Thom polynomials for Lagrange, Legendre, and critical point function singularities

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Abstract

We define Thom polynomials for Lagrange, Legendre and critical point function singularities. Our approach is based on the notion of classifying space of singularities. This approach provides a universal method of computing Thom polynomials. Characteristic classes of complex Lagrange and Legendre singularities of small codimensions are computed. These expressions reduced modulo 2 agree with those obtained by Vassiliev for the real case.

1 Introduction

The natural way to study the global properties of isolated hypersurface singularities is to include the hypersurface into a generic family. Formally this can be described as follows. Consider a smooth embedded hypersurface in the total space of a smooth locally trivial complex analytic fibration:

$$\begin{array}{ccc} H & \hookrightarrow & W \\ & & \downarrow \pi \\ & & B \end{array}$$

(The case when π is the trivial bundle is already interesting enough.) We consider H as a family of (possibly singular) hypersurfaces $H_b \subset W_b$, $W_b = \pi^{-1}(b)$, $H_b = H \cap W_b$, $b \in B$. Let $M \subset H$ be the union of all singular points of H_b's. Generically M is smooth and has the codimension $n = \dim W - \dim B$ in H. It can be identified with the zero locus of a certain section of some vector bundle. Namely, the bundle is $\mathrm{Hom}(V, I)$, where $V \subset TW$ is the subbundle of vectors tangent to the fibres of π and I is the normal bundle of H. The section is given by the natural projection $V \subset T_H W \to T_H W/TH = I$. The genericity condition means that this section is transversal to the zero section of the same bundle. This condition is open but in complex situation it is not necessary dense. Similarly below by genericity for some smooth map we mean the transversality of its jet extension to a certain stratification on the jet space.

Let Ω be any class of isolated hypersurface singularities (an algebraic subvariety in some jet space of function germs $\mathbb{C}^n, 0 \to \mathbb{C}, 0$ which is invariant with respect to the group of left-right changes of variables). Define the locus $\Omega(H) \subset M$ as the locus consisting of the points at which

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the hypersurface H_b belongs to the given singularity class Ω. According to the general principle of Thom the cohomology class Poincaré dual to the locus $\Omega(H)$ is independent on H (provided H is generic) and can be expressed as a universal polynomial in Chern classes of W, B, H. We claim that this polynomial can be expressed in terms of some particular combinations of these classes. Namely, let $u = c_1(I) = c_1(TW - TH)$ be the restriction to M of the class of the divisor H. Denote $c_i = c_i(V) = c_i(TW - \pi^*TB)$, and define classes $a_i = c_i(V^* \otimes I - V)$ as homogeneous components in the expansion

$$1 + a_1 + a_2 + \ldots = \frac{(1+u)^n - (1+u)^{n-1}c_1 + (1+u)^{n-2}c_2 - \ldots \pm c_n}{1 + c_1 + c_2 + \ldots + c_n}. \tag{1}$$

These classes satisfy relations

$$(1 + a_1 + a_2 + \ldots)\left(1 - \frac{a_1}{1+u} + \frac{a_2}{(1+u)^2} - \ldots\right) = 1, \tag{2}$$

(following from the identity $U + U^* \otimes I = 0$, where U is the formal difference $U = V^* \otimes I - V$). These relations allow to expand the squares of classes a_i and hence any polynomial in $u, a_1, a_2, \ldots$ can be expressed as a linear combination of monomials $u^{i_0} a_1^{i_1} a_2^{i_2} \ldots$ with $i_0 \geq 0$, $i_k \in \{0, 1\}$ $(k > 0)$.

Theorem 1. *For any isolated hypersurface singularity class Ω the cohomology class in $H^*(M)$ Poincaré dual to the locus $\Omega(H)$ can be expressed as a universal polynomial P_Ω in $u, a_1, a_2, \ldots$. This polynomial* (*called* Thom polynomial) *is independent on n* (*we use the same letter Ω for the class of function germs $\mathbb{C}^{n'}, 0 \to \mathbb{C}, 0$, $n' \neq n$, stably equivalent to the functions from Ω*).

For the singularity classes of codimension not greater than 6 the Thom polynomials are represented in Table 1.

To determine the cohomology class dual to the locus $\Omega(H)$ in H or in W we apply the push-forward formula $i_*(i^*a\ b) = a\ i_*(b)$ to the embeddings $M \overset{i}{\hookrightarrow} H \overset{j}{\hookrightarrow} W$ and the classes $a = P_\Omega$, $b = 1$. For instance, since $i_*(1) = [M] = c_n(\mathrm{Hom}(V, I)) = u^n - u^{n-1}c_1 + \ldots \pm c_n$, we get that the dual of $\Omega(H)$ considered as a locus in H is equal to

$$[\Omega(H)] = (u^n - u^{n-1}c_1 + \ldots \pm c_n)\ P_\Omega \in H^*(H).$$

Similarly, the homomorphism $j_* : H^*(H) \to H^*(W)$ on the class above is given by the multiplication by u.

To prove Theorem 1 we relate the problem to the theory of Lagrange and Legendre singularities and their characteristic classes. Namely, we consider the hypersurface H as the 'generating family' for the Legendre immersion $M \to PT^*B$.

In the simplest case when the bundle I is trivial (and hence $u = 0$) the problem is reduced to the study of Lagrange singularities. Lagrange singularities are those of the projection of a Lagrange submanifold to the base of the cotangent bundle. Singularity loci of this projection could define cohomology classes on this manifold. The simplest example is Arnold-Maslov class which is dual to the total critical set of the projection. The theory of characteristic classes related to the *real* Lagrange singularities was developed by V.Vassiliev. In his book [18] a cochain complex (so called Vassiliev universal complex of singularity classes) was constructed whose generators correspond to the singularity classes. The cohomology groups of this complex are well defined characteristic classes. Vassiliev has computed the cohomology of this complex

Table 1: Thom polynomials of isolated hypersurface singularities of codim ≤ 6

$$
\begin{aligned}
A_2 &= \mathbf{a_1} \\
A_3 &= \mathbf{3a_2} + ua_1 \\
A_4 &= \mathbf{3a_1a_2} + \mathbf{6a_3} + 4ua_2 + u^2a_1 \\
D_4 &= \mathbf{a_1a_2} - \mathbf{2a_3} - ua_2 \\
A_5 &= \mathbf{27a_1a_3} + \mathbf{6a_4} + u(16a_1a_2 - 12a_3) - 4u^2a_2 + u^3a_1 \\
D_5 &= \mathbf{6a_1a_3} - \mathbf{12a_4} + u(4a_1a_2 - 14a_43 - 4u^2a_2 \\
A_6 &= \mathbf{87a_2a_3} + \mathbf{54a_1a_4} + \mathbf{78a_5} + \\
&\qquad u(127a_1a_3 - 53a_4) + u^2(59a_1a_2 - 126a_3) - 41u^3a_2 + u^4a_1 \\
D_6 &= \mathbf{12a_2a_3} - \mathbf{24a_5} + u(14a_1a_3 - 40a_4) + u^2(8a_1a_2 - 30a_3) - 8u^3a_2 \\
E_6 &= \mathbf{9a_2a_3} - \mathbf{12a_1a_4} + \mathbf{6a_5} + 3ua_4 + u^2(3a_1a_2 - 6a_3) - 3u^3a_2 \\
A_7 &= \mathbf{135a_1a_2a_3} + \mathbf{465a_2a_4} + \mathbf{264a_1a_5} + \mathbf{522a_6} + u(516a_2a_3 - 16a_1a_4 + 485a_5) + \\
&\qquad u^2(305a_1a_3 - 70a_4) + u^3(190a_1a_2 - 440a_3) - 165u^4a_2 + u^5a_1 \\
D_7 &= \mathbf{24a_1a_2a_3} - \mathbf{24a_2a_4} + \mathbf{48a_1a_5} - \mathbf{144a_6} + u(8a_2a_3 + 44a_1a_4 - 224a_5) + \\
&\qquad u^2(48a_1a_3 - 172a_4) + u^3(20a_1a_2 - 88a_3) - 20u^4a_2 \\
E_7 &= \mathbf{9a_1a_2a_3} + \mathbf{6a_2a_4} - \mathbf{42a_1a_5} + \mathbf{36a_6} + u(21a_2a_3 - 61a_1a_4 + 80a_5) + \\
&\qquad u^2(43a_4 - 6a_1a_3) + u^3(7a_1a_2 - 8a_3) - 7u^4a_2 \\
P_8 &= \mathbf{a_1a_2a_3} - \mathbf{6a_2a_4} + \mathbf{6a_1a_5} - \mathbf{4a_6} + \\
&\qquad u(7a_1a_4 - 4a_2a_3 - 10a_5) + u^2(2a_1a_3 - 8a_4) - 2u^3a_3
\end{aligned}
$$

in the codimension not exceeded 6 and found the expressions for all these classes (except A_7) in terms of Stiefel-Whitney classes.

In the paper [9] we suggested an approach to this problem based on the study of classifying space of Lagrange singularities. This has led to understanding the geometrical meaning of Vassiliev complex and to introducing new characteristic classes. In this paper we develop this approach. In particular, we complete computing characteristic classes dual to singularity classes in terms of multiplicative generators the ring of characteristic classes (i.e. Thom polynomials) and describe also the complex version of the theory.

Any classification problem in singularity theory can be considered as a problem of classifying orbits for an action of some Lie group G on some vector space V. For instance, for left-right equivalence of maps V is the space of map germs $(\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$ (or jets of maps of fixed order) and G is the group of the left-right changes which is homotopy equivalent to $\mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(m, \mathbb{R})$. The theory of characteristic classes of this classification problem is therefore the theory of characteristic classes of the given Lie group G. The *classifying space* BV of this classification problem is defined using standard Borel's construction, $BV = V \times_G BE = (V \times BE)/G$, where $BE \to BG$ is the classifying principle G-bundle. The classifying space BV is homotopy equivalent to BG (since the bundle $BV \to BG$ has contractible fibres). Any invariant algebraic subset $\Sigma \subset V$ gives rise to a subset $B\Sigma = \Sigma \times_G BE \subset V$ of the same codimension. With this approach the 'Theorem about existence of Thom polynomials' is evident; Thom polynomial of $\Sigma \subset V$ is just the element represented by the fundamental cycle of Σ in the equivariant cohomology group $H^*_G(V) \cong H^*_G(\mathrm{pt}) \cong H^*(BG)$, or, which is equivalent, the element, represented by the fundamental cycle of $B\Sigma$ in the usual cohomology group $H^*(BV) \cong H^*(BG)$.

The classifying spaces BG and BV have infinite dimensions, but they always have very nice finite dimensional approximations that can be used as well for 'stable' problems, where the maps of manifolds of different dimensions are considered. For example, for the theory of singularities of $(\mathbb{R}^*, 0) \to (\mathbb{R}^{*+k}, 0)$ the classifying space is the space of germs (or jets of high order) of n-manifolds in $(\mathbb{R}^N, 0)$, $N \gg n \gg 0$. This space is homotopy equivalent to the usual Grassmannian $G_{n,N}$ and stratified according to singularities of the projection to the fixed coordinate $(n+k)$-subspace. In a similar way, for the classifying space of Lagrange singularities one can take the space of all germs (or jets of high order) of Lagrange submanifolds in the symplectic space $(\mathbb{R}^{2n}, 0)$, $n \gg 0$. The same constructions can be used for studying complex spaces and holomorphic maps.

Note that if $\Sigma \subset V$ is an orbit then $B\Sigma = \Sigma \times_G EG \cong BG_\Sigma$ is the classifying space of the 'symmetry group' G_Σ of the singularity Σ (the stationary subgroup of any point $x \in \Sigma$). A similar description of $B\Sigma$ exists even if Σ consists of many orbits.

In [13] Szücs and Rimányi used an alternative approach to the definition of the classifying space of singularities based on Szücs's idea of gluing the classifying spaces of symmetry groups of singularities. They considered only simple singularities, and very clear topology of the classifying space does not follow from their construction. It should be noticed nevertheless that their construction works as well for the case of multisingularities, see [14, 11, 12] for some applications. It is an interesting problem to find an *a priori* construction for the classifying space of multisingularities and to describe its topology (the work [13] implies that it should be related to cobordism theory).

The group of Lagrange characteristic classes is the cohomology group of Lagrange Grassmannian $\Lambda^{\mathbb{R}}$ (or its complex analogue $\Lambda^{\mathbb{C}}$). The cohomological information about adjacencies of singularities is translated into properties of the spectral sequence constructed by the filtration on the classifying space by the codimension of singularities [8]. Let us describe this spectral sequence for the classification of (complex) Lagrange singularities [9]. The initial term of this sequence is $E_2^{*,*} = \bigoplus_\Sigma H^*(BG_\Sigma)$, where G_Σ is the symmetry group of the singularity Σ. The symmetry groups of all singularities of small codimensions are well known. They are all finite (some extension of S_3 for the singularity D_4 and cyclic groups for all other singularities). In fact, Vassiliev proved recently that the symmetry group of any critical point singularity of finite multiplicity is finite, see the Russian translation of [18]. This gives the complete description of the groups $E_2^{p,q} \cong E_\infty^{p,q}$ for small p. They are all torsion groups for $q > 0$ and the free generators of the raw $E_2^{*,0}$ correspond to the fundamental cycles of singularity classes. Hence we immediately arrive without any calculation to the following conclusion.

Theorem. *The classes of complex Lagrange singularities $A_2, A_3, \ldots, E_7, P_8$ form a basis in the group of Lagrange characteristic classes $H^{\leq 12}(\Lambda^{\mathbb{C}}, \mathbb{Q})$. In case of integer coefficients these classes generate freely subgroups of finite indices* $1, 3, 12, 360, \ldots$ *respectively in $H^{2d}(\Lambda^{\mathbb{C}}, \mathbb{Z})$, $d \leq 6$.*

Of course, this result follows also from the explicit form of these classes represented in Table 1 (one should set $u = 0$; the corresponding terms are marked in boldface). The presents of even numbers in the sequence $1, 3, 12, 360, ...$ implies that in case of real singularities there are some relations between these classes. These relations have been found by Vassiliev in [18]: $A_4 - D_4 = D_5 = D_6 = A_6 - E_6 = D_7 = E_7 - P_8 = 0 \pmod 2$.

We describe two methods for computing Thom polynomials, both based on the concept of the classifying space. The idea of the first method is the following. Consider a cellular partition of the classifying space such that both Schubert cycles and singularity classes are some combinations of cells. Then the problem is reduced to linear algebra in some finitely generated cochain complex. To obtain such a partition one can consider a classification of critical point singularities with respect to a smaller group of equivalence consisting of diffeomorphism germs with identical linear terms. Such classification has no simple singularities but the number of modules is always finite. The main property of this classification is that the symmetry group of every its singularity class is trivial, and hence, the induced partition of the classifying space is a cellular partition.

A realization of the program above is possible though it requires a great deal of computations. To reduce the amount of computations we use as a kind of compromise another detailed classification of singularities with smaller but not trivial symmetry groups. The corresponding classes are called *marked* singularity classes. A number of relations between these marked singularities is sufficient to reduce the problem of finding Thom polynomials to those singularity classes for which this problem can be solved by classical methods like resolutions of singularities. To describe these relations we introduce the notion of *adjacency exponent* for a pair of singularities of (complex) neighbour codimensions which is an analogue of the incidence coefficient in real case, see [18].

For the simplicity we present the computations of Thom polynomials for the classes of function singularities. All steps of our computations can easily be reformulated in terms of Lagrange (or Legendre) singularities.

Another method of finding Thom polynomials uses basically the idea of R.Rimányi. Any example when both Chern classes and the class represented by singularity locus can be computed produces some relations between the coefficients of Thom polynomial. When the number of computed examples is high enough these relations could be sufficient to determine Thom polynomial completely. Rimányi noticed that a lot of examples can be produced by considering tubular neighbourhoods of singularity loci in the classifying space. To see when this could give the result let us look again at the characteristic spectral sequence converging to the group of characteristic classes. Its second term is $E_2^{*,*} = \bigoplus_\Sigma H^*(BG_\Sigma)$. In complex case all topology is often concentrated in even dimensions and the spectral sequence converges in the second term. This means that any characteristic class is completely determined by the collection of its images in the groups $H^*(BG_\Sigma)$. In fact, these images belong not to the group $H^*(BG_\Sigma) = H^*(B\Sigma)$ itself but to the (isomorphic to it) cohomology group $H^*(T\Sigma)$ of Thom space $T\Sigma$ of the normal bundle to the submanifold $B\Sigma$ of the classifying space of singularities. Therefore, to apply Rimányi's method we need that the homomorphisms $\varphi : H^*(T\Sigma) \to H^*(B\Sigma)$ induced by inclusion of the zero section of the normal bundle to its Thom space would be monomorphic.

If G_Σ is trivial then the homomorphism φ (given by the multiplication by the top Chern class of the bundle) is trivial. In a similar way φ is trivial if G_Σ is finite and we consider the cohomology with coefficients in a field of characteristic zero. Therefore *Rimányi's method of finding Thom polynomials can be applied only when every singularity has a continuous group of symmetry.* This is not the case for Lagrange singularities. Nevertheless this is true in case of *Legendre* or *twisted Lagrange* singularities. In the classification of Legendre singularities the right equivalence of functions is replaced by V- (or K- depending on the terminology) classification when one allows to multiply a function by another nowhere vanishing function. In this classification any quasihomogeneous singularity has an obvious $U(1)$-symmetry. Thus the Rimányi's method gives Thom polynomials of all Legendre singularities in small codimensions and as a particular case Thom polynomials of Lagrange singularities.

Reducing the coefficients modulo two we get real Legendre characteristic classes (Thom polynomials for them were not computed in [18]).

The paper is organised as follows. In Section 2 we present two independent methods of computing Thom polynomials of Table 1. Their existence is proved in Section 3 where the characteristic classes of Lagrange and Legendre singularities are studied.

I appreciate the hospitality of the I. Newton Institute, Cambridge, where the work on the paper was completed.

2 Resolutions and adjacencies of function singularities

In this section we compute the Thom polynomials listed in Theorem 1 for the case of bundle map problem considered here. The proof of their existence is postponed until Section 3.2. Everywhere in this section we use the same notations for the singularity loci and for the cohomology classes represented by these loci.

2.1 Bundle map problem

Before computing Thom polynomials of Theorem 1 we formulate a slightly different but, in fact, an equivalent problem. Consider two complex vector bundles V, I of ranks $\operatorname{rk} V = n$, $\operatorname{rk} I = 1$ over a smooth base M. We do not assume any complex structure on the base M. Consider a smooth bundle map

$$\begin{array}{ccc} V & \xrightarrow{f} & I \\ & \searrow \ \swarrow & \\ & M & \end{array}$$

whose restriction $f_w : V_w \to I_w$, $w \in M$ to each fibre is a complex polynomial of some fixed degree $N \gg 0$ with a critical point at the origin. We may think of f as a generic section $f = f_{(2)} + f_{(3)} + \dots$ of the vector bundle $S^2V^* \otimes I \oplus S^3V^* \otimes I \oplus \cdots$. With any function singularity class Ω we associate the locus $\Omega(f) \subset M$ consisting of the points $w \in M$ such that the polynomial f_w has the prescribed singularity type Ω at the origin. Theorem 1 is a particular cases of the following more general one.

Theorem 2. *For any generic f the cohomology class Poincaré dual to the locus $\Omega(f)$ can be expressed as a universal polynomial P_Ω in classes $u = c_1(I)$ and $a_i = c_i(V^* \otimes I - V)$. For singularity classes of codimension ≤ 6 these polynomials are those listed in Theorem 1.*

Proof of Theorem 1. Let as explain how the problem of studying the hypersurface singularities discussed at the introduction can be reduced to a bundle map problem considered here. Let the diagram $H \subset W \xrightarrow{\pi} B$ and the critical set $M \subset H$ be as in the introduction. Assume that the divisor H is given locally by the equation $g = 0$ where g is some function given in a neighbourhood of the given point $w \in M$. Denote by f_w the N-jet of the restriction of g to the fibre of π through w. These function jets f_w form together a section in the bundle J whose structure group is the Lie group formed by the N-jets of right changes in $\mathbb{C}^n, 0$ and multiplications by the N-jets of non-vanishing functions. This group is contractible to the subgroup $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C})$ of linear changes. Hence, the structure group of J can be reduced to this subgroup. The possibility of such reduction means the possibility of introducing the complex linear structures 'up to the order N' on the germs W_w, C^∞-smoothly depending on the point $w \in M$. After such reduction the N-jets f_w can be considered as polynomial maps $f_w : V_w \to I_w$, where $V_w = T_wW_w$, $I_w = T_wW/T_wH$. By Tugeron's finite determinacy theorem the singularity type of the hypersurface H_w is that of the function germ f_w. Therefore the partition on M by different singularity loci of H_w coincides with that of the polynomial bundle map $f : V \to I$. $\square$

2.2 The Gysin homomorphism

Some part of Thom polynomials of Theorem 1 is computed using the method of resolutions of singularities. The computations of this method use a formula for the Gysin homomorphism proved in [10]. Consider vector bundles $V, I \to M$ of ranks $n, 1$ respectively and a *quadratic* bundle map $f_{(2)} : V \to I$. The map $f_{(2)}$ is considered as a section of the bundle $S^2V^* \otimes I$ or as a linear self-adjoined bundle map $f_{(2)} : V \to V^* \otimes I$. Denote by $\mathbf{F}_r = \mathbf{F}_r(V)$, $r \le n$ the flag bundle over M whose total space is formed by all possible flags

$$F_w^1 \subset F_w^2 \subset \dots F_w^r \subset V_w, \qquad \dim F_w^i = i, \quad w \in M,$$

in the fibres of V. The cohomology ring of $\mathbf{F}_r$ is generated by the cohomology of M and by the classes $t_i = -c_1(F^i/F^{i-1}) = c_1((F^i/F^{i-1})^*)$, where F^i are the corresponding tautological vector bundles. Denote by $Z_r \subset \mathbf{F}_r$ the locus defined by the condition $F_w^r \subset \ker f_{(2)w}$, $w \in M$. Generically Z_r is smooth and $\dim_{\mathbb{C}} Z_r = \dim_{\mathbb{C}} M - r$. We study the Gysin (or push-forward or transfer) homomorphism

$$p_{r*} : H^*(Z_r) \to H^{*+2r}(M)$$

corresponding to the natural projection $p_r : Z_r \to M$.

Theorem ([10]). *Assume that the bundle I is trivial. Then for any monomial $t_1^{s_1} \cdots t_r^{s_r}$ we have*

$$p_{r*}(t_1^{s_1} \cdots t_r^{s_r}) = Q_{s_1+1,\dots,s_r+1} \in H^{2\sum(s_i+1)}(M),$$

where $Q_{\lambda_1,\dots,\lambda_r} = Q_{\lambda_1,\dots,\lambda_r}(a_1, a_2, \dots)$ are polynomials in classes $a_k = c_k(V^* - V)$ defined as follows:

- *for $r = 1$ we have $Q_k = a_k$;*
- *for $r = 2$ we set*

$$Q_{k,l} = -Q_{l,k} = \sum_{i=1}^{\infty}(-1)^i \begin{vmatrix} a_{k+i} & a_{k-i} \\ a_{l+i} & a_{l-i} \end{vmatrix};$$

- *for any even $r \ge 4$ we set*

$$Q_{\lambda_1,\dots,\lambda_r} = \mathrm{Pf}\, |Q_{\lambda_i,\lambda_j}|_{1 \le i,j \le r};$$

- *for any odd $r \ge 3$ we set*

$$Q_{\lambda_1,\dots,\lambda_r} = \sum_{k=1}^{r}(-1)^{k-1} a_{\lambda_k} Q_{\lambda_1,\dots,\lambda_{k-1},\lambda_{k+1},\dots\lambda_r}.$$

Here Pf is the Pfaffians of a skew-symmetric matrix. Remark that in the infinite sum for $Q_{k,l}$ only first $\max(k,l)$ terms may be different from zero.

The computation of the homomorphism p_{r*} in generic case can be reduced to the case considered above.

Corollary. *The formula of previous Theorem holds true in the generic case if it is applied to the classes* $\hat{t}_i = -c_1(F^i/F^{i-1}) + u/2$ *instead of* t_i *and to the classes* $\hat{a}_k$ *instead of* a_k*, where* $u = c_1(I)$ *and*

$$1 + \hat{a}_1 + \hat{a}_2 + \ldots = \frac{1 - \frac{c_1}{1+u/2} + \frac{c_2}{(1+u/2)^2} - \ldots}{1 + \frac{c_1}{1-u/2} + \frac{c_2}{(1-u/2)^2} + \ldots}, \quad c_i = c_i(V).$$

P r o o f. We use the following trick borrowed from [7]. Consider first the case when $I = J^{\otimes 2}$, where J is another line bundle with $c_1(J) = c_1(I)/2 = u/2$. Then $S^2V^*\otimes I = S^2(V\otimes J^*)^*$ so $f_{(2)}$ can be treated as a self-adjoined bundle map $V \otimes J^* \to (V \otimes J^*)^*$ and we can apply previous Theorem to the classes $\hat{t}_i = -c_1((F^i/F^{i-1})\otimes J^*) = -c_1(F^i/F^{i-1}) + u/2$ and $\hat{a}_k = c_k((V\otimes J^*)^* - V\otimes J^*)$. The formulas obtained in this way can be applied to any line bundle I since they are universal, cf. [7, 10].

Remark. The classes $\hat{a}_i$ are related to the integer classes $a_i = c_i(V^* - V\otimes I)$ via

$$1 + \hat{a}_1 + \hat{a}_2 + \ldots = 1 + \frac{a_1}{1 - u/2} + \frac{a_2}{(1 - u/2)^2} + \ldots.$$

The formula of Corollary above uses the division by powers of 2. Of course the direct image under p_{r*} of any monomial in $t_i = c_1(F_i/F_{i-1})$ is an integer class and so it can be expressed as a universal polynomial with integer coefficients in $u, a_1, a_2, \ldots$. The Corollary allows to find these polynomials for any particular monomial but these expressions have no nice closed form similar to that in case $I = \mathbb{C}$.

Example. The image $p_1(Z_1) \subset M$ is the closure of the locus $A_2(f)$. Therefore,

$$A_2 = p_{1*}(1) = Q_1 = a_1.$$

This is the first formula of Table 1.

2.3 Marked singularities and their resolutions

In this section we compute the Thom polynomials for the classes A_2, A_3, D_4, D_5, E_6, P_8 using their resolutions. The other classes from the list of Theorem 1 have no good resolutions but the partial resolution of these classes simplifies the computation of Thom polynomials for them as well.

Let $f : V \to I$ be a polynomial bundle map such that the restrictions $f_w : V_w \to I_w$ have a critical point at the origin for any $w \in M$ as in Section 2.1. The second differential $f_{(2)w} = d^2 f_w : V_w \to I_w$ is a well defined at any point $w \in M$. We consider $f_{(2)}$ as a twisted self-adjoined map of vector bundles $f_{(2)} : V \to V^*\otimes I$ on M.

Let $\mathbf{F}_1 = P(V)$ be the projectivisation of the vector bundle $V \to M$. Consider the submanifold $Z_1 \subset \mathbf{F}_1$ formed by the pairs of the form (a point $w \in M$, a line $l \subset K_w$, where $K_w = \ker f_{(2)w}$ is the kernel of the second differential of f_w. We would like to represent the cycles of fibre singularities of f as the direct images of certain cycles on Z_1 under the natural projection

$$p_1 : Z_1 \to M.$$

Definition. Let $g : \mathbb{C}^n, 0 \to \mathbb{C}, 0$ be a germ of a critical point singularity and $K = \ker d^2 g \subset T_0\mathbb{C}^n$ be the kernel of its second differential. The line $l \subset K$ is called *distinguished* if the cubic form $g_{(3)}$ given by the third-order terms in the Taylor expansion of g vanishes on

l. A *marked critical point singularity* is a pair (g, l), where g is a function germ and l is a distinguished direction.

Remark that the cubic form $g_{(3)}$ is well defined on K (in a sense that it is independent on the choice of coordinates on $\mathbb{C}^n, 0$). Consider the classification of marked critical point singularities.

If g has a singularity of type A_k, $k \geq 3$, or E_k, $k = 6, 7, 8$, then the distinguished direction is unique. The corresponding marked singularity is denoted by $\widetilde{A}_k, \widetilde{E}_k$ respectively.

In case of singularity D_k, the distinguished direction may be either a simple or a double zero of the cubic form $g_{(3)}|_K$. We denote the two cases by $\widetilde{D}'_k$ and $\widetilde{D}''_k$ respectively. So for $k = 4$ there are 3 directions of $\widetilde{D}'_4$-type and for $k > 4$ there are two distinguished directions, one $\widetilde{D}'_k$ and one $\widetilde{D}''_k$.

In case of singularity P_8 the distinguished directions form a cubic curve in $\mathbb{C}P^2 = P(K)$. We use the notation $\widetilde{P}'_8$ if this is a generic point of the cubic and $\widetilde{P}''_8$ if this is one of the 9 inflection points on it.

We get the following classification of marked function singularities.

	codim
$\widetilde{A}_3$	1
$\widetilde{A}_4, \widetilde{D}'_4$	2
$\widetilde{A}_5, \widetilde{D}'_5, \widetilde{D}''_5$	3
$\widetilde{A}_6, \widetilde{D}'_6, \widetilde{D}''_6, \widetilde{E}_6, \widetilde{P}'_8$	4
$\widetilde{A}_7, \widetilde{D}'_7, \widetilde{D}''_7, \widetilde{E}_7, \widetilde{P}''_8$	5
$\dots$	

The codimension is counted in the space of function germs $g : \mathbb{C}^n, 0 \to \mathbb{C}, 0$ such that $dg(0) = 0$ and $\ker d^2 g$ contains some fixed direction $l_0 \subset T_0\mathbb{C}^n$.

The classification of marked singularities produces the corresponding classification of the points on the manifold Z_1. Namely, we say that the point $(w, l) \in Z_1 \subset P(V)$ has marked singularity $\widetilde{\Omega}$ if the function germ f_w has singularity type Ω and the line $l \subset \ker f_{(2)w}$ is distinguished.

Lemma. *The cohomology classes on M dual to the singularity loci of codimension ≤ 6 can be determined as the direct images of the following cycles on Z_1:*

$$
\begin{aligned}
A_k &= p_{1*}\widetilde{A}_k, \quad (k \geq 3); \\
D_4 &= \frac{1}{3} p_{1*}\widetilde{D}'_4; \qquad D_k \;=\; p_{1*}\widetilde{D}'_k \;=\; p_{1*}\widetilde{D}''_k, \quad (k \geq 5); \\
E_k &= p_{1*}\widetilde{E}_k, \quad (k = 6, 7, 8); \\
P_8 &= \frac{1}{9} p_{1*}\widetilde{P}''_8 \;=\; \frac{1}{3} p_{1*} t_1 \widetilde{P}'_8.
\end{aligned}
$$

P r o o f. Just note that every point $w \in M$ such that the singularity type of f_w is A_k, E_k, or $D_{>4}$ has a unique preimage in the corresponding cycle on Z_1. For the singularity D_4 there are 3 such preimages (corresponding to three zero lines of $f_{(3)w}$ on K_w) and for the singularity P_8 there are 9 preimages on $\widetilde{P}''_8$ (corresponding to 9 inflection points of the cubic $f_{(3)w} = 0$ on $P(K)$). The restriction of $t_1 = -c_1(F^1)$ to any fibre of $P(V) \to M$ is the class of a hyperplane. Hence the three preimages of a generic point $w \in P_8$ on a cycle representing $t_1\widetilde{P}'_8$ correspond to 3 points of the intersection of the cubic $f_{(3)w} = 0$ with a generic projective line in $P(K_w)$. $\square$

Since the homomorphism p_{1*} is known by results of Section 2.2, the last lemma reduces the problem to the computation of cohomology classes represented by certain cycles on Z_1.

Example. The third differential $d^3 f_w = f_{(3)w} : F^1 \to I$ is a section of the line bundle $(F^1)^{*\otimes 3} \otimes I$ on Z_1, where F_1 is the tautological line bundle on $Z_1 \subset P(V)$. The closure of the cycle $\widetilde{A}_3$ is the zero locus of this section. Therefore, $\widetilde{A}_3 = c_1((F^1)^{*\otimes 3} \otimes I) = 3t_1 + u$ and

$$A_3 = p_{1*}(3t_1 + u), \qquad t_1 = -c_1(F_1), \quad u = c_1(I).$$

which gives after applying the formulas of Section 2.2 the Thom polynomial for A_3.

Similar arguments can be used for some other marked singularity classes if we construct resolutions with help of flags instead of projective spaces. The following Lemma uses notations of Section 2.2.

Lemma. *The following relations hold for the Gysin homomorphism $p_{1*} : H^*(Z_1) \to M^{*+2}$.*

$$\begin{aligned}
p_{1*}(\widetilde{A}_3) &= p_{1*}(3t_1 + u) \\
p_{1*}(\widetilde{D}'_4) &= p_{2*}(3t_1 + u) \\
p_{1*}(\widetilde{D}''_5) &= p_{2*}((3t_1 + u)(2t_1 + t_2 + u)) \\
p_{1*}(\widetilde{D}'_5) &= p_{2*}((3t_1 + u)2(t_1 + 2t_2 + u)) \\
p_{1*}(\widetilde{E}_6) &= p_{2*}((3t_1 + u)(2t_1 + t_2 + u)(t_1 + 2t_2 + u)) \\
p_{1*}(\widetilde{P}'_8) &= p_{3*}((3t_1 + u)(2t_1 + t_2 + u)) \\
p_{1*}(\widetilde{P}''_8) &= p_{3*}((3t_1 + u)(2t_1 + t_2 + u)(t_1 + 2t_2 + u))
\end{aligned} \tag{3}$$

Applying formulas for the homomorphisms $p_{r*} : H^*(Z_r) \to M$ described in Section 2.2 we complete the computations of the Thom polynomials for singularities A_3, D_4, D_5, E_6, P_8.

Proof. The case of $\widetilde{A}_3$ is considered above. In the case of singularities $\widetilde{D}'_4, \ldots, \widetilde{E}_6$ the kernel $K_w = \ker f_{(2)w}$ has dimension 2. We can associate with any of these cycles the corresponding cycles $\widetilde{\widetilde{D}}'_4, \ldots, \widetilde{\widetilde{E}}_6 \subset Z_2 \subset \mathbf{F}_2$ by letting F^2_w to be the kernel K_w. Then we get $p_{1*}\widetilde{D}'_4 = p_{2*}\widetilde{\widetilde{D}}'_4$ and the same for $\widetilde{D}'_5, \widetilde{D}''_5, \widetilde{E}_6$. The third differential $f_{(3)w}$ can be written as

$$f_{(3)w} = a_0 x^3 + a_1 x^2 y + a_2 x y^2 + a_3 y^3$$

with an appropriate choice of coordinates (x, y) on F^2 such that the line F^1 is given by $y = 0$. The coefficients a_i are globally defined sections of the bundles $(F^1)^{*\otimes(3-i)} \otimes (F^2/F^1)^{*\otimes i} \otimes I$ with the first Chern classes $(3-i)t_1 + it_2 + u$, $i = 0, 1, 2, 3, 4$ (every section in this sequence is defined only on the zero locus of the previous one). The closures of the considered cycles are given by the equations

$$\begin{aligned}
\widetilde{\widetilde{D}}'_4 :& \quad a_0 = 0, \\
\widetilde{\widetilde{D}}''_5 :& \quad a_0 = a_1 = 0, \\
\widetilde{\widetilde{E}}_6 :& \quad a_0 = a_1 = a_2 = 0, \\
\widetilde{\widetilde{D}}'_5 :& \quad a_0 = a_2^2 - 4a_1 a_3 = 0
\end{aligned}$$

respectively which gives the formulas of Lemma (the expression $a_2^2 - 4a_1 a_3$ is a well defined section of $((F^1)^* \otimes (F^2/F^1)^{*\otimes 2} \otimes I)^{\otimes 2}$ on the zero locus of a_1).

The singularities $\widetilde{P}'_8, \widetilde{P}''_8$ are resolved in a similar way using flags $F^1_w \subset F^2_w \subset F^3_w$, where $F^3_w = K_w$, PF^2_w is the tangent line to the cubic $f_{(3)w}|F^3_w = 0$. The tangency condition is equivalent to the condition that $f_{(3)w}|F_w^2$ has a double zero on F_w^1 and the condition that the tangent line is a point of inflection is equivalent to the condition that $f_{(3)w}|F_w^2$ has a triple zero on F_w^1. Arguing as above we arrive to the expressions for the direct images of $\widetilde{\widetilde{P}}{}'_8, \widetilde{\widetilde{P}}{}''_8$. $\square$

2.4 Adjacency exponents

The Thom polynomials for classes not considered in previous Section are computed using relations between them. These relations are applied not to the classes themselves but to the classes of their partial resolutions.

Lemma (basic relations). *The cohomology classes of cycles $\widetilde{A}_3, \widetilde{A}_4, \ldots \widetilde{P}''_8 \subset Z_1$ are subject to the following relations.*

$$\begin{aligned}
(4t_1+u)\,\widetilde{A}_3 &= \widetilde{A}_4 - \widetilde{D}'_4 \\
(5t_1+u)\,\widetilde{A}_4 &= \widetilde{A}_5 - 2\widetilde{D}''_5 \\
(6t_1+2u)\,\widetilde{D}'_4 &= -\widetilde{D}'_5 + 4\widetilde{D}''_5 \\
(6t_1+u)\,\widetilde{A}_5 &= \widetilde{A}_6 - 2\widetilde{D}''_6 - 3\widetilde{E}_6 \\
(8t_1+3u)\,\widetilde{D}'_5 &= -\widetilde{D}'_6 + 12\widetilde{E}_6 + 4\widetilde{P}'_8 \\
(4t_1+u)\,\widetilde{D}''_5 &= \widetilde{D}''_6 - \widetilde{P}'_8 \\
(7t_1+u)\,\widetilde{A}_6 &= \widetilde{A}_7 - 2\widetilde{D}''_7 - 5\widetilde{E}_7 \\
(10t_1+4u)\,\widetilde{D}'_6 &= -\widetilde{D}'_7 + 16\widetilde{E}_7 \\
(5t_1+u)\,\widetilde{D}''_6 &= \widetilde{D}''_7 - \widetilde{E}_7 \\
(4t_1+u)\,\widetilde{E}_6 &= \widetilde{E}_7 - \widetilde{P}''_8
\end{aligned} \tag{4}$$

F i n a l c o m p u t a t i o n s o f p o l y n o m i a l s o f T h e o r e m 1. A part of Thom polynomial are computed in previous section. Using relations (3) and (4) we get subsequently

$$\begin{aligned}
\widetilde{A}_4 &= (4t_1+u)\widetilde{A}_3 + \widetilde{D}'_4, \\
p_{1*}(\widetilde{A}_4) &= p_{1*}((4t_1+u)(3t_1+u)) + p_{2*}(3t_1+u); \\
\widetilde{A}_5 &= (5t_1+u)\widetilde{A}_4 + 2\widetilde{D}''_5, \\
p_{1*}(\widetilde{A}_5) &= p_{1*}((5t_1+u)(4t_1+u)(3t_1+u)) + p_{2*}((9t_1+2t_2+3u)(3t_1+u));
\end{aligned}$$

and so on. Continuing this way we get expressions for the direct images of all cycles on Z_1 of codimension less or equal to 5. Applying formulas for the Gysin homomorphisms from Section 2.2 we obtain all Thom polynomials of Theorem 1. Note that the expressions for $p_{1*}(\widetilde{D}'_k)$ are not necessary for computing Thom polynomials. We use these expressions only to verify our computations. $\square$

The reason for the basic relations is as follows. Consider a germ of some marked singularity, say $\widetilde{A}_k$. In some coordinate system it can be written as

$$z = x^{k+1} + Q,$$

where Q is a nondegenerate quadratic form in the remaining variables. Such a coordinate system is not unique. Another choice of the coordinate system results in the multiplication of the tangent vector $\partial/\partial x$ to the distinguished line by some complex number c and the simultaneous multiplication of the tangent vector $\partial/\partial z$ to the target space by c^{k+1}. Therefore the tensor

$s = dx^{\otimes(k+1)} \otimes \partial/\partial z$ is invariantly defined. We obtain that the bundle $(F^1)^{*\otimes(k+1)} \otimes I$ restricted to the cycle $\widetilde{A}_k \subset Z_1$ admits a canonical nowhere vanishing section s.

Lemma. *The restrictions of the line bundles listed in the table below to the corresponding singularity loci in Z_1 are trivial.*

	f	bundle Y	$c_1(Y)$
$\widetilde{A}_k$	x^{k+1}	$(F^1)^{*\otimes k+1} \otimes I$	$(k+1)t_1 + u$
$\widetilde{D}'_k$	$y^{k-1} + yx^2$	$(F^1)^{*\otimes(2k-2)} \otimes I^{\otimes(k-2)}$	$(2k-2)t_1 + (k-2)u$
$\widetilde{D}''_k$	$x^{k-1} + xy^2$	$(F^1)^{*\otimes(k-1)} \otimes I$	$(k-1)t_1 + u$
$\widetilde{E}_6$	$x^4 + y^3$	$(F^1)^{*\otimes 4} \otimes I$	$4t_1 + u$
$\widetilde{E}_7$	$x^3y + y^3$	$(F^1)^{*\otimes 9} \otimes I^{\otimes 2}$	$9t_1 + 2u$
$\widetilde{E}_8$	$x^5 + y^3$	$(F^1)^{*\otimes 5} \otimes I$	$5t_1 + u$

The canonical sections for the bundles from this lemma are chosen so that for the (marked) functions in the normal form above the coordinate of this section is equal to 1 (the distinguished line is the x-axis).

The proof for all cases is the same as for the case of singularity $\widetilde{A}_k$. The symmetry group of all these singularities acts on the lines F^1 and I by quasi-homogeneous homoteties. Therefore, the required bundle can be chosen in the form $(F^1)^{*\otimes\alpha} \otimes I^{\otimes\beta}$, where α/β is equal to the quotient of quasi-homogeneous weights of the function and of the variable x respectively. $\square$

Let $\Omega \subset Z_1$ be a cycle of some marked singularity from the last Lemma, $Y \to Z_1$ be the corresponding line bundle whose restriction to Ω is trivialised. This trivialisation of Y can not be extended to the closure of Ω since the bundle Y is not trivial. Let $\Theta \subset Z_1$ be a singularity class of neighbouring complex codimension, $\operatorname{codim}\Theta = \operatorname{codim}\Omega + 1$. Choose some point $w \in \Theta$ and a $(\operatorname{codim}\Theta)$-dimensional transversal slice T to Θ at this point. The singularity locus Ω cut out a number of curves $\gamma_1, \gamma_2, ...$ on T. Let $(\mathbb{C}, 0) \to (T, w)$ be a normalisation (= parameterisation) of one of these curves γ_i. Then the canonical section s of Y on $\gamma_i \subset \Omega$ can be written (using some local trivialisation of Y near w) in the form $s = \tau^{k_i} h_i$ where τ is a parameter on the curve and h_i is a germ of a holomorphic non-vanishing function.

Definition. The *adjacency exponent* $[\Omega, \Theta]$ is the sum of the exponents k_i over all curves γ_i of singularity Ω in the transversal to the singularity Θ.

Lemma. *The following equality holds in the cohomology of Z_1,*

$$c_1(Y)\,\Omega = \sum [\Omega, \Theta]\,\Theta,$$

where the sum is taken over all classes of marked singularities $\Theta \subset Z_1$ with $\operatorname{codim}\Theta = \operatorname{codim}\Omega +$ 1.

P r o o f. Let $Q \subset Z_1$ be a test compact cycle of real dimension $2(\operatorname{codim}_{\mathbb{C}}\Omega + 1)$. Without loss of generality we assume that Q intersects Ω transversally so that $D = Q \cap \Omega$ is a real surface without some finite set $\operatorname{Sing} D$ of points corresponding to intersections of Q with singularity classes of (complex) codimension $\operatorname{codim}_{\mathbb{C}}\Omega + 1$. Then

$$(c_1(Y)\,\Omega, Q) = (c_1(Y), \overline{D}).$$

The last number can be computed using the restriction of the section s to the cycle D. It is equal to the sum of indices $\operatorname{ind}_x(s)$ of this section over all points of $\operatorname{Sing} D$. But every such index

$\mathrm{ind}_x(s)$ is equal, by definition, to the adjacency exponent of singularities Ω and $\Theta \ni x$. Hence, the sum of the indices is equal to the intersection number of Q with the linear combination of cycles $\sum[\Omega,\Theta]\,\Theta$ over all classes $\Theta \subset Z_1$ with $\operatorname{codim}\Theta = \operatorname{codim}\Omega + 1$.

Thus both sides of the equality of Lemma take the same values on the elements of homology group of the complement dimension. This proves the equality of Lemma modulo torsion. In fact, this equality holds for any group of coefficients since the group of characteristic classes of complex vector bundles is torsion free. $\square$

To complete the proof of basic relations (4) we need to compute the adjacency exponents for marked singularity classes. Finding adjacency exponents is a part of the proof which really requires a lot of computations. One should find all possible adjacencies of classes of neighbour codimensions and to compute the adjacency exponents. In these computations the methods and results from [1, 18, 15] are used.

Lemma. *The following lists and the comments below exhaust all possible adjacencies of marked singularity classes of* codim ≤ 6. (*The distinguished direction in the lists below is the diection of the x-axis. The functions of the family corresponding to the adjacency $\Theta \to \Omega$ have singularity Ω at the origin for all parameter values $\tau \neq 0$ and Θ for $\tau = 0$. In the table below s is the canonical section, $c \in \mathbb{C}$ is a constant.*)

$\Theta \to \Omega$	f	s	$[\Omega:\Theta]$	*notes*
$\widetilde{A}_{k+1} \to \widetilde{A}_k$	$x^{k+2} + \tau x^{k+1}$	$c\tau$	1	
$\widetilde{D}'_{k+1} \to \widetilde{D}'_k$	$y^k + x^2y + \tau y^{k-1}$	$c\tau^{-1}$	-1	
$\widetilde{D}''_{k+1} \to \widetilde{D}''_k$	$x^k + xy^2 + \tau x^{k-1}$	$c\tau$	1	
$\widetilde{D}'_4 \to \widetilde{A}_3$	$y^3 + x^2y + \tau y^2$	$c\tau^{-1}$	-1	
$\widetilde{D}''_5 \to \widetilde{D}'_4$	$x^4 + xy^2 + \tau x^2y$	$c\tau^4$	4	
$\widetilde{D}''_5 \to \widetilde{A}_4$	$(x^2+\tau y)^2 + xy^2$	$c\tau^{-2}$	-2	
$\widetilde{D}''_6 \to \widetilde{A}_5$	$x^5 - xy^2 + \tau(y \pm x^2)^2$	$c\tau^{-1}$	-2	(i)
$\widetilde{E}_6 \to \widetilde{A}_5$	$y^3 + (x^2+\tau y)^2$	$c\tau^{-3}$	-3	
$\widetilde{E}_6 \to \widetilde{D}'_5$	$x^4 + y(y+\tau x)^2$	$c\tau^{12}$	12	
$\widetilde{E}_6 \to \widetilde{D}''_5$	$x^4 + (y+\tau x)y^2$	c	0	
$\widetilde{P}'_8 \to \widetilde{D}'_5$	$x^2y + z^3 - zy^2 + \tau z^2$	$c\tau$	4	(ii, iii)
$\widetilde{P}'_8 \to \widetilde{D}''_5$	$y^2x + z^3 - zx^2 + \tau z^2$	$c\tau^{-1}$	-1	(iii)
$\widetilde{D}''_7 \to \widetilde{A}_6$	$x^6 + xy^2 - \tau^2x^5 + (\tau^2x^2 + \tau y)^2$	$c\tau^{-2}$	-2	
$\widetilde{E}_7 \to \widetilde{A}_6$	$y^3 + (x^2 - 4\tau y)(xy + \tau x^2 - 4\tau^2 y)$	$c\tau^{-5}$	-5	
$\widetilde{E}_7 \to \widetilde{D}'_6$	$x^3y + (y+\tau x)^2y + \tau x^4$	$c\tau^{16}$	16	
$\widetilde{E}_7 \to \widetilde{D}''_6$	$x^3y + (y+\tau x)y^2$	$c\tau^{-1}$	-1	
$\widetilde{E}_7 \to \widetilde{E}_6$	$x^3y + \tau y^4 + y^3$	$c\tau$	1	
$\widetilde{P}''_8 \to \widetilde{E}_6$	$y^3 + x^2z + xz^2 + \tau z^2$	$c\tau^{-1}$	11	(iii)

Comments. (i) There are 2 curves realizing the adjacency $\widetilde{D}''_6 \to \widetilde{A}_5$ corresponding to the two possible signs of $\pm$. For both of them we have $s = c\tau^{-1}$ so $[\widetilde{A}_5 : \widetilde{D}''_6] = -2$.

(ii) There are 4 curves realizing the adjacency $\widetilde{P}'_8 \to \widetilde{D}'_5$ (see below). For all of them we have $\lambda = c\tau$ so $[\widetilde{D}'_5 : \widetilde{P}'_8] = 4$.

(iii) The singularity $\widetilde{P}''_8$ has a module (of a plane cubic). Similarly $\widetilde{P}'_8$ has two modules (a plane cubic and a point on it). In the formulas above we used some particular values of modules. The adjacency exponent does not depend on the choice of modules.

(iv) The critical point function singularity P_8 is not adjacent neither to D_6, nor to A_6, see [15].

(v) There are no adjacencies $\widetilde{D}'_{k+1} \to \widetilde{A}_k$ $(k > 3)$, $\widetilde{D}''_{k+1} \to \widetilde{D}'_k$ $(k > 4)$, $\widetilde{D}'_{k+1} \to \widetilde{D}''_k$.

(vi) There is no adjacency $\widetilde{P}'_8 \to \widetilde{A}_5$.

P r o o f. The method of finding the adjacencies is described in details in [18]. The condition that a function has a singularity of certain type is reformulated as a system of algebraic equations on the coefficients of the Taylor expansion of the function. These equations may be explicitly solved which gives the formulas above. Most of these formulas (except those related to adjacencies of 'new' singularity type P'_8, see below) are taken from [1, 18, 15] (sometimes with a minor change of variables).

Essentially new part of our calculations is finding the adjacency exponents. As an example we show the computation of the asymptotic $s = c\tau^{-5}$ for the adjacency $\widetilde{E}_7 \to \widetilde{A}_6$ above. Consider the family of function germs

$$f(x, y; \tau) = y^3 + (x^2 - 4\tau y)(xy + \tau x^2 - 4\tau^2 y).$$

For $\tau \neq 0$ the partial derivative

$$f_y = 32\tau^3 y - 8\tau^2 x^2 - 8\tau xy + 3y^2 + x^3$$

has no critical point at the origin. Therefore by parametric Morse Lemma this function is stably equivalent to its restriction to the smooth curve $f_y = 0$. This equation defines implicitly y as a function in x

$$y = \frac{x^2}{4\tau} + \frac{x^3}{32\tau^3} + \frac{x^4}{512\tau^5} - \frac{x^5}{1024\tau^7} + o(x^5).$$

After substitution to f we get

$$f|_{f_y=0} = \frac{x^7}{512\tau^5} + o(x^7).$$

This means that the singularity type of f is A_6 for $\tau \neq 0$ and

$$s = \frac{1}{512\tau^5}; \qquad [\widetilde{A}_6 : \widetilde{E}_7] = -5.$$

The cases of other singularities are treated in a similar way.

Let us describe in more details adjacencies of the singularity $\widetilde{P}'_8$. The function f_0 realizing this singularity is a cubic form in three variables x, y, z. The distinguished direction $P \in \mathbb{C}P^2$ of the x-axis belongs to the cubic $C \in \mathbb{C}P^2$ given by $f_0 = 0$. The codimension of the class $\widetilde{P}'_8$ is 3. A possible transversal is given by the family $f_0 + Q$ where $Q = \lambda_1 y^2 + 2\lambda_2 yz + \lambda_3 z^2$ is the family of quadratic forms having the direction $\partial/\partial x$ in the kernel. By homogeneity all functions of the family

$$f = f_0 + \tau Q_0$$

are right equivalent to each other for $\tau \neq 0$ for any fixed quadratic form Q_0. Hence any adjacency is realized by a family of this type. The function f has the singularity D_5 iff $Q_0 = l^2$

where $l = 0$ is the equation of the tangent to the cubic C. If the tangency point is P then the distinguished direction P is of $\widetilde{D}_5''$ type in our classification. It is possible also that the line $l = 0$ passes through P and is tangent to C at another point. Then the distinguished direction P has $\widetilde{D}_5'$ type. Generically there are 4 such lines. So there are 4 curves of singularity $\widetilde{D}_5'$ in the transversal to $\widetilde{P}_8'$.

Now let us prove the equality $[\widetilde{A}_5 : \widetilde{P}_8''] = 0$. It is sufficient to show that $f_0 = xy^2 + z^3 - zx^2$ is not adjacent to $\widetilde{A}_5$. This would imply that no singularity of type $\widetilde{P}_8'$ close to f_0 is adjacent to $\widetilde{A}_5$ and neither are $\widetilde{P}_8'$-singularities from a Zarisski open set in the space of modules and hence $[\widetilde{A}_5 : \widetilde{P}_8''] = 0$. So assume that a function of the form $f_0 + Q_0$ with $Q_0 = \lambda_1 y^2 + 2\lambda_2 yz + \lambda_3 z^2$ has singularity A_5 at the origin. Then the form Q_0 is non-degenerate and f is stably equivalent to its restriction $f|_{f_y=f_z=0}$. Resolving the system $f_y = f_z = 0$ we get

$$f|_{f_y=f_z=0} = c_4x^4 + c_5x^5 + o(x^5), \qquad c_4 = \frac{\lambda_1}{4(\lambda_2^2 - \lambda_1\lambda_3)}, \qquad c_5 = \frac{\lambda_2^2}{4(\lambda_2^2 - \lambda_1\lambda_3)^2}.$$

It is clear that the system $c_4 = c_5 = 0$ has no solution that is the function $f_0 + Q_0$ cannot be of A_5-type. □

Combining two last Lemmas we complete the proof of the basic relations Lemma formulated at the beginning of this Section. □

2.5 Symmetries and Thom polynomials

In this section we describe a method of computing Thom polynomials which is based on Rimányi's idea of using symmetries. Unlike the direct method for computing Thom polynomials described in previous sections this method uses an a priori Theorem 1 about the existence of these polynomials. This method is less geometric but it uses less computations. The idea is the following. We know that the class dual to some singularity locus Ω is given by a certain polynomial P_Ω in Chern classes so we need to compute the coefficients of this polynomial. Every example where both the cohomology class dual to the singularity locus of $\Omega \subset M$ and the classes u, a_i can be computed explicitly gives rise to a number of linear relations on the coefficients of this polynomial P_Ω. If the number of examples is high enough then these relations could determine the polynomial completely. A number of examples are produced in the following way. Consider some quasihomogeneous family of function germs realizing a transversal to some singularity class, say

$$f(x, y, a_1, \ldots, b_2) = y^5 + x^2y + a_1y^4 + a_2y^3 + a_3y^2 + b_1x^2 + b_2xy.$$

Consider some line bundle $\xi \to B$ with the first Chern class $t = c_1(\xi) \in H^2(B)$. With any variable $x, y, a_1, \ldots, b_2$ we associate a line bundle $\xi^{\otimes l}$ where l is the quasihomogeneous weight of the variable. Then the family f may be interpreted as a quasihomogeneous bundle map

$$f : \xi^{\otimes 2} \oplus \xi \oplus \xi \oplus \cdots \oplus \xi^{\otimes 2} \to \xi^{\otimes 5}, \qquad x \oplus y \oplus a_1 \oplus \cdots \oplus b_2 \mapsto f(x, y, a_1, \ldots, b_2).$$

Now define M to be the total space of the bundle $\pi : \xi \oplus \cdots \oplus \xi^{\otimes 2} \to B$ corresponding to the parameters $a_1, \ldots, b_2$ of the family f; $V \to M$ to be the rank 2 bundle $\pi^*\xi^{\otimes 2} \oplus \pi^*\xi$ over M corresponding to variables x, y; and $I = \pi^*\xi^{\otimes 5}$. So we constructed vector bundles $V, I \to M$ and a smooth bundle map $f : V \to I$ as in Section 2.1. The characteristic classes in this example are

$$u = c_1(I) = 5t, \quad a = c(V^* \otimes I - V) = \frac{(1 + (5-1)t)(1 + (5-2)t)}{(1+t)(1+2t)} = 1 + 4t - 2t^2 + \ldots.$$

On the other hand we may compute the classes dual to the singularity loci. This gives the following relations on the coefficients of the Thom polynomials $P_\Omega(u, a_1, a_2, \ldots)$:

- If $\Omega = D_6$ then $\Omega(f)$ is the zero section of the bundle $M \to B$ and its dual cohomology class is the top Chern class $e = c_5(M \to B)$ of this bundle, so

$$P_{D_6}(5t, 4t, -2t^2, \ldots) = t \cdot 2t \cdot 3t \cdot t \cdot 2t = 12t^5.$$

- If $\Omega = A_6, E_6$ or Ω is any singularity class of greater codimension then $\Omega(f) = \varnothing$ and so $P_\Omega(5t, 4t, -2t^2, \ldots) = 0$.

(We may compute the classes $P_{D_k}(5t, 4t, -2t^2, \ldots)$, $k < 6$ in a similar way but these extra relations are redundant.) The characteristic classes for quasihomogeneous deformations of other singularities of codimension ≤ 6 are given in the following table.

Ω	u	$a = 1 + a_1 + a_2 + \ldots$	e
A_k	$(k{+}1)\,t$	$\dfrac{1+kt}{1+t}$	$(k{-}1)!\,t^{k-1}$
D_k	$2(k{-}1)\,t$	$\dfrac{(1+2(k{-}2)t)(1+kt)}{(1+2t)(1+(k{-}2)t)}$	$(k{-}2)!!\,t^{k-1}$
E_6	$12\,t$	$\dfrac{(1+9t)(1+8t)}{(1+3t)(1+4t)}$	$6!\,t^5$
E_7	$9\,t$	$\dfrac{(1+7t)(1+6t)}{(1+2t)(1+3t)}$	$3 \cdot 5!\,t^6$
P_8	$3\,t$	$\dfrac{(1+2t)^3}{(1+t)^3}$	t^6

Relations arising from these examples are sufficient to compute all Thom polynomials of Theorem 1. □

3 Characteristic classes of Lagrange and Legendre singularities

The correspondence between Lagrange (Legendre) singularities and critical point singularities described in this section is valid for both real and complex cases. We assume some familiarity of the reader with the theory of Lagrange and Legendre singularities, see, eg. [4, 2]. Our definitions should not be considered for the introduction to symplectic or contact geometry.

3.1 Lagrange singularities and characteristic classes

A Lagrange singularity is a projection singularity of a Lagrange submanifold in the space of cotangent bundle to the base of the bundle.

Definition. A submanifold $M \subset T^*B$, $\dim M = \dim B$ is called *Lagrangian*, if the *standard symplectic 2-form* $\omega = \sum dp_i \wedge dq_i$ vanishes on L, where q_i are some local coordinates on the base B, and p_i a the corresponding coordinates on the fibres of the cotangent bundle $T^*B \to B$.

With any germ of Lagrange submanifold in the cotangent bundle one can associate a critical point function singularity. Namely, any Lagrange germ $M\subset T^*\mathbb{C}^n$ may be given by a germ of its generating family of functions $F(x,q)$ according to the rule (cf. [4, 2])

$$L=\Big\{(p,q)\in T^*\mathbb{C}^n\;\Big|\;\exists x,\;\partial F/\partial x=0,\;p=\partial F/\partial q\Big\}.$$

Here x is the coordinate on the fibres of an auxiliary bundle $(\mathbb{C}^{n+m},0)\to(\mathbb{C}^n,0)$. We associate with the Lagrange germ M the the initial function germ of its generating family $f(x)=F(x,0)$. The generating family is not unique but another choice of the family or of the coordinates on the base lead to R_{st}-equivalent function germs. (Recall that two function germs in spaces of possibly different dimensions are called R_{st}-*equivalent* (stably right equivalent) if after adding suitable non-degenerate quadratic forms in new variables each of these functions can be brought into another by a change of variables.)

Example. Let V be a vector space. With any function germ $f:V,0\to\mathbb{C},0$ we associate a Lagrange germ $L(f)\subset T^*V$, the graph of the differential $-df$ (it is convenient to put the sign '$-$' here). The natural isomorphisms $T^*V\cong V\oplus V^*\cong TV^*$ allows to consider $L_w(f)$ also as a submanifold in T^*V^*. The symplectic structures induced on $V\oplus V^*$ by this isomorphisms differ by sign. Hence, $L(f)$ is also Lagrange as a submanifold in T^*V^*. The critical point function singularity corresponding to the germ $L(f)\subset T^*V^*$ is the singularity of the function f itself. Indeed, $L(f)\subset T^*V^*$ may be given by the following generating family

$$F(x,q)=\langle x,q\rangle+f(x),\qquad q\in V^*,\quad x\in V.$$

Here x is considered as an additional variable and $\langle\cdot,\cdot\rangle:V\times V^*\to\mathbb{C}$ is the natural coupling.

Definition. The *classifying space of Lagrange singularities* $\mathcal{L}=\mathcal{L}_N^{\mathbb{C}}$ is the space of all K-jets of Lagrange germs $M\subset(T^*\mathbb{C}^N,0)=(\mathbb{C}^{2N},0)$, where $K,N\gg0$ are some large integers. This space is homotopy equivalent to the *Lagrange Grassmannian* $\Lambda=\Lambda_N^{\mathbb{C}}$ consisting of Lagrange planes in $\mathbb{C}^{2N}$ since the natural projection $\mathcal{L}\to\Lambda$ sending a Lagrange germ to its tangent plane have contractible fibres. The ring of *Lagrange characteristic classes* is the limit cohomology ring $\lim\limits_{N\to\infty}H^*(\Lambda_N^{\mathbb{C}})$.

The topology of real Lagrange Grassmannian is well studied (see [6]). Its $\mathbb{Z}_2$-cohomology ring $H^*(\Lambda_N^{\mathbb{R}})$ is generated by Stiefel-Whitney classes α_i of the tautological bundle, the generators α_i satisfy relations $\alpha_i^2{=}0$. Similar description exists for the (integer) cohomology ring of the complex Lagrange Grassmannian.

Theorem ([17, 10]). *The ring of Lagrange characteristic classes is isomorphic to the quotient of polynomial ring in variables $a_1,a_2,\dots$ of degrees $2,4,\dots$ over the ideal generated by elements*

$$a_i^2-2a_{i+1}a_{i-1}+2a_{i+2}a_{i-2}-2a_{i+3}a_{i-3}+\dots. \tag{5}$$

The group of Lagrange characteristic classes is torsion free and the monomials $a_1^{i_1}\dots a_n^{i_n}$, $i_k\in\{0,1\}$ form a free additive basis.

For the generators we choose the classes $a_i=c_i(\mathbb{C}^{2N}/L)=c_i(L^*)=(-1)^ic_i(L)\in H^{2i}(\Lambda_N^{\mathbb{C}})$, where L is the tautological bundle $L\to\Lambda_N^{\mathbb{C}}$. To express an element of this ring in terms of the additive basis one should apply repeatedly relation (5) to every monomial which contains squares of generators. This will require a finite number of steps since every newly appeared monomial has the degree strictly less than the original one if one uses the 'strange' filtration with the degree of a_i equal i^2.

The meaning of the relations is the following. The symplectic form induces the canonical isomorphism $\mathbb{C}^{2N}/L \cong L^*$. By Whitney formula we have $c(L)c(L^*) = 1$ or

$$(1 + a_1 + a_2 + \ldots)(1 - a_1 + a_2 - \ldots) = 1$$

which is equivalent to (5).

By construction above the points of $\mathcal{L}$ are classified according to the $R_{\rm st}$-classification of function germs. With any singularity class Ω (given as an R-invariant algebraic subset in some jet space of function germs) we associate the corresponding subvariety $\Omega(\mathcal{L}) \subset \mathcal{L}$.

Definition. The *Thom polynomial* associated with an $R_{\rm st}$-class Ω of critical point singularity is the universal Lagrange characteristic class $P_\Omega \in H^*(\Lambda)$ (expressed in terms of multiplicative generators a_i) represented by the intersection with the variety $\Omega(\mathcal{L}) \subset \mathcal{L}_N$.

This definition is independent on K, N provided these numbers are large enough (N must be larger than the codimension of the singularity and K is chosen so that the K-jet of the singularity is sufficient, see [4, 3]).

Proof of Theorem 2 for the trivial bundle I. Consider vector bundles $V, I \to M$ and a fibre bundle map $f : V \to I$ as in Section 2.1. Assume that the bundle I is *trivial*, $I = \mathbb{C}$.

For each point $w \in W$ we define the Lagrange germ $L_w(f) \subset T^*V_x$ as above. We would like to extend this correspondence between the critical point singularities and the Lagrange singularities and to construct a classifying map $M \to \mathcal{L}$ which preserves the Lagrange (or critical point) singularity type at considered points.

The construction is as follows. Consider a bundle $U \to M$ such that $V \oplus U$ is the trivial bundle $\mathbb{C}^N$. Then $V \oplus V^* \times U \oplus U^* \cong \mathbb{C}^{2n}$ is also trivial. Hence all Lagrange germs $\kappa(w) = L_w(f) \times 0 \oplus U^*_w \subset V_w \oplus V^*_w \times U_w \oplus U^*_w = T^*(V^*_w \times U^*_w) = T^*\mathbb{C}^N$ belong to the same symplectic space $\mathbb{C}^{2N}$. One can see that the critical point singularity corresponding to $\kappa(w)$ is the same as for $L_w(f)$, i.e. f_w. Thus constructed map

$$\kappa : M \to \mathcal{L}_N, \qquad w \mapsto L_w(f) \times 0 \oplus U^*,$$

induces both the classes dual to the loci of Lagrange singularities and the characteristic classes $a_i \in H^*(\mathcal{L}_N)$. By definition, $\kappa^* a_i = c_i((V \oplus U^*)^*) = c_i(V^* - V)$.

We have proved, therefore, that *the cohomology class on M Poincaré dual to the locus $\Omega(f) \subset M$ is equal to the defined above polynomial P_Ω evaluated on the classes $a_i = c_i(V^* - V)$.* This proves Theorem 2 of Section 2.1 in case when the bundle I is trivial. □

Theorem. *The Thom polynomials of $R_{\rm st}$-singularities of function of codimension ≤ 6 are obtained from the polynomials of the list of Table 1 by setting $u = 0$.*

Proof. There are two possible proofs of this theorem. First we observe simply that the homomorphism $\mathbb{Z}[a_1, a_2, \ldots] \to \mathbb{Z}[c_1, c_2, \ldots]$ which sends the generator a_i to the ith homogeneous term of the expansion $(1 - c_1 + c_2 - \ldots)(1 + c_1 + c_2 + \ldots)^{-1}$ induces an injective homomorphism of the ring of Lagrange characteristic classes to the polynomial ring $\mathbb{Z}[c_1, c_2, \ldots]$ (see [10]). It follows that the formulas for the characteristic classes found for the case of fiber singularities can be applied to the case of Lagrange singularities.

In another proof we oserve that all steps of our computations made in Sections 2.2–2.4 (including resolutions, the formula for the Gysin homomorphism, markings, adjacensy exponents and basic relations) can be carried out directly for the case of Lagrange singularities. For instance, the kernel of the second differential $f_{(2)}$ of a function germ corresponds to the intersection of the thangent plane of a Lagrange germ $L \subset \mathbb{C}^{2N}$ with the fixed Lagrange plane

$\mathbb{C}^N \subset \mathbb{C}^{2N}$, zeroes of the third-order terms $f_{(3)}$ on $\ker f_{(2)}$ correspond to the lines of higher order of tangency of Lagrange submanifolds ect. In fact, our original computation of Thom polynomials of Theorem 1 was performed on the languaue of Lagrange (or Legendre) singularities and only later we translated it to the language of fiber singularities of functions. $\square$

Example (characteristic classes of Lagrange submanifolds in T^*B). Let $M \to T^*B$ be a Lagrange immersion. A similar construction exists for the map $\kappa : M \to \Lambda_N$ which preserves critical point singularity types associated with Lagrange germs (see also [5]). The characteristic classes induced by this map are $a_i = c_i(T_M B - TM)$. Again, the classes dual to different singularity loci of Lagrange projection $M \to T^*B \to B$ are given by universal Thom polynomials evaluated on the classes $c_i(T_M B - TM)$.

Remark. The most general situation where Lagrange characteristic classes appear is the following. Let $E \to M$ be a vector bundle of even rank equipped with symplectic bilinear forms on its fibers (given as a nowhere degenerating section of $\Lambda^2 E^*$). Let $L_1, L_2 \subset E$ be two Lagrange subbundles (in a sense that the fibers of L_1, L_2 are Lagrange planes in the fibers of E. Then the *relative Chern classes* (cf. [16]) of the triple (E, L_1, L_2) are defined as $a_i = c_i(L_2^* - L_1)$. The equalities $c(L_1 + L_1^*) = c(L_2 + L_2^*) = c(E)$ imply the identity 5 for these classes.

The situations considered above fit into this pattern. In case of fiber bundle map $f : V \to \mathbb{C}$ we take $E = V \oplus V^*$, $L_1 = V \oplus 0$, and $L_2 \cong L_1$ is the bundle of Lagrange planes tangent to the germs of $L_w(f) \subset V_w \oplus V_w^*$.

In case of Lagrange immersion $M \to T^*B$ we set $E = T_M(T^*B)$, $L_1 = TM$, and $L_2 \cong (T^*B)|_M$ is the bundle of 'vertical' tangent vectors to T^*B, corresponding to the kernel of the differential of the projection $T^*B \to B$. $\square$

3.2 Legendre singularities and characteristic classes

The theory of Legendre characteristic classes is a twisted version of the theory of Lagrange ones. Consider vector spaces V, I such that $\dim I = 1$. The space $V \oplus V^* \otimes I$ has the natural nondegenerate skew-symmetric bilinear form with values in I. After any isomorphism $I \cong \mathbb{C}$ this form turns into the standard symplectic form on $V \oplus V^* \cong T^*V$. The Grassmannian of Lagrange planes in $V \oplus V^* \otimes I$ with respect to this form is isomorphic to the usual Lagrange Grassmannian Λ_n, where $n = \dim V$.

Definition. The *Legendre Grassmannian* $\widetilde{\Lambda} = \widetilde{\Lambda}_N^{\mathbb{C}}$ is the total space over $\mathrm{BU}(1)$ whose fibers are formed by the Grassmannians of Lagrange subspaces in the twisted symplectic fibers of the bundle $\mathbb{C}^N \oplus \mathbb{C}^N \otimes \xi$, where $\mathrm{BU}(1) \cong \mathbb{C}\mathrm{P}^\infty$ is the classifying space of one-dimensional vector bundles (or some its finite-dimensional approximation $\mathbb{C}P^{N'}$, $N' \gg 0$), and $\xi \to \mathrm{BU}(1)$ is the canonical line bundle. The ring of *Legendre characteristic classes* is the cohomology ring of the stable Legendre Grassmannian $H^*(\widetilde{\Lambda}) = \lim\limits_{n\to\infty} H^*(\widetilde{\Lambda}_N)$.

Theorem (cf. [10]). *The ring of Legendre characteristic classes is given by generators $u, a_1, a_2, \ldots$, and relations which are homogeneous components of the equality*

$$(1 + a_1 + a_2 + a_3 + \ldots)\left(1 - \frac{a_1}{1+u} + \frac{a_2}{(1+u)^2} - \frac{a_3}{(1+u)^3} + \ldots\right) = 1. \qquad (6)$$

The class $u = c_1(\xi)$ is the standard generator of $H^*(\mathrm{BU}(1))$ and the classes a_i are defined as $a_i = c_i(-L)$, where L is the tautological bundle over the Grassmannian. The relation above comes from the isomorphism $(\mathbb{C}^N \oplus \mathbb{C}^N \otimes \xi)/L \cong L^* \otimes \xi$, or, formally, $L + (L^* - \mathbb{C}^N) \otimes \xi = 0$.

Remark. The description above is valid for both complex case and integer coefficients (with $\deg a_i = 2i$) and real case and $\mathbb{Z}_2$-coefficients (with $\deg a_i = i$). Note also that the monomials $u^{i_0} a_1^{i_1} a_2^{i_2}$, $i_0 \geq 0$, $i_k \in \{0,1\}$ for $k > 0$, form a free additive basis. Nevertheless even for the case of $\mathbb{Z}_2$-coefficients it is not isomorphic to $\mathbb{Z}_2[u] \otimes \Lambda_{\mathbb{Z}_2}(a_1, a_2, \ldots)$ (The multiplicative structure in the ring of Legendre characteristic $\mathbb{Z}_2$-classes is wrongly computed in [9].) Indeed, the relation of degree 2 is $a_1 u + a_1^2 = 0 \pmod 2$ and so the square of none element of degree 1 vanishes. On the other hand one can show that for the cohomology with coefficients in any field K of characteristic different from 2 there is an isomorphism $H^*(\widetilde{\Lambda}, K) \cong K[u] \otimes H^*(\Lambda, K)$.

P r o o f. The class u generates the cohomology of the base and the classes $a_1, a_2, \ldots$ generate the cohomology of each fibre. It follows that the spectral sequence of the bundle $\widetilde{\Lambda} \to \mathrm{BU}(1)$ degenerate at the second term and the classes $u, a_1, a_2, \ldots$ generate the whole cohomology ring of $\widetilde{\Lambda}$. We know already some set of relations and comparing the dimensions we see that there are no other relations between the generators. $\square$

Now we explain the relationship between the definition above and the theory of Legendre singularities. Let V, I be smooth manifolds (not necessary vector spaces) of dimensions $n, 1$ respectively. *The space* $J^1(V, I)$ *of* 1*-jets of maps* $V \to I$ is the total space of the bundle $T^*V \otimes TI$ over $V \times I$. This space carries the natural contact structure (a codimension 1 subbundle in the tangent bundle). If $z : I \to \mathbb{C}$ is a local coordinate on I then we get $J^1(V, \mathbb{C}) \cong T^*V \times \mathbb{C}$. The contact structure on $J^1(V, \mathbb{C}) = T^*V \times \mathbb{C}$ is given by the field of kernels of the 1-form

$$\alpha = dz - \lambda,$$

where λ is the Liouville form on T^*V (written as $\lambda = \sum p_i \, dq_i$ in canonical coordinates). Another choice of the coordinate z leads to a multiplication of α by a nonzero function so the field of kernels of α is invariantly defined.

Definition. A submanifold $M \subset J^1(V, I)$, is called *Legendrean*, if it is tangent to the contact field at every point.

With any germ of Legendre submanifold $L \subset J^1(V, I)$ one can associate a critical point function singularity. To do that, we choose a local coordinate z on I. Observe that the image of L under the natural projection $J^1(V, \mathbb{C}) = T^*V \times \mathbb{C} \to T^*V$ is Lagrangian. Then we apply the construction of previous section. Another choice of the coordinate on $\mathbb{C}$ may lead to another function but *the class of* V_{st}*-equivalence of the critical point function singularity is well defined* (see [4, 2]). Recall that two function germs in spaces of possibly different dimensions are called V_{st}*-equivalent*[1] if after adding suitable non-degenerate quadratic forms in new variables and multiplication by non-vanishing functions they can be brought on into another by a change of variables.

Remark that the correspondence between Lagrange germs in $(T^*\mathbb{C}^n, 0)$ and Legendre germs in $(J^1(\mathbb{C}^n, \mathbb{C}) = T^*\mathbb{C}^n \times \mathbb{C}, 0)$ is bijective. Indeed, the z-coordinate is uniquely determined by the condition $dz = p\, dq$ since the restriction of the form $p\, dq$ to a Lagrange germ is closed (and hence, exact).

Example. Let V and I be vector spaces. Then we have the following bijections

$$\begin{array}{ccccccc} \text{Legendre} & & \text{Lagrange} & & \text{Lagrange} & & \text{Legendre} \\ \text{germs in} & \overset{1}{\longleftrightarrow} & \text{germs in} & \overset{2}{\longleftrightarrow} & \text{germs in} & \overset{3}{\longleftrightarrow} & \text{germs in} \\ (J^1(V, I), 0) & & (T^*V, 0) & & (T^*V^*, 0) & & (J^1(V^* \otimes I, I), 0) \end{array}$$

[1] Sometimes in the literature this equivalence is called contact equivalence. We prefer following [4, 2] to keep the notion of contact equivalence for contact diffeomorphisms of the ambient space.

where 2 is induced by the isomorphism $T^*V = V \oplus V^* = T^*V^*$. The correspondences 1 and 3 depend on the choice of coordinate on $I \cong \mathbb{C}$. Nevertheless the resulting correspondence between Legendre submanifolds in $J^1(V,I)$ and $J^1(V^*\otimes I, I)$ is invariantly defined. Moreover, this correspondence is given by the global (nonlinear) contactomorphism of these spaces. This contactomorphism

$$h : J^1(V,I) \longrightarrow J^1(V^*\otimes I, I)$$

is called the *hodograph transform*. It is given by

$$h : (v,u,z) \longmapsto (v,u,\langle v,u\rangle - z), \quad v \in V, \quad u \in V^*\otimes I, \quad z \in I$$

where we identify $J^1(V,I) = V \times V^*\otimes I \times I = J^1(V^*\otimes I, I)$. (Remark that the two contact structures induced on $V \times V^*\otimes I \times I$ are different.)

It is easy to verify that if $L(f) \subset J^1(V,I)$ is a germ of Legendre submanifold given as the 1-jet extension of the function $f : V,0 \to I,0$ then the class of V-equivalence of function singularities associated with $h(L(f)) \subset J^1(V^*\otimes I, I)$ is represented by the function germ f itself.

Definition. The *classifying space of Legendre singularities* $\widetilde{\mathcal{L}} = \widetilde{\mathcal{L}}_N^{\mathbb{C}}$ is the total space of the bundle over (a finite dimensional approximation of) BU(1) with the fibre over $x \in \mathrm{BU}(1)$ consisting of all K-jets of Legendre germs in $(J^1(\mathbb{C}^N,\xi_x),0) \cong (J^1(\mathbb{C}^N\otimes\xi_x,\xi_x),0)$, where ξ is the canonical line bundle over BU(1). This space is homotopy equivalent to the Legendre Grassmannian $\widetilde{\Lambda} = \widetilde{\Lambda}_N^{\mathbb{C}}$

As it is explained above, the points of $\widetilde{\mathcal{L}}$ are classified according to the V_{st}-classification of function germs. With any singularity class Ω (given as a V-invariant algebraic subset in some jet space of function germs) we associate the corresponding subvariety $\Omega(\widetilde{\mathcal{L}}) \subset \widetilde{\mathcal{L}}$.

Definition. The *Thom polynomial* associated with a V-class Ω of critical point singularity is the universal Legendre characteristic class $P_\Omega \in H^*(\widetilde{\mathcal{L}})$ (expressed in terms of the multiplicative generators u, a_i) represented by the intersection with the variety $\Omega(\widetilde{\mathcal{L}}) \subset \widetilde{\mathcal{L}}_N$.

Similar to the Lagrange case, this definition is independent on K, N provided these numbers are large enough.

Proof of Theorem 2. Consider vector bundles $V, I \to M$ and a fibre bundle map $f : V \to I$ as in Section 2.1.

With each point $w \in W$ we associate the Legendre germ $L_w(f) \subset J^1(V,I)$ given as the 1-graph of $f_w : V_w \to I_w$. Using the hodograph transform we may consider this germ as a germ $L_w(f) \subset J^1(V_w^*\otimes I_w, I_w)$. The V-singularity class associated with this germ is the class of the germ f_w. We would like to extend this correspondence between the critical point singularities and the Legendre singularities and to construct a classifying map $M \to \widetilde{\mathcal{L}}$ which preserves the Legendre singularity type at considered points.

The construction is similar to that considered for Lagrange case in previous section. Consider a vector bundle $U \to M$ such that $V \oplus U$ is the trivial bundle $\mathbb{C}^N$. Then for each $w \in M$ the germ $L_w(f)$ defines the germ $L_w(f) \times 0 \oplus U_w^*\otimes I_w \subset J^1(V_w^*\otimes I_w \oplus U_w^*\otimes I_w, I_w) = J^1(\mathbb{C}^N\otimes I_w, I_w)$. It remains to observe that the spaces $J^1(\mathbb{C}^N\otimes I_w, I_w)$ form a $U(1)$-bundle that can be induced from the universal one. The universal bundle is, by definition, the space $\widetilde{\mathcal{L}}_N$. The correspondence used in this construction preserves the V-singularity class associated to Legendre germs. The characteristic classes induced by this construction are $u = c_1(I)$ and $a_i = c_i(-(V \oplus U^*\otimes I)) = c_i(V^*\otimes I - V)$. $\square$

Theorem. *The Thom polynomials of V_{st}-singularities of functions of codimension ≤ 6 are those from the list of Theorem 1.*

Proof repeats the arguments used for the proof(s) of similar theorem in Lagrange case of previous section. $\square$

Example (characteristic classes of Legendre submanifolds in PT^*B). The space PT^*B of projectivised cotangent bundle is formed by pairs (a point $b \in B$, a hyperplane $h \subset T_bB$). Such pairs are called *contact elements*. The space PT^*B carries the natural contact structure that can be defined as follows. Represent (locally) the base B as $B = M \times I$. Denote by $P_0 \subset PT^*B$ the open set formed by contact elements that are transversal to the lines $\{\mathrm{pt}\} \times I \subset M \times I$. Every such contact element $h \in T_{w,z}(M \times I) = T_wM \oplus T_zI$ can be considered as a linear map $h : T_wM \to T_zI$. This allows to identify $P_0 \cong J^1(M, I)$. The contact structure on PT^*B is independent on the presentation $B = M \times I$.

With every germ of Legendre submanifold $L \subset J^1(M, I) = P_0 \subset PT^*B$ we can associate the class of V_{st}-equivalence of function singularities. This class is also independent of the local representation $B = M \times I$. It is not difficult to construct a map $\kappa : L \to \Lambda_N$ which preserves V_{st}-singularity types associated with Legendre germs. The characteristic classes induced by this map are the following: $u = c_1(I)$, where I is the conjugate of the tautological line bundle on PT^*B (it can be also defined as the normal line bundle of the contact structure); for the classes a_i we have $a_i = c_i(T_MB - TM - I)$ (see below). Similar to the Lagrange case, the classes dual to different singularity loci of Legendre projection $M \to T^*B \to B$ are given by universal Thom polynomials evaluated on the classes $c_1(I)$, $c_i(T_MB - TM - I)$.

Remark. The most general situation where Legendre characteristic classes appear is the following. Let $E \to M$ be a vector bundle of even rank. Assume that the fibres of this bundle are equipped with symplectic bilinear forms that take values in the fibres of some line bundle I (i.e. we are given a nowhere degenerating section of $\Lambda^2E^* \otimes I$). Let $L_1, L_2 \subset E$ be two Lagrange subbundles with respect to this twisted symplectic form. Then the *relative Chern classes* of the quadruple (E, I, L_1, L_2) are defined as $u = c_1(I)$, $a_i = c_i(L_2^* \otimes I - L_1)$. The equalities $c(L_1 + L_1^* \otimes I) = c(L_2 + L_2^* \otimes I) = c(E)$ imply the identity (6) for these classes.

The situations considered above fit into this pattern. In case of fibre bundle map $f : V \to I$ we take $E = V \oplus V^* \otimes I$, $L_1 = V \oplus 0$, and $L_2 \cong L_1$ is the bundle of the tangent planes to the graphs of the differentials of the maps $f_w : V_w \to I_w$, $w \in M$.

In case of Legendre immersion $M \to PT^*B$ we set $E \subset T_M(PT^*B)$ to be the bundle of contact planes, $I = T_M(PT^*B)/E$ to be the normal bundle of the contact structure. The symplectic form is given by $(\xi, \eta) \mapsto [\xi, \eta] \pmod{E}$, where ξ, η are vector fields tangent to E and $[\cdot, \cdot]$ is the commutator of vector fields. (This is the invariant definition of the linear symplectic structure on contact planes given by $d\alpha|_E$, where the contact structure E is the field of kernels of the 1-form α.)

The subbundles L_1, L_2 are TM, and the bundle of 'vertical' tangent vectors to PT^*B, corresponding to the kernel of the differential of the projection $PT^*B \to B$. Since E_w/L_{2w} is the hyperplane of the contact element $w \in M \subset PT^*B$, we have $L_2^* \otimes I = E - L_2 = T_MB - I$, i.e.

$$a_i = c_i(L_2^* \otimes I - L_2) = c_i(T_MB - TM - I). \qquad \square$$

3.3 Real Lagrange and Legendre singularities and characteristic classes

The theorems on complex characteristic classes have usually a real analogue where the complex manifolds, maps and bundles are replaced by the real ones, integer cohomology by $\mathbb{Z}_2$-cohomology, Chern classes by Stiefel-Whitney classes etc. This principle can be applied to the problems studied in this paper as well. The notions of Lagrange and Legendre characteristic classes, the correspondence between singularities of functions and Lagrange (Legendre) singularities, the definitions of classifying spaces and Thom polynomials repeat word-by-word the corresponding notions defined in this paper for complex case. The main difference is that the homomorphism (1) of the ring of Legendre characteristic classes to the polynomial ring in variables $u, c_1, c_2, \dots$ (which is injective over $\mathbb{Z}$) has a big kernel over $\mathbb{Z}_2$. For instance, it is trivial in Lagrange case when $u = 0$ (indeed, the total Stiefel-Whitney class $\omega(V^* - V)$ is trivial for any real bundle V). It follows that the characteristic classes of fibre singularities of real-valued functions are trivial, see [8]. Nevertheless these characteristic classes are not trivial if they are applied directly to the cycles of Lagrange singularities on Lagrange submanifolds in T^*B (respectively, Legendre singularities of Legendre submanifolds in PT^*B).

Theorem. *The Thom polynomials of real Legendre singularities of real codimension ≤ 6 are obtained from the list of Theorem 1 by replacing the Chern classes a_i by the corresponding Stiefel-Whitney classes α_i and reducing the coefficients modulo 2,*

$$\begin{aligned}
A_2 &= \alpha_1 \\
A_3 &= \alpha_2 \quad + u\alpha_1 \\
A_4 &= \alpha_1\alpha_2 \quad + u^2\alpha_1 \\
D_4 &= \alpha_1\alpha_2 \quad + u\alpha_2 \\
A_5 &= \alpha_1\alpha_3 \quad + u^3\alpha_1 \\
D_5 &= 0 \\
A_6 &= \alpha_2\alpha_3 \quad + u(\alpha_1\alpha_3 + \alpha_4) + u^2\alpha_1\alpha_2 + u^3\alpha_2 + u^4\alpha_1 \\
D_6 &= 0 \\
E_6 &= \alpha_2\alpha_3 \quad + u\alpha_4 + u^2\alpha_1\alpha_2 + u^3\alpha_2 \\
A_7 &= \alpha_1\alpha_2\alpha_3 + \alpha_2\alpha_4 \quad + u\alpha_5 + u^2\alpha_1\alpha_3 + u^4\alpha_2 + u^5\alpha_1 \\
D_7 &= 0 \\
E_7 &= \alpha_1\alpha_2\alpha_3 \quad + u(\alpha_2\alpha_3 + \alpha_1\alpha_4) + u^2\alpha_4 + u^3\alpha_1\alpha_2 + u^4\alpha_2 \\
P_8 &= \alpha_1\alpha_2\alpha_3 \quad + u\alpha_1\alpha_4
\end{aligned}$$

In particular, let $M \subset PT^*B$ be a real Legendre immersion. Then the $\mathbb{Z}_2$-cohomology class dual to some cycle of Legendre singularities of codim ≤ 6 is equal to the corresponding Thom polynomial evaluated on the classes $u = \omega_1(I)$ and $\alpha_i = \omega_i(T_M B - TM - I)$, where I is the normal line bundle of contact structure on PT^*B.

Theorem (cf. [18]). *The Thom polynomials of real Lagrange singularities of real codimension ≤ 6 are obtained from the list of previous theorem by setting $u = 0$.*

In particular, let $M \subset T^*B$ be a real Lagrange immersion. Then the $\mathbb{Z}_2$-cohomology class dual to some cycle of Legendre singularities of codim ≤ 6 is equal to the corresponding Thom polynomial evaluated on the classes $\alpha_i = \omega_i(T_M B - TM)$.

These theorems can be proved applying step by step the real versions of all constructions used in Section 2 in the proof of the corresponding formulas for complex case. $\square$

Remark. The formulas for classes of real Lagrange singularities of codim ≤ 6 (except A_7) were obtained in [18] by a different method. The expression $A_7 = \alpha_1\alpha_2\alpha_3 + \alpha_2\alpha_4$ as well as all classes of real Legendre singularities are new.

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