studies,* **149**, Princeton Univ. Press, Princeton, NJ, 2001.

[12] M. Goresky and R. MacPherson, *Stratified Morse Theory,* Springer, Berlin a.o., 1988

[13] M. Goussarov, M. Polyak, and O. Viro, Finite type invariants of classical and virtual knots, *Topology* **39**:5 (2000), 1045–1068.

[14] A. Hatcher, Spaces of knots, **http://math.cornell.edu/~hatcher**

[15] Kontsevich, M. (1993) Vassiliev's knot invariants, in *Adv. in Sov. Math.,* **16:2,** AMS, Providence RI, 137–150.

[16] M. Kontsevich, Formal (non-)commutative symplectic geometry. In: L. Corvin, I. Gel'fand, J. Lepovsky (eds.), *The I.M. Gel'fand's mathematical seminars 1990–1992,* 1993, Birkhäuser, Basel, 173–187.

[17] J. Lannes, Sur les invariants de Vassiliev de degreé inferieur ou égal à 3. *L'Enseignement Mathématique* **39** (1993), 295–316.

[18] A.B. Merkov, Vassiliev invariants classify plane curves and doodles, Preprint, 1998, **http://www.pdmi.ras.ru/~arnsem/papers**, to appear in Math. Sbornik.

[19] A.B. Merkov, Segment–arrow diagrams and invariants of ornaments, *Mat. Sbornik* **191** (2001), 1635–1666.

[20] S.A. Piunikhin, *Combinatorial expression for universal Vassiliev link invariant*, Harvard Univ. preprint **hep-th/9302084**, 1993.

[21] M. Polyak and O. Viro, Gauss diagram formulas for Vassiliev invariants, *Internat. Math. Res. Notes* **11** (1994), 445–453.

[22] M. Polyak and O. Viro, On the Casson knot invariant, *Tel Aviv University and Uppsala University preprint,* 1996, **math.GT/9903158**

[23] D. Sinha, The topology of spaces of knots, **math.AT/0202287**, 2002.

[24] D. Thurston, Integral expressions for the Vassiliev knot invariants, **math.QA/9901110**, 1995.

[25] V. Tourtchine, Sur l'homologie des espaces des nœuds non-compacts, **math.QA/0010017**, 2000.

[26] V. Tourtchine, Sur les questions combinatoires de la théorie spectrale des nœuds, These de Doctorate, Universite Paris-7, 2002.

[27] S.D. Tyurina, On the Lannes and Viro-Polyak type formulas for finite type invariants, *Math. Notes*, **66** (1999), No.3–4, 525–530.

[28] V.A. Vassiliev, *Complements of discriminants of smooth maps: topology and applications, Revised ed.*, Translations of Math. Monographs, AMS, Providence RI, 1994.

[29] V.A. Vassiliev, Cohomology of knot spaces, Theory of Singularities and its Applications (V. I. Arnold, ed.), Advances in Soviet Math. Vol. **1** (1990), p. 23–69 (AMS, Providence, RI).

[30] V.A. Vassiliev, Complexes of connected graphs, in: L. Corvin, I. Gel'fand, J. Lepovsky (eds.), *The I. M. Gel'fand's mathematical seminars 1990–1992,* 1993, Birkhäuser, Basel, 223–235.

[31] V.A. Vassiliev, *Topology of complements of discriminants,* Phasis, Moscow (in Russian), 1997.

[32] V.A. Vassiliev, On invariants and homology of spaces of knots in arbitrary manifolds, in: B. Feigin and V. Vassiliev, eds. *Topics in Quantum Groups and Finite-Type Invariants. Mathematics at the Independent University of Moscow.* AMS Translations. Ser. 2. Vol. **185**. Advances in the Mathematical sciences. AMS, Providence RI, 1998, p. 155–182.

[33] V.A. Vassiliev, Topology of two-connected graphs and homology of spaces of knots, in: S. L. Tabachnikov (ed.), *Differential and Symplectic Topology of Knots and Curves,* AMS Transl., Ser. 2, **190**, AMS, Providence RI, 1999, 253–286.

[34] V.A. Vassiliev, On combinatorial formulas for cohomology of spaces of knots, *Moscow Math. J.*, **1**:1 (2001), 91–123.

[35] V.A. Vassiliev, Topology of plane arrangements and their complements, Russian Math. Surveys, **56**:2 (2001), 167–203.

[36] V.A. Vassiliev, Combinatorial computation of combinatorial formulas for knot invariants, **http://www.pdmi.ras.ru/~arnsem/papers**, 2001.

[37] S. Yuzvinsky, Small rational model of subspace complement, Translations AMS; **http://xxx.lanl.gov/abs/math.CO/9806143**, 1999.

[38] G.M. Ziegler and R.T. Živaljević, Homotopy type of arrangements via diagrams of spaces, *Math. Ann.* **295** (1993), 527–548.