Spaces of Hermitian operators with simple spectra and their finite-order cohomology

V. A. Vassiliev*

Abstract

V. I. Arnold [3] studied the topology of spaces of Hermitian operators in $\mathbf{C}^n$ with non-simple spectra in a relation with the theory of adiabatic connections and the quantum Hall effect. The natural filtration of these spaces by the sets of operators with fixed numbers of eigenvalues defines the spectral sequence, providing interesting combinatorial and homological information on this stratification.

We construct a different spectral sequence, also counting the homology groups of these spaces and based on the universal techniques of *topological order complexes* and *conical resolutions* of algebraic varieties, generalizing the combinatorial inclusion-exclusion formula and similar to the construction of finite-order knot invariants.

This spectral sequence degenerates at the term E_1, is (conjecturally) multiplicative, and as n grows then it converges to a stable spectral sequence counting the cohomology of the space of infinite Hermitian operators without multiple eigenvalues, all whose terms $E_r^{p,q}$ are finitely generated. It allows us to define the finite-order cohomology classes of this space, and to apply the well-known facts and methods of the topological theory of flag manifolds to the problems of geometrical combinatorics, especially concerning the continuous partially ordered sets of subspaces and flags.

1 Introduction

Denote by $\mathcal{H}(n)$ the space of Hermitian operators in $\mathbf{C}^n$; this is a n^2-dimensional real vector space. The *discriminant variety* $\Sigma \equiv \Sigma(n) \subset \mathcal{H}(n)$ consists of operators with at least one eigenvalue of multiplicity ≥ 2; this is a subvariety of codimension 3. The Alexander duality

$$\bar{H}_i(\Sigma(n)) \simeq \tilde{H}^{n^2-i-1}(\mathcal{H}(n) \setminus \Sigma(n)) \tag{1}$$

relates its Borel–Moore homology groups (i.e. homology groups of locally finite chains) with the standard cohomology groups of the complementary space of matrices with simple spectra.

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V. I. Arnold [3] studied these homology groups, considering the natural filtration of the space $\Sigma(n)$ in accordance with the numbers of different eigenvalues of operators. Although the answer is known (there is a ring isomorphism

$$H^*(\mathcal{H}(n) \setminus \Sigma(n)) \cong H^*(\mathbf{CP}^{n-1} \times \mathbf{CP}^{n-2} \times \cdots \times \mathbf{CP}^1)), \tag{2}$$

this study gives interesting information on the topology of the discriminant set. In particular, the spectral sequence, defined by this filtration, degenerates at the second term E^2 (see [9]), and all its groups $E^1_{p,q}$ consist of homology groups of certain complex flag manifolds.

We consider a different spectral sequence, also converging to the same group (1), also built of homology groups of flag manifolds, and based on a *conical resolution* of Σ. This sequence (and the related filtration in the ring (2)) seems to be interesting, because

1. it degenerates at the *first* term $E^1 \equiv E^\infty$ (at least in characteristic 0);
2. it is compatible very much with the inclusions $\mathcal{H}(n) \hookrightarrow \mathcal{H}(n+1) \hookrightarrow \cdots$, thus stabilizing to a similar spectral sequence, converging to the cohomology ring of the space of infinite Hermitian matrices without multiple eigenvalues, all whose terms $E_r^{p,q}$ are finitely generated;
3. it is (conjecturally) multiplicative;
4. it is closely related with the shift operator of the spectrum, and the corresponding filtration in the cohomology is invariant under this operator;
5. it allows us to apply the (more or less easy or well–known) facts on the topology of flag manifolds to the problems of geometrical combinatorics, especially concerning topological partially ordered sets and order complexes;
6. its algebraic presentation is very similar to the theory of finite-order invariants of knots (the stratum of operators with eigenvalues of multiplicities $a_1, \ldots, a_l$ corresponds to that of smooth maps $S^1 \to \mathbf{R}^3$ with l selfintersection points of the same multiplicities), thus inventing to the ring (2) useful structures and notions of this theory. In particular, it allows us to define the orders of cohomology classes, and, for any class of order p, its *symbol* (or *generalized residue*) at the strata of $\Sigma(n)$ of complexity p in the same way as it was done in [14], [13] for knot invariants and strata of singular knots.

2 Hermitian matrices with simple spectra

If all eigenvalues λ_i of a Hermitian operator $\mathbf{C}^n \to \mathbf{C}^n$ are different, then they (and the corresponding one-dimensional complex eigenspaces) can be ordered by $\lambda_1 < \cdots < \lambda_n$. These eigenspaces form n line bundles over the space $\mathcal{H}(n) \setminus \Sigma(n)$ of all such operators; let $c^1, \ldots, c^n$ be the first Chern classes of these bundles.

Proposition 1 *(see [5]). There is canonical ring isomorphism*

$$H^*(\mathcal{H}(n) \setminus \Sigma(n)) \simeq \mathbf{Z}[c^1, \dots, c^n]/\{Sym\}, \tag{3}$$

where $\{Sym\}$ is the ideal generated by all symmetric polynomials of positive degrees. □

It is convenient to consider all such rings for all n simultaneously, i.e. to consider the ring

$$\mathbf{Z}[[a^0, a^1, a^{-1}, a^2, a^{-2}, \dots]]/\{Sym\} \tag{4}$$

of formal power series of infinitely many (in both directions) two-dimensional variables a^j factored through the ideal spanned by symmetric series of positive degrees. This ring is called the cohomology ring of the space of *infinite* Hermitian matrices with simple spectra.

The ring (3) can be identified in many ways with a quotient ring of (4): we can choose any number $i = 0, 1, -1, \dots$ and factor (4) additionally through all elements a^j with $j \leq i$ and $j > i + n$. Identifying then the variables $c^1, \dots, c^n$ with $a^{i+1}, \dots, a^{i+n}$ respectively, we get an isomorphism between this quotient ring and (3).

The obvious operator, mapping any variable a^i to a^{i+1}, acts on the algebra (4); it is called the *shift operator.*

3 Topological posets and conical resolutions

Suppose that we have a stratified variety and wish to calculate its homology groups. There are two main approaches to this problem. The obvious method of *open strata* is as follows: we filter the variety by unions S_i of smooth strata of "complexity $\geq i$", and consider the corresponding spectral sequence (whose term $E^1_{p,q}$ is the group $H_{p+q}(S_i, S_{i+1})$, Poincaré dual to a cohomology group of the smooth manifold $S_i \setminus S_{i+1}$).

A different approach, modelling the combinatorial formula of inclusions and exclusions, is as follows: first we consider a singularity resolution $\pi : \tilde{V} \to V$ of entire variety V (thus changing it over the singular set), and then improve it over the *closures* of strata of smaller and smaller dimensions in such a way that at the last step we get a space V' with a *proper* projection onto V and contractible preimages of all points. Then V' is homotopy equivalent to the original space; in all important cases this space V' has a very transparent topological structure, in particular a natural filtration, whose spectral sequence degenerates very fast, see e.g. [11], [13], [15].

EXAMPLE. The group $H^2(\mathcal{H}(n)\setminus\Sigma(n))$ is $(n-1)$-dimensional and consists of all sequences $(\alpha_1, \dots, \alpha_n)$ of integer numbers (i.e., of corresponding sums $\sum \alpha_i c^i$) factored through the constant sequences.

The spectral sequence from [3] has on the corresponding line $\{p + q = n^2 - 3\}$ unique nontrivial term E_∞^{n-1, n^2-n-2}, isomorphic to $\mathbf{Z}^{n-1}$ and canonically generated by linking numbers with all $n - 1$ smooth strata of maximal dimension in $\Sigma(n)$, i.e. by "δ'-like" sequences of the form $(0,\ldots,0,1,-1,0,\ldots,0)$. Our spectral sequence defined below has $n-1$ one-dimensional groups on the same line, and the p-th term of the corresponding filtration consists of all integer-valued polynomial sequences of degree $\leq p$ (modulo the constants). The stable filtration in the 2-dimensional component of (4) also is finitely generated: it is defined by polynomials of any degrees, in particular is invariant under the shift operators.

The explicit realization of our method is based on the notion of the *topological order complex* and on the techniques of *conical resolutions*. It generalizes the method of *simplicial resolutions*, see e.g. [10], [13], applicable in the case, when all essential singularities of the variety V are finite (self-)intersections only.

Instead of formal definitions, we present here the following illustration, important for our further calculations.

3.1 Determinant variety and homology of the group $U(n)$.

Consider the space $Mat(\mathbf{C}^n)$ of all complex linear operators $\mathbf{C}^n \to \mathbf{C}^n$; the variety in question is its *determinant subvariety Det* consisting of all degenerate operators. Consider all possible kernels of degenerate operators, i.e. all Grassmann manifolds $G_1(\mathbf{C}^n)$, $\ldots, G_{n-1}(\mathbf{C}^n), G_n(\mathbf{C}^n)$. The incidence of subspaces, corresponding to their points, makes the disjoint union of these Grassmannians a *poset* (:=partially ordered set) with unique maximal element $\{\mathbf{C}^n\} \subset G_n(\mathbf{C}^n)$. Then we take the join $G_1(\mathbf{C}^n) * \ldots * G_n(\mathbf{C}^n)$, i.e., roughly speaking, the union of all simplices (of different dimensions), whose vertices belong to different Grassmannians. Such a simplex is called *coherent*, if all subspaces in $\mathbf{C}^n$ corresponding to its vertices form a flag. Finally, the *topological order complex* $\Theta(n)$ is the subset of our join, defined as the union of all coherent simplices. It is contractible: indeed, it is a cone with the vertex $\{\mathbf{C}^n\}$. Its base $\partial\Theta(n)$ is defined in a similar way as the union of all coherent simplices of the join $G_1(\mathbf{C}^n) * \ldots * G_{n-1}(\mathbf{C}^n)$.

Proposition 2 *(see [11], [13]). The space $\partial\Theta(n)$ is PL-homeomorphic to S^{n^2-2}.* $\square$

Remark. Probably this assertion (and its generalizations) is assumed in Remark 1.4 of the work [6]. I thank M. M. Kapranov for this reference.

For any subspace $L \subset \mathbf{C}^n$ we define the cone $\Theta(L)$ as the union of coherent simplices subordinate to L, i.e. such that all subspaces corresponding to their vertices belong to L. In particular, $\Theta(n) \equiv \Theta(\{\mathbf{C}^n\})$. The subspace $\chi(L) \subset Mat(\mathbf{C}^n)$ is defined as the space of all operators, whose kernels contain L.

The desired *conical resolution* Δ_n of *Det* is a subset of $Mat(\mathbf{C}^n) \times \Theta(n)$. Namely, for any subspace $L \subset \mathbf{C}^n$ of any positive dimension we consider the space $\chi(L) \times \Theta(L) \subset Mat(\mathbf{C}^n) \times \Theta(n)$ and define Δ_n as the union of such spaces over all possible L. This space is naturally filtered: the term F_i of this filtration is the union of sets $\chi(L) \times \Theta(L)$ over all L of dimensions $\leq i$.

Proposition 3 *The obvious map $\Delta_n \to Det$ (defined by the projection $Mat(\mathbf{C}^n) \times \Theta(n) \to Mat(\mathbf{C}^n)$) induces a homotopy equivalence of one-point compactifications of these spaces, in particular an isomorphism of their Borel–Moore homology groups.*

Now consider the spectral sequence, calculating these groups and generated by our filtration $\{F_i\}$ in Δ_n. By definition,

$$E^1_{p,q} \simeq \bar{H}_{p+q}(F_p \setminus F_{p-1}) \tag{5}$$

$F_p \setminus F_{p-1}$ is the space of a fiber bundle, whose base is $G_p(\mathbf{C}^n)$ and the fiber over the point $\{L\}$ is diffeomorphic to the direct product $\mathbf{C}^{n(n-p)} \times (\Theta(L) \setminus \partial\Theta(L))$. By Proposition 2 the last factor of this product is homeomorphic to the open disc of dimension $p^2 - 1$, thus the group (5) is isomorphic to $H_t(G_p(\mathbf{C}^n))$, $t = p + q - (2n^2 - 2np + p^2 - 1)$.

This spectral sequence degenerates at the first step: $E^1 \equiv E^\infty$.

The Alexander dual cohomological spectral sequence, defined by the identity $E_r^{p,q} \equiv E^r_{-p,2n^2-q-1}$, converges to the cohomology group of the complementary space $GL(n, \mathbf{C}) \sim U(n)$. Its degeneration gives us immediately the *Miller splitting formula*

$$H^m(U(n)) \cong \bigoplus_{p=0}^{n} H^{m-p^2}(G_p(\mathbf{C}^n)). \tag{6}$$

Similar splittings hold also for other classical Lie groups $O(n)$ and $Sp(n)$, cf. [11], [13].

This cohomological spectral sequence is multiplicative. Indeed, the ring $H^*(U(n))$ is an exterior algebra with n generators $\alpha_1, \alpha_3, \dots, \alpha_{2n-1}$ of corresponding dimensions. Our spectral sequence induces a filtration in this ring, whose i-th term coincides with the subalgebra generated by all products of $\le i$ elements α_j.

4 Conical resolution of the space of Hermitian matrices with multiple spectra

Consider again the discriminant variety $\Sigma(n) \subset \mathcal{H}(n)$. The corresponding conical resolution, order complexes etc. are constructed as follows.

Let $A = (a_1 \ge a_2 \ge \cdots \ge a_l)$ be a sequence of naturals with $a_l \ge 2$ and $\sum a_i \le n$. The *complexity* of the index A is the number $\sum_{i=1}^{l}(a_i - 1)$; its *length* $\#A$ is the number of its elements a_i (denoted above by l), and *liberty* $\delta(A)$ is the number $n - \sum a_i$. Also set $|A| = \sum a_i$.

Denote by $\Gamma_A(n)$ the space of *unordered* collections of $\#A$ pairwise Hermitian–orthogonal subspaces in $\mathbf{C}^n$ of complex dimensions $a_1, \dots, a_{\#A}$. If all numbers a_i are different, then it can be identified with the flag manifold $\mathcal{F}_{a_1, a_1+a_2, \dots, a_1+\dots+a_{\#A}}$, and if some of a_i coincide then with the quotient space of this flag manifold under the (free) action of corresponding permutation group $S(A)$. In any case, it is a complex compact manifold.

The disjoint union of all spaces $\Gamma_A(n)$ is a partially ordered set: a point $\gamma \in \Gamma_A(n)$ is subordinate to the point $\gamma' \in \Gamma_{A'}(n)$ if any of $\#A$ subspaces forming the point γ belongs to some of $\#A'$ subspaces forming γ'. In this case we say also that the points γ and γ' are *incident* to one another. This poset has unique maximal element: the point $\{\mathbf{C}^n\} \subset \Gamma_{(n)}(n)$.

The corresponding topological order complex $\Xi(n)$ is defined as in the previous section: we consider the join of spaces $\Gamma_A(n)$ over all possible multiindices A with $|A| \leq n$, and take the union of all coherent simplices in it, i.e. of simplices, all whose vertices are incident to one another, thus forming a monotone sequence. It is obviously a cone with the vertex at the point $\{\mathbf{C}^n\} \subset \Gamma_{(n)}(n)$.

To any point $\gamma \in \Gamma_A(n)$ there corresponds the subcomplex $\Xi(\gamma) \subset \Xi(n)$: the union of all coherent simplices, all whose vertices are subordinate to γ. In particular $\Xi(n) \equiv \Xi(\{\mathbf{C}^n\})$.

Also define the *link* $\partial\Xi(\gamma)$ as the union of all coherent simplices forming $\Xi(\gamma)$, which *do not* contain the maximal vertex $\{\gamma\}$. The *open cone* $\breve{\Xi}(\gamma)$ is the difference $\Xi(\gamma) \setminus \partial\Xi(\gamma)$. Their homology groups are related by the boundary isomorphism $\partial : \bar{H}_*(\breve{\Xi}(\gamma)) \xrightarrow{\sim} \tilde{H}_{*-1}(\partial\Xi(\gamma))$.

Theorem 1 *For any n, the reduced homology group $\tilde{H}_i(\partial\Xi(n), \mathbf{C})$ is trivial in all even dimensions if n is even and in all odd dimensions if n is odd.*

EXAMPLES. For $n = 3, 4$ and 5, the Poincaré polynomials of groups $\tilde{H}_*(\partial\Xi(n), \mathbf{C})$ are equal to $t^2(1+t^2)$, $t^3(1+t^4)(1+t^2+t^4)$ and $t^4(1+t^2+t^4+t^6)(1+t^2+t^4+t^6+t^8+t^{10})$, respectively. These groups are encoded also in the very right nontrivial columns in three tables of Fig. 1.

A recurrent method for calculating these homology groups will be described in the end of § 5.4, however I do not know any compact expression for them.

For any index A and any $\gamma \in \Gamma_A(n)$ denote by $\chi(\gamma)$ the subspace in $\mathcal{H}(n)$ consisting of all Hermitian operators such that any of spaces of dimension a_i, forming the point γ, belongs to an eigenspace of this matrix. This is a real vector space of dimension $\#A + (\delta(A))^2$.

The conical resolution $\sigma(n)$ of the discriminant variety $\Sigma(n) \subset \mathcal{H}(n)$ is a subset of the direct product $\mathcal{H}(n) \times \Xi(n)$, namely, the union of products $\chi(\gamma) \times \Xi(\gamma)$ over all possible indices A and all points $\gamma \in \Gamma_A(n)$.

Proposition 4 *The space $\sigma(n)$ is semialgebraic. The obvious map $\sigma(n) \to \Sigma(n)$ (defined by the projection $\mathcal{H}(n) \times \Xi(n) \to \mathcal{H}(n)$) is proper and induces a homotopy equivalence of one-point compactifications of these spaces, in particular an isomorphism of their Borel–Moore homology groups.* $\square$

There is *standard filtration* $\sigma_1(n) \subset \cdots \subset \sigma_{n-1}(n) \equiv \sigma(n)$ in the space $\sigma(n)$: its term $\sigma_i(n)$ is the union of products $\chi(\gamma) \times \Xi(\gamma)$ over all $\gamma \in \Gamma_A(n)$, where A is some multiindex of complexity $\leq i$.

The *main spectral sequence,* calculating the Borel–Moore homology group of $\sigma(n)$, is that generated by this filtration; by definition its term $E^1_{p,q}$ is isomorphic to $\bar{H}_{p+q}(\sigma_p(n) \setminus \sigma_{p-1}(n))$.

Theorem 2 *The main spectral sequence, calculating the group (1) with complex coefficients, degenerates at the first term:* $E^1 \equiv E^\infty$.

These spectral sequences for $n = 3, 4$ and 5 are shown in Fig. 1.

Over the integers, the similar statement is not true: see the end of § 5.3.

Now, let us describe the terms $E^1_{p,q}$ of this spectral sequence. For any point $\gamma = (\gamma_1, \dots, \gamma_{\#A}) \in \Gamma_A(n)$ denote by $\gamma^\perp$ the Hermitian-orthogonal complement of the linear span of all subspaces γ_i.

Proposition 5 *Any space* $\sigma_p(n) \setminus \sigma_{p-1}(n)$ *is a disjoint union (over all indices* A *of complexity exactly* p*) of spaces of fiber bundles, whose bases are the spaces* $\Gamma_A(n)$ *and the fiber over the point* $\gamma \in \Gamma_A(n)$ *splits into the direct sum of three spaces*

$$\mathbf{R}^{\#A} \times \mathcal{H}(\delta(A)) \times \breve{\Xi}(\gamma). \tag{7}$$

Namely, the factor $\mathbf{R}^{\#A}$ is defined by eigenvalues of operators $\Lambda \subset \chi(\gamma)$ on all subspaces γ_i forming the point γ; the factor $\mathcal{H}(\delta(A))$ is defined by restrictions of such operators to the orthogonal subspaces $\gamma^\perp$, and the sign $\breve{}$ over $\Xi(\gamma)$ is due to the fact that the subspace $\chi(\gamma) \times \partial\Xi(\gamma)$ lies in the smaller term $\sigma_{p-1}(n)$ of the filtration.

Denote the spaces of these fiber bundles by $\beta_A(n)$. So, $\beta_A(n)$ is the space of a fibered product of three bundles over $\Gamma_A(n)$, whose fibers are three factors in (7).

Proposition 6 *The second bundle over* $\Gamma_A(n)$*, formed by spaces isomorphic to* $\mathcal{H}(\delta(A))$*, is orientable.*

Indeed, it is induced from a similar bundle over the simply-connected Grassmannian manifold $G_{|A|}(\mathbf{C}^n)$ by the obvious map (sending any collection γ of subspaces to their linear span).

On the other hand, the bundle of first factors in (7) is nonorientable if some of numbers a_i coincide. Any such factor splits into the sum of 1-dimensional subspaces associated with all $\#A$ subspaces forming the basepoint γ; a path in $\Gamma_A(n)$ providing an odd permutation of these subspaces violates the orientation of this factor.

The following proposition reduces the homology of the space $\breve{\Xi}(A)$ to these for indices A' of length 1.

Recall the rigorous definition of the join $X * Y$ of two topological spaces X, Y as the quotient space of the product $X \times [-1, 1] \times Y$ by the equivalence relations $(x, -1, y) \sim (x, -1, y')$ and $(x, 1, y) \sim (x', 1, y)$ for any $x, x' \in X$ and any $y, y' \in Y$.

Proposition 7 *Suppose that* $A = A' \cup A''$, $\gamma \in \Gamma_A(n)$*, and the collection of subspaces in* $\mathbf{C}^n$ *defining the point* γ *splits into the union of similar collections defining some points* $\gamma' \in \Gamma_{A'}(n)$ *and* $\gamma'' \in \Gamma_{A''}(n)$*. Then there is a standard homeomorphism of the complex* $\Xi(A)$ *to the join* $\Xi(\gamma') * \Xi(\gamma')$*, sending the vertex* $\{\chi(\gamma)\}$ *to the point* $\{\chi(\gamma')\} \times 0 \times \{\chi(\gamma'')\}$ *and the link* $\partial\Xi(\gamma)$ *to the image (under the above-described factorization map) of the boundary* $(\Xi(\gamma') \times [-1, 1] \times \partial\Xi(\gamma'')) \cup (\partial\Xi(\gamma') \times [-1, 1] \times \Xi(\gamma''))$ *of the prism* $\Xi(\gamma') \times [-1, 1] \times \Xi(\gamma'')$.

q \ p	1	2
5	$\mathbf{Z}$	
4	0	$\mathbf{Z}$
3	$\mathbf{Z}$	0
2	0	$\mathbf{Z}$
1	$\mathbf{Z}$	0
0	0	

q \ p	1	2	3
12	$\mathbf{C}$		
11		$\mathbf{C}$	
10	$\mathbf{C}$		$\mathbf{C}$
9		$\mathbf{C}^2+\mathbf{C}$	
8	$\mathbf{C}^2$		$\mathbf{C}$
7		$\mathbf{C}^2$	
6	$\mathbf{C}$		$\mathbf{C}^2$
5		$\mathbf{C}^2+\mathbf{C}$	
4	$\mathbf{C}$		$\mathbf{C}$
3		$\mathbf{C}$	
2			$\mathbf{C}$
1		$\mathbf{C}$	

q \ p	1	2	3	4
21	$\mathbf{C}$			
20		$\mathbf{C}$		
19	$\mathbf{C}$		$\mathbf{C}$	
18		$\mathbf{C}^2+\mathbf{C}$		$\mathbf{C}$
17	$\mathbf{C}^2$		$\mathbf{C}^2+\mathbf{C}$	
16		$\mathbf{C}^3+\mathbf{C}$		$\mathbf{C}^2$
15	$\mathbf{C}^2$		$\mathbf{C}^4+\mathbf{C}^2$	
14		$\mathbf{C}^4+\mathbf{C}^2$		$\mathbf{C}^3$
13	$\mathbf{C}^2$		$\mathbf{C}^5+\mathbf{C}^3$	
12		$\mathbf{C}^4+\mathbf{C}^2$		$\mathbf{C}^4$
11	$\mathbf{C}$		$\mathbf{C}^6+\mathbf{C}^4$	
10		$\mathbf{C}^3+\mathbf{C}^3$		$\mathbf{C}^4$
9	$\mathbf{C}$		$\mathbf{C}^5+\mathbf{C}^4$	
8		$\mathbf{C}^2+\mathbf{C}^2$		$\mathbf{C}^4$
7			$\mathbf{C}^4+\mathbf{C}^3$	
6		$\mathbf{C}+\mathbf{C}^2$		$\mathbf{C}^3$
5			$\mathbf{C}^2+\mathbf{C}^2$	
4		$\mathbf{C}$		$\mathbf{C}^2$
3			$\mathbf{C}+\mathbf{C}$	
2		$\mathbf{C}$		$\mathbf{C}$

Figure 1: Main spectral sequences for $n = 3$ (left top), $n = 4$ (left bottom) and $n = 5$ (right)

The proof is immediate. $\square$

COROLLARY. *Define the graded group* $h_*(\gamma)$ *by the equation* $h_i(\gamma) \equiv \bar{H}_{i-1}(\breve{\Xi}(\gamma), \mathbf{C})$. *Then for any* $A = (a_1, \dots, a_{\#A})$ *and* $\gamma \in \Gamma_A(n)$ *there is* almost *canonical isomorphism*

$$h_*(\gamma) \simeq \bigotimes_{i=1}^{\#A} h_*(\gamma_i), \tag{8}$$

where all γ_i *are subspaces of dimensions* a_i *forming the point* γ *and considered as the points of corresponding Grassmannian manifolds* $\Gamma_{(a_i)}(n)$.

"Almost" here means the following. The isomorphism (8) depends on the order of subspaces γ_i forming the collection γ. A reordering of such subspaces (of equal dimensions a_i) takes this isomorphism to itself multiplied by $(-1)^s$, where s is the parity of this permutation. In particular, we cannot realize the isomorphism (8) by a natural construction depending continuously on the point $\gamma \in \Gamma_A(n)$: a path in $\Gamma_A(n)$ providing such an odd permutation will violate this construction.

Besides this bundle $\beta_A(n) \to \Gamma_A(n)$, it is convenient to consider also its "universal covering"

$$\beta!_A(n) \to \Gamma!_A(n). \tag{9}$$

Here $\Gamma!_A(n)$ is the universal covering space of $\Gamma_A(n)$. This is a principal covering, whose group $S(A)$ is the direct product of permutation groups S_μ, where μ are multiplicities, with which equal numbers a_i meet in the sequence A. The space $\Gamma!_A(n)$ is diffeomorphic to the flag manifold $\mathcal{F}_{a_1, a_1+a_2, \dots, |A|}$. The bundle (9) is again the fibered product of three bundles with fibers indicated in (7); in this case the first bundle with fibers $\mathbf{R}^{\#A}$ is trivial, and the decomposition (8) of the homology of the third bundle can be realized so that it depends continuously on the basepoint $\gamma! \in \Gamma!_A(n)$. The group of the covering $\Gamma!_A(n) \to \Gamma_A(n)$ acts naturally on the space $\beta!_A(n)$, permuting corresponding fibers, and $\beta_A(n)$ can be considered as the quotient space under this action.

5 Examples

In this section we calculate our spectral sequences for $n \leq 5$, and determine our filtration in the groups $H^2(\mathcal{H}(n) \setminus \Sigma(n))$.

5.1 The marginal columns of the spectral sequence.

For any n, the first term $\sigma_1(n)$ of $\sigma(n)$ is the *canonical resolution* of $\Sigma(n)$, i.e. the space of pairs {a 2-plane in $\mathbf{C}^n$; an operator whose restriction on this plane is scalar}.

Its Borel–Moore homology group coincides with the usual homology group of the Grassmannian manifold $G_2(\mathbf{C}^n)$ with grading shifted by $(n-2)^2+1$, see the columns $\{p = 1\}$ of all three tables in Fig. 1.

Now, let us consider the very right columns $\{p = n-1\}$. The unique index A of complexity $n-1$ is equal to (n). The space $\Gamma_A(n)$ in this case consists of unique point $\{\mathbf{C}^n\}$, thus by Proposition 5 we have the isomorphism of homology groups,

$$\bar{H}_i(\sigma_{n-1}(n) \setminus \sigma_{n-2}(n)) \equiv \bar{H}_i(\breve{\Xi}(n) \times \mathbf{R}^1) \simeq \tilde{H}_{i-2}(\partial\Xi(n)). \tag{10}$$

5.2 Cases $n = 2$ and 3.

If $n = 2$, then $\Sigma(n)$ consists of scalar matrices $\binom{\lambda\ 0}{0\ \lambda}$, $\lambda \in \mathbf{R}$, thus $\mathcal{H}(2) \setminus \Sigma(2) \sim S^2$, see [3]. The ingredients of the construction of § 4 are as follows. The unique admissible index A is equal to (2), the corresponding space $\Gamma_A(2)$ is a point O, the space $\Xi(O) \equiv \breve{\Xi}(O)$ also is a point, and $\chi(O)$ is a real line. Thus the (homological) spectral sequence consists of unique element $E_{1,0} \simeq \mathbf{Z}$.

Now let be $n = 3$. In this case both columns are described in § 5.1. Namely, the column $\{p = 1\}$ again contains the homology groups of the canonical resolution $\sigma_1(n) \to \Sigma(n)$. This resolution is a local diffeomorphism over all points of $\Sigma(n)$ other than the scalar operators, and such operators are "blown up" to spaces $G_2(\mathbf{C}^3) \equiv \partial\Xi(3)$. The term $\sigma_2(n)\setminus\sigma_1(n)$ consists of open cones $\breve{C}(G_2(\mathbf{C}^3)) \sim \breve{\Xi}(3)$, spanning all these fibers $\partial\Xi(3)$. Thus we get the table in the left upper part of Fig. 1.

5.3 The calculations for $n = 4$.

For $n = 4$ there are exactly two indices A of complexity 2, namely (3) and (2, 2). The corresponding blocks $\beta_A(4)$ are as follows.

For $A = (3)$, the space $\Gamma_3(n)$ is the Grassmannian manifold $G_3(\mathbf{C}^4) \cong \mathbf{CP}^3$, and for any point γ of this space, the space $\partial\Xi(\gamma)$ is equal to $\mathbf{CP}^2$. Thus the spectral sequence of the fiber bundle $\beta_{(3)}(4) \to \Gamma_{(3)}(4)$ is as follows: its term $\mathcal{E}^2_{a,b}$ is isomorphic to $H_a(\mathbf{CP}^3, \tilde{H}_{b-3}(\mathbf{CP}^2))$, where the number 3 in the lower index $b-3$ is $\#A + (\delta(A))^2 +$ (the loss of dimensions in the boundary homomorphism). This spectral sequence obviously degenerates in this term $\mathcal{E}^2$ and gives us the direct summand of the column $\{p = 2\}$ written in its left part.

For $A = (2,2)$, the space $\Gamma_A(4)$ is the quotient space of the Grassmannian manifold $G_2(\mathbf{C}^4)$ under the involution sending any 2-plane into its Hermitian–orthogonal plane. It is easy to calculate that the complex homology group of this manifold is isomorphic to $\mathbf{C}$ in dimensions 0, 4 and 8 and is trivial in other dimensions. Three factors of the fiber (7) of the bundle $\beta_A(4) \to \Gamma_A(4)$ in this case are equal respectively to $\mathbf{R}^2$, one point, and the open interval. (For any $\gamma \in \Gamma_{(2,2)}(4)$ this intervals consists of two segments joining the point γ with two subordinate points of $G_2(\mathbf{C}^4)$. Their endpoints lying in $G_2(\mathbf{C}^4)$ are of smaller filtration and hence are removed.) The generator of the group $\pi_1(\Gamma_A(4)) \sim \mathbf{Z}_2$ violates the orientations of both the bundle of spaces $\mathbf{R}^2$ and that of open intervals, hence $\bar{H}_i(\beta_A(4)) \simeq H_{i-3}(\Gamma_A(4))$, which gives us the right-hand part of column $\{p = 2\}$.

Finally, let us calculate the column $\{p = 3\}$, i.e. the homology group of the link $\partial\Xi(4)$.

Consider the subset Ω of this link, swept out by all coherent segments connecting points of manifolds $G_2(\mathbf{C}^4)$ and $G_3(\mathbf{C}^4)$.

Lemma 1 *The group $\tilde{H}_a(\Omega)$ is isomorphic to $\mathbf{Z}$ in dimensions 7, 9, 11, and is trivial in all other dimensions.*

Proof. Consider the space $\Theta(4)$ studied in Proposition 2 (and, accordingly to this proposition, homeomorphic to S^{14}). The space Ω is canonically embedded in this sphere, and by the construction is *Spanier–Whitehead dual* (see [11]) to the space $\mathbf{CP}^3$ (also embedded in it). Hence their reduced homology groups are related by the Alexander duality in S^{14}, and our lemma is proved. $\square$

Further, the space $\partial\Xi(4) \setminus \Omega$ is the space of a nonorientable fiber bundle over the space $\Gamma_{(2,2)}(4)$, whose fibers are open intervals.

The complex Borel–Moore homology group of this space M participates in the (splitting) Smith exact sequence of the double covering $G_2(\mathbf{C}^4) \to \Gamma_{(2,2)}(4)$,

$$\cdots \stackrel{0}{\to} \bar{H}_{i+1}(M,\mathbf{C}) \to H_i(G_2(\mathbf{C}^4),\mathbf{C}) \to H_i(\Gamma_{(2,2)}(4),\mathbf{C}) \stackrel{0}{\to} \bar{H}_i(M,\mathbf{C}) \to \cdots$$

and hence can easily be calculated: it is equal to $\mathbf{C}$ in dimensions 3, 5 and 7, and is trivial in all other dimensions.

Since $\bar{H}_*(M) \equiv \tilde{H}_*(\partial\Xi(4),\Omega)$, the homology group of $\partial\Xi(4)$ follows from the exact sequence of this pair. The result is encoded in the right-hand column of the lower left table in Fig. 1.

Remark. The similar integer-valued spectral sequence does nod degenerate at the term E^1 : the right-hand part of its column $\{p=2\}$ contains a 2-torsion, which disappears in the limit homology group.

5.4 The case $n=5$.

Here we prove that the term E^1 of the main spectral sequence calculating $\bar{H}_*(\Sigma(5),\mathbf{C})$ is as shown in the right-hand table of Fig. 1. The column $\{p=1\}$ is already justified: it coincides with the homology group of $G_2(\mathbf{C}^5)$ up to a shift of dimensions.

The left part of the column $\{p=2\}$ corresponds to the block $\beta_{(3)}(5)$: its term on the level q coincides with the $(q-4)$-dimensional component of the graded group $H_*(G_3(\mathbf{C}^5)) \otimes \tilde{H}_*(G_2(\mathbf{C}^3))$. (Here $4 = \#A + (\delta(A))^2 + 1 - p$.)

The right-hand part of the same column contains the homology of the space $\beta_{(2,2)}(5)$. Its base space $\Gamma_{(2,2)}(5)$ is fibered over $\mathbf{CP}^4$ with fiber $\Gamma_{(2,2)}(4)$. The spectral sequence of this fibration obviously degenerates at the second term, so that $H_*(\Gamma_{(2,2)}(5)) \simeq H_*(\mathbf{CP}^4) \otimes H_*(\Gamma_{(2,2)}(4))$. The fiber (7) of the bundle $\beta_{(2,2)}(5) \to \Gamma_{2,2}(5)$ is the direct product of $\mathbf{R}^3$ and an open interval, and this fibration is orientable. Therefore by the Thom isomorphism we get the right-hand part of the column $\{p=2\}$.

The left part of the column $\{p = 3\}$ contains the homology groups of the space $\beta_{(4)}(5)$, which is fibered over $\Gamma_{(4)}(5) \equiv \mathbf{CP}^4$ with the fiber $\mathbf{R}^2 \times \breve{\Xi}(4)$. The homology groups of the last space $\breve{\Xi}(4)$ are calculated in the previous subsection (see the column $\{p = 3\}$ of the corresponding spectral sequence). It follows from this calculation, that terms $\mathcal{E}^2_{a,b}$ of the spectral sequence of the fibration $\beta_{(4)}(5) \to \mathbf{CP}^4$ can be nontrivial only if a is even and b is odd. Therefore this sequence degenerates and gives us the left part of the column $\{p = 3\}$.

The right-hand part of the same column contains the homology of the space $\beta_{(3,2)}(5)$. Its basespace $\Gamma_{(3,2)}(5)$ coincides with $G_3(\mathbf{C}^5)$, and the three factors of the fiber (7) over some its point are respectively $\mathbf{R}^2$, one point, and the open cone over the suspension $\Sigma(\mathbf{CP}^2)$, cf. Proposition 7. Therefore the column $\{p = 3\}$ also is justified.

Finally, the shape of the last column $\{p = 4\}$ follows from formula (2) (providing all groups $\bar{H}_i(\Sigma(n), \mathbf{C})$), Theorem 2 (stating that any such group is the direct sum of groups $E^1_{p,i-p}$), and the columns calculated previously (providing all other elements of these sums).

A similar consideration allows us to calculate all groups $\partial\Xi(n)$ if we know similar groups with smaller values of n, and also the homology groups of all spaces $\beta_A(n)$ with A of complexities $\leq n-2$. Concerning the calculation of these groups, see § 6 below.

5.5 2-dimensional cohomology classes of $\mathcal{H}(n) \setminus \Sigma(n)$.

It is convenient to replace our homological spectral sequence $E^r_{p,q} \to \bar{H}_{p+q}(\Sigma(n))$, defined in § 4, by its "Alexander dual" cohomological sequence, obtained from it by the formal inversion of indices,

$$E_r^{p,q} \equiv E^r_{-p,n^2-q-1}. \tag{11}$$

This spectral sequence lies in the second quadrant in the wedge $\{p \leq 0, p+q \geq 0\}$ and converges exactly to the group $H^*(\mathcal{H}(n) \setminus \Sigma(n))$.

Proposition 8 *For any n the integer-valued cohomological spectral sequence has exactly $n-1$ nontrivial terms $E_1^{p,q}$ with $p+q=2$, namely $E^{-1,3}$, $E^{-2,4}$, $E^{-n+1,n+1}$, all of which are isomorphic to $\mathbf{Z}$. The corresponding filtration in the group $H^2(\mathcal{H}(n) \setminus \Sigma(n))$ is as follows: its term F_p consists of all sums $\sum_{i=1}^n \alpha_i c^i$, where the sequence $\{\alpha_i\}$ coincides with a polynomial of degree $\leq p$ taking integer values at integer points.*

So, the quotient group $E^{-p,p+2}$ is generated by the basic polynomial sequence of degree p, $\alpha_i = i(i-1)\cdots(i-p+1)/p!$, or, if we are interested only in $\mathbf{C}$-valued cohomology, then just by the monomial i^p.

Proposition 9 *For any n the group $E^{-1,5}$ or first-order elements of the group $H^4(\mathcal{H}(n) \setminus \Sigma(n), \mathbf{Z})$ is one-dimensional and is generated by the series $\sum_{i=1}^n i(c^i)^2$.*

The proofs are immediate. □

6 Proof of Theorems 1, 2

We shall prove these theorems by induction over n.

Lemma 2 *Suppose that Theorem 1 is true for all $n \leq n_0$, and all elements a_i of the multiindex $A = (a_1, \cdots, a_{\#A})$ do not exceed n_0; let $\beta_A(N)$ be the corresponding block of the resolution of the discriminant space $\Sigma(N) \subset \mathcal{H}(N)$, and $\beta!_A(N)$ its universal covering space, see the end of § 4. Then*

a) the spectral sequence of the fiber bundle $\beta!_A(N) \to \Gamma!_A(N)$, converging to the group $\bar{H}_*(\beta!_A(N))$, degenerates at the second term;

b) the groups $\bar{H}_*(\beta!_A(N), \mathbf{C})$ and $\bar{H}_*(\beta_A(N), \mathbf{C})$ are trivial in all even (respectively, odd) dimensions if N is even (respectively, odd).

Proof. $\beta!_A(N)$ is the space of a fiber bundle over the simply-connected manifold $\Gamma!_A(N)$ (all whose odd-dimensional homology groups are trivial), with the fiber (7), all whose ($N -$ *even*)-dimensional Borel–Moore homology groups also are trivial by our assumption. This implies statement a) of the Lemma, and statement b) follows from it because the complex homology group of the base of a finite covering is "not greater" than that of its space. □

Proposition 10 *If Theorem 1 is true for all $n < N$, then Theorem 2 is true for $n = N$, i.e. the spectral sequence calculating the group $\bar{H}_*(\Sigma(N), \mathbf{C})$ degenerates at the first term.*

This proposition implies the assertion of Theorem 1 for $n = N$ and thus completes the step of induction. Indeed, by this proposition the group $\bar{H}_i(\Sigma(N), \mathbf{C})$ splits into the direct sum of groups $E^1_{p,i-p}$. By the formulas (1), (2) this group is trivial in all dimensions comparable with N mod 2. Hence also the group $E^1_{N-1,i-(N-1)} \equiv \bar{H}_i(\beta_{(N)}(N), \mathbf{C}) \equiv \bar{H}_i(\mathbf{R}^1 \times \check{\Xi}(N), \mathbf{C}) \cong \tilde{H}_{i-2}(\partial\Xi(N), \mathbf{C})$ is trivial for such i.

In the proof of Proposition 10 we shall use the following version of the Poincaré duality in the singular semialgebraic sets like $\overline{\beta_A(N)}$, $\overline{\beta!_A(N)}$, and $\overline{\sigma(N)}$, cf. [7].

Given a compact Whitney stratified semialgebraic variety V, we embed it into some sphere S^M as an absolute retract of some its neighborhood $U \subset S^M$ such that its boundary ∂U is a smooth manifold. Then a *quasicycle* in V is any relative cycle of the pair $(U, \partial U)$, generic with respect to the stratification of V. Any complex-valued cohomology class of the space V can be realized as the intersection index with some such quasicycle. The *support* of a quasicycle is its intersection set with V.

Proof of Proposition 10. We shall prove the dual (and thus equivalent) statement concerning the cohomological spectral sequence calculating the Borel–Moore *cohomology* of $\Sigma(N)$ (i.e. the reduced cohomology of its one-point compactification $\overline{\Sigma(N)}$). Its term $e_1^{p,q}$ is equal to $\tilde{H}^{p+q}\left(\overline{(\sigma_p(N) \setminus \sigma_{p-1}(N))}, \mathbf{C}\right) \equiv \oplus_{|A|-\#A=p} \tilde{H}^{p+q}\left(\overline{\beta_A(N)}, \mathbf{C}\right)$, summation over all indices A of complexity p. Thus we need to prove that for any element ω of such a group $\tilde{H}^{p+q}\left(\overline{\beta_A(N)}, \mathbf{C}\right)$, all its differentials $d^i(\omega) \in e^{p+i,q-i+1}$ are trivial.

Let $\omega!$ be the lifting of ω to the cohomology of $\beta!_A(N)$, and $[\omega!]$ the compact quasicycle in $\beta!_A(N)$ Poincaré dual to this class. Its projection $[\omega] \subset \beta_A(N)$ is Poincaré dual to some nonzero integer multiple of the class ω.

Consider the open subset $reg\beta!_A(N) \subset \beta!_A(N)$, consisting of all pairs of the form {a Hermitian operator Λ; a point $\xi \in \breve{\Xi}(\gamma!)$} (where $\gamma! \in \Gamma!_A(N)$) such that the restriction of Λ onto the orthogonal plane $(\gamma!)^\perp \subset \mathbf{C}^N$ is an operator with simple (i.e. $\delta(A)$-element) spectrum.

Lemma 3 *For any class $\omega! \in \bar{H}^*(\beta!_A(N), \mathbf{C})$, the Poincaré dual quasicycle $[\omega!]$ can be chosen so that it support lies in $reg\beta!_A(N)$.*

Proof. This assertion is equivalent to the following one: the homomorphism

$$\bar{H}^*(\beta!_A(N), \mathbf{C}) \to \bar{H}^*(reg\beta!_A(N), \mathbf{C}), \tag{12}$$

induced by the identical embedding, is monomorphic. To prove it, denote by $\Upsilon_A(N)$ the total space of the subbundle of the fiber bundle $\beta!_A(N) \to \Gamma!_A(N)$, whose fibers are the products of only first and third factors in (7). Then we have the commutative diagram of fiber bundles

$$\begin{array}{ccc} \beta!_A(N) & \leftarrow & reg\,\beta!_A(N) \\ \downarrow & & \downarrow \\ \Upsilon_A(N) & \stackrel{Id}{\leftarrow} & \Upsilon_A(N) \end{array} \tag{13}$$

with fibers $\mathcal{H}((\gamma!)^\perp)$ and $\mathcal{H}((\gamma!)^\perp) \setminus \Sigma((\gamma!)^\perp)$ respectively.

The left bundle is an orientable $(\delta(A))^2$-dimensional vector bundle, hence the second term of the corresponding spectral sequence, calculating the Borel–Moore cohomology of $\beta!_A(N)$, has unique nontrivial row $q = (\delta(A))^2$, which coincides with the graded group $\bar{H}^*(\Upsilon_A(N), \mathbf{C})$; in particular this sequence degenerates at this term. By the Corollary of Proposition 7 and Theorem 1 (which we assume to be true for all values of n equal to elements a_i of the multiindex A) this graded group trivial in dimensions comparable with $|A| \bmod 2$.

The similar spectral sequence for the right-hand bundle in (13) also degenerates at the second term: this again follows from dimensional reasons, because the group $\bar{H}^*(\mathcal{H}(\gamma!^\perp) \setminus \Sigma(\gamma!^\perp), \mathbf{C})$ is nontrivial only in dimensions, comparable with $\delta(A) \bmod 2$.

The homomorphism of these second terms of spectral sequences is an isomorphism of rows $\{q = (\delta(A))^2\}$, and is zero operator for all other q. This implies our lemma.

Further, consider the subset $perf\beta!_A(N) \subset reg\beta!_A(N)$, consisting of pairs of the form {an operator Λ; a point $\xi \in \breve{\Xi}(\gamma!)$}, where $\gamma! = (\gamma_1, \dots, \gamma_{\#A})$, such that additionally the eigenvalues of Λ on the space γ_1 (respectively, γ_2, ..., $\gamma_{\#A}$, $\gamma!^\perp$) lie in the interval $(0, 1)$ (respectively, $(2, 3), \dots, (2\#A - 2, 2\#A - 1), (2\#A, +\infty)$).

Lemma 4 *For any class $\omega! \in \bar{H}^*(\beta!_A(N), \mathbf{C})$, the Poincaré dual quasicycle $[\omega!]$ can be chosen so that it support lies in $perf\,\beta!_A(N)$.*

Proof. On the space $\beta!_A(N)$ there acts the group $\mathbf{R}^{\#A} \oplus \mathbf{R}$, where $\mathbf{R}^{\#A}$ consists of the shifts along the first (trivial) factor in (7), and $\mathbf{R}$ of addings the operators which are scalar on the spaces $\gamma!^{\perp}$ and trivial on all γ_i. (If $\delta(A) = 0$, then the last action of $\mathbf{R}$ is trivial.) This action preserves all fibers of the bundle $\beta!_A(N) \to \Gamma!_A(N)$ and the subspace $reg\beta!_A(N)$, and takes quasicycles to quasicycles.

Any compact subset in $\beta!_A(N)$ obviously can be moved to the domain $perf\beta!_A(N)$ by the action of this group and the additional action of the group $\mathbf{R}_+$ of dilations in $\mathbf{C}^n$. $\Box$.

So, our class $\omega \in H^*(\beta_A(N), \mathbf{C})$ can be realized by the projection $[\omega]$ of some compact quasicycle $[\omega!] \subset perf\beta!_A(N)$. But the projection of the set perf $\beta!_A(N)$ to $\beta_A(N)$ does not meet the closures of any other blocks $\beta_{A'}(N)$ of greater filtration (i.e., with $|A'| - \#A' > |A| - \#A$). Thus our quasicycle $[\omega]$ is a quasicycle in entire space $\overline{\sigma(n)}$, the dual Borel–Moore cohomology class in $\sigma(n)$ is well defined, and Proposition 10 is proved. $\Box$

7 Stabilization

Definition. The *order* of an element of the cohomology group (1) is the filtration of the Alexander dual homology class in $\Sigma(n)$, i.e. the smallest number i such that it can be realized by a locally finite chain inside σ_i.

Consider again the cohomological spectral sequences $E_r^{p,q}(n) \to H^{p+q}(\mathcal{H}(n) \setminus \Sigma(n))$, see (11).

Let $n < N$ be any two naturals, s one of numbers $0, \dots, N-n$, and $i_s : \mathcal{H}(n) \to \mathcal{H}(N)$ an embedding, sending an operator $\Lambda : \mathbf{C}^n \to \mathbf{C}^n$ to the operator $\Lambda + \Lambda'$, where Λ' acts on the orthogonal complement of $\mathbf{C}^n$ in $\mathbf{C}^N$, depends smoothly on Λ and has a simple spectrum, exactly s (respectively, $N - n - s$) elements of which are below (respectively, above) the spectrum of Λ. In particular $\Sigma(n) = i_s^{-1}(\Sigma(N))$.

Proposition 11 *Any such inclusion induces a homomorphism of cohomological spectral sequences,*

$$E_r^{p,q}(N) \to E_r^{p,q}(n), \tag{14}$$

$r \geq 1$ *which* does not depend *on s and on the inclusion i_s, is epimorphic for any p and q, and provides an isomorphism of terms $E_r^{p,q}$ if n is sufficiently large with respect to $|p| + |q|$.*

Thus we can define the *stable cohomological spectral sequence*, whose term $E_r^{p,q}$ is the inductive limit of $E_r^{p,q}(n)$ over all possible such homomorphisms. By the last statement of the proposition any such stable term is finitely generated. This sequence converges to entire ring (4), thus defining the notion of the order also for elements of this ring. In easy terms, an element of the ring (4) is *of order i* if for any sufficiently large n and any projection of (4) onto $H^*(\mathcal{H}(n) \setminus \Sigma(n), \mathbf{C})$, described in § 2, it becomes an element of order i. Certainly, not all elements of the ring (4) have some finite order, however the spaces of finite-order elements *weakly converge* to this ring in the following precise sense: for any n any cohomology class on the space $\mathcal{H}(n) \setminus \Sigma(n)$ coincides with the restriction of some stable class of finite order. For the similar theory of knot invariants the similar statement (the completeness of finite-order invariants) is not proved.

Here is an estimate for the convergence of groups $E_r^{p,q}$. For any multiindex A, the obvious inclusions of flag manifolds, $\Gamma!_A(n) \hookrightarrow \Gamma!_A(n+1) \hookrightarrow \cdots$, induce the maps of their cohomology groups. For any i denote by $stab(A,i)$ the smallest number n at which all these cohomology groups of dimensions $\leq i$ stabilize, i.e. all further inclusions induce isomorphisms in these dimensions.

Proposition 12 *For any biindex (p,q) with $p \leq 0, p+q \geq 0$, the groups $E_1^{p,q}(n)$ stabilize no later than at the instant $n = \max_{|A|-\#A=-p} stab(A, p+q-2\#A)$.*

The precise construction of the homomorphism mentioned in Proposition 11 is as follows (cf. [12], [13]). The embedding $i_s : (\mathcal{H}(n), \Sigma(n)) \to (\mathcal{H}(N), \Sigma(N))$ can be tautologically lifted to the filtration-preserving map of resolutions, $i_s : \sigma(n) \to \sigma(N)$. Its image admits in $\sigma(N)$ an open neighborhood U homeomorphic to $\sigma(n) \times \mathbf{R}^{N^2-n^2}$, where the image of any space $* \times \mathbf{R}^{N^2-n^2}$ lies completely in one and the same block $\beta_A(N)$. Then we get the map $\bar{H}_*(\sigma(N)) \to \bar{H}_{*-(N^2-n^2)}(\sigma(n))$ as the composition of the restriction homomorphism $\bar{H}_*(\sigma(N)) \to \bar{H}_*(U)$ and the Künneth isomorphism $\bar{H}_*(\sigma(n) \times \mathbf{R}^{N^2-n^2}) \to \bar{H}_{*-(N^2-n^2)}(\sigma(n))$. By the construction, it is compatible with the map of Alexander dual groups, $i_s^* : H^*(\mathcal{H}(N) \setminus \Sigma(N)) \to H^*(\mathcal{H}(n) \setminus \Sigma(n))$ induced by the embedding i_s. The homomorphism of terms E_1 of our spectral sequences is defined by the restrictions of this construction on any block $\beta_A(N)$.

For any A, set $U_A \equiv U \cap \beta_A(N)$. The homeomorphism $\beta_A(n) \times \mathbf{R}^{N^2-n^2} \to U_A$ is compatible with the structure of the fiber bundle in spaces $\beta_A(\cdot)$, mentioned in Proposition 5. Namely, the embedding $\mathbf{C}^n \hookrightarrow \mathbf{C}^N$ defines the embeddings $\Gamma_A(n) \hookrightarrow \Gamma_A(N)$ and $\Gamma!_A(n) \hookrightarrow \Gamma!_A(N)$; there is a tubular neighborhood W of $\Gamma_A(n)$ in $\Gamma_A(N)$, whose bundle is orientable and fibers are homeomorphic to $\mathbf{R}^{2|A|(N-n)}$. We get the diagrams of bundles

$$\begin{array}{ccccc} \beta!_A(n) & \hookrightarrow & U! & \subset & \beta!_A(N) \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma!_A(n) & \hookrightarrow & W! & \subset & \Gamma!_A(N), \end{array} \qquad \begin{array}{ccccc} \beta_A(n) & \hookrightarrow & U & \subset & \beta_A(N) \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma_A(n) & \hookrightarrow & W & \subset & \Gamma_A(N). \end{array} \tag{15}$$

For any point $\gamma \in \Gamma_A(n)$ or $\gamma! \in \Gamma!_A(n)$, the fibers $\mathbf{R}^{\#A}$ and $\breve{\Xi}(\gamma)$ of the first and third factors of the fibered products $\beta_A(n) \to \Gamma_A(n)$, $\beta!_A(n) \to \Gamma!_A(n)$ over this point (see (7)) go identically to the corresponding fibers of similar bundles over $\Gamma_A(N)$ and $\Gamma!_A(N)$, while the fibers $\mathcal{H}(n-|A|)$ of the second (vector) bundle become embedded into similar fibers $\mathcal{H}(N-|A|)$ of the second bundle over $\Gamma!_A(N)$ as subspaces, whose quotient spaces form an orientable $(N^2-n^2-2|A|(N-n))$-dimensional vector bundle over the image of $\Gamma_A(n)$ or $\Gamma!_A(n)$. The permutation group $S(A)$ acts on the left-hand diagram (15), and the right-hand diagram is formed by the spaces of its orbits; thus all complex homology homomorphisms induced by the arrows in the latter diagram are determined by similar homomorphisms for the former one.

But these homomorphisms in the complex Borel–Moore homology groups are very easy: the bottom homomorphism

$$\bar{H}_*(\Gamma !_A(n), \mathbf{C}) \leftarrow \bar{H}_{*+2|A|(N-n)}(\Gamma !_A(N), \mathbf{C}) \qquad (16)$$

is nothing but the standard cohomology map, induced by the obvious embedding and rewritten in the terms of Poincaré dual groups. The Borel–Moore homology groups of spaces of both bundles of this diagram are just the tensor products of corresponding groups (16) of their bases and the homology groups of fibers, which coincide up to a shift of dimensions by $N^2 - n^2 - 2|A|(N-n)$. The corresponding map $\bar{H}_*(\beta !_A(n), \mathbf{C}) \leftarrow \bar{H}_{*+(N^2-n^2)}(\beta !_A(N), \mathbf{C})$ of these tensor products decomposes into these actions on their factors.

The map (16) is always epimorphic, thus also the map (14) is.

Proof of Proposition 12. Let us fix some multiindex A and dimension i and estimate the greatest dimension $J = J(A,i)$ such that the group $\bar{H}_J(\beta !_A(n))$ depends on the group $H^i(\Gamma !_A(n))$. By Proposition 5 and formula (8) this dimension J does not exceed the sum of 4 numbers: $\dim \Gamma !_A(n) - i$, $\#A$, $(n-|A|)^2$ and $\#A - 1 + \sum_{i=1}^{\#A} \dim_h \breve{\Xi}(a_i)$, where $\dim_h(\cdot)$ is the maximal dimension i such that $\bar{H}_i(\cdot)$ is nontrivial.

Define $s_2(A)$ as the second symmetric polynomial of numbers $a_1, \dots, a_{\#A}$, $s_2(A) = a_1a_2 + a_1a_3 + \cdots + a_{\#A-1}a_{\#A}$, then $\dim \Gamma !_A(n) = 2(s_2(A) + |A|(n-|A|))$. By § 5.1 the group $\bar{H}_i(\breve{\Xi}(a), \mathbf{C})$ is a direct summand of the group $\bar{H}_{i+1}(\Sigma(a))$. Since $\dim \Sigma(a) = a^2 - 3$, we have $\dim_h \breve{\Xi}(a) \leq a^2 - 4$. Thus the sum of four above numbers does not exceed $2(s_2(A) + |A|(n-|A|)) - i + \#A + (n-|A|)^2 + \#A - 1 + \sum a_i^2 - 4\#A \equiv n^2 - 1 - 2\#A - i$. Therefore by (11) any cell $E_1^{p,q}$ depends only on such groups $H^i(\Gamma !_A(n), \mathbf{C})$ that $|A| - \#A = -p$ and $p + q \geq i + 2|A|$. $\square$

Residues of cohomology classes at strata of $\Sigma(n)$.

Definition. Given a class $\omega \in H^*(\mathcal{H}(n) \setminus \Sigma(n))$ of order p and a multiindex A of complexity p, the corresponding *symbol* $s(A,\omega) \in \bar{H}_*(\beta_A(n))$ is the restriction of the corresponding homology class in $\bar{H}_*(\Sigma)$ to the block $\beta_A(n)$. (In a more detailed way, we realize the latter class by a cycle lying in $F_p(\sigma(n))$ and then reduce modulo the union of σ_{p-1} and all other blocks $\beta_{A'}(n)$ of complexity p, $A' \neq A$.) Similarly, the *!-symbol* $s!(A,\omega) \in \bar{H}_*(\beta !_A(n))$ is the homology class of the complete preimage of this class in the covering $\beta !_A(n)$.

By Lemma 2, any such !-symbol is the element of the cohomology group of the flag manifold $\Gamma !_A(n)$ with coefficients in the homology group of the topological order complex $\partial \Xi(A)$. In a similar way, the usual symbol $s(A,\omega)$ is the element of the cohomology group of the manifold $\Gamma_A(n)$ with coefficients in a local system of groups with the same fibers.

8 Problems

Problem 1. Our cohomological spectral sequence defines filtrations ("orders") in the rings

(3), (4). The problem is to find the explicit expression of elements of this filtration. I hope very much that it coincides with or is closely related to some classical structure in these rings.

In some sense, the term F_{-p} of this filtration should consist of power series, whose coefficients are "constructive functions of complexity $\leq p$" of corresponding exponents, cf. § 5.5. What is the precise sense of this "complexity"?

Does there exist any natural (and physically motivated) presentation of elements of this filtration in terms of differential forms in $\mathcal{H}(n) \setminus \Sigma(n)$, cf. [3]?

Here are two related subproblems, which can also be investigated independently.

Problem 2. The shift operator $\mathbf{S}$ (see the end of § 2) preserves our filtration: $\mathbf{S}(F_p) = F_p$. Is it true, that the corresponding derivative, sending the cohomology class α to $\mathbf{S}(\alpha) - \alpha$, maps any F_p to F_{p-1}?

Problem 3: Is there true the following MULTIPLICATIVITY CONJECTURE: *Our cohomological spectral sequence is multiplicative, i.e.* $F_{p'} \smile F_{p''} \subset F_{p'+p''}$.

If it is true, the next problem is to define the corresponding maps $E_1^{p',q'} \otimes E_1^{p'',q''} \to E_1^{p'+p'',q'+q''}$ explicitly.

There is some obvious multiplication of this sort (in some sense modelling the Kontsevich's formula for multiplication of knot invariants in the terms of chord diagrams, see e.g. [13].) Namely, consider any multiindices $A' = (a'_1, \dots, a'_{\#A'})$ and $A'' = (a''_1, \dots, a''_{\#A''})$ and set $A = A' \cup A''$. Below we define the natural map

$$\bar{H}_{n^2-i-1}(\beta_{A'}(n), \mathbf{C}) \otimes \bar{H}_{n^2-j-1}(\beta_{A''}(n), \mathbf{C}) \to \bar{H}_{n^2-i-j+1}(\beta_A(n), \mathbf{C}); \tag{17}$$

the desired multiplication will follow from these by the Alexander duality (11) and the decomposition $\sigma_p(n) \setminus \sigma_{p-1}(n) = \bigsqcup_{|A|-\#A=p} \beta_A(n)$.

Indeed, define the group $\bar{H}_*(\breve{\Xi}(A))$ as $\bar{H}_*(\breve{\Xi}(\gamma))$ for any $\gamma \in \Gamma_A(n)$. Then the map (17) is the composition of following maps of complex homology groups:

$$\begin{array}{ccccccccc}
\bar{H}_*(\beta_{A'}(n)) & \overset{\triangle}{\to} & \bar{H}_*(\beta!_{A'}(n)) & \overset{\sim}{\to} & H_*(\Gamma_{A'}(n)) & & \bar{H}_*(\mathbf{R}^{2\#A'-1+(n-|A'|)^2}) & & \bar{H}_*(\breve{\Xi}(A')) \\
\otimes & & \otimes & & \otimes & \otimes & \otimes & \otimes & \otimes \\
\bar{H}_*(\beta_{A''}(n)) & \overset{\triangle}{\to} & \bar{H}_*(\beta!_{A''}(n)) & \overset{\sim}{\to} & H_*(\Gamma_{A''}(n)) & & \bar{H}_*(\mathbf{R}^{2\#A''-1+(n-|A''|)^2}) & & \bar{H}_*(\breve{\Xi}(A'')) \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
\bar{H}_*(\beta_A(n)) & \overset{\nabla}{\leftarrow} & \bar{H}_*(\beta!_A(n)) & \overset{\sim}{\leftarrow} & H_*(\Gamma_A(n)) & \otimes & \bar{H}_*(\mathbf{R}^{2\#A-1+(n-|A|)^2}) & \otimes & \bar{H}_*(\breve{\Xi}(A)).
\end{array}$$

Here the maps $\overset{\triangle}{\to}$ are the liftings, representing cycles in $\beta_A(n)$ as projections of some cycles from $\beta!_A(n)$; all identities $\overset{\sim}{\to}$, $\overset{\sim}{\leftarrow}$ follow from Proposition 5 and statement a) of Lemma 2. The vertical map $H_*(\Gamma!_{A'}(n)) \otimes H_*(\Gamma!_{A''}(n)) \to H_*(\Gamma!_A(n))$ (attention! it does not preserves degrees!) is the composition of a) Poincaré isomorphisms in $\Gamma!_{A'}(n)$, $\Gamma!_{A''}(n)$, b) the Künneth isomorphism $H^*(\Gamma!_{A'}(n)) \otimes H^*(\Gamma!_{A''}(n)) \to H^*(\Gamma!_{A'}(n) \times \Gamma!_{A''}(n))$, c) the map in cohomology induced by the tautological embedding of $\Gamma!_A(n)$ into this product, and finally d) the Poincaré isomorphism in $\Gamma_A(n)$. The vertical map $\bar{H}_*(\mathbf{R}^{2\#A'-1+(n-|A'|)^2}) \otimes \bar{H}_*(\mathbf{R}^{2\#A''-1+(n-|A''|)^2}) \to \bar{H}_*(\mathbf{R}^{2\#A-1+(n-|A|)^2})$ (also not preserving the grading) maps the product of canonical generators of two first groups into the canonical generator of the last one. The vertical maps $\bar{H}_*(\breve{\Xi}(A')) \otimes \bar{H}_*(\breve{\Xi}(A'')) \to \bar{H}_*(\breve{\Xi}(A))$ are described in the Proposition 7 (they increase the grading by 1), finally the map $\stackrel{\nabla}{\leftarrow}$ is the obvious projection. It is easy to check that this composition maps a (n^2-i+1)-dimensional cycle in $\beta_{A'}(n)$ and a (n^2-j+1)-dimensional cycle in $\beta_{A''}(n)$ into a $(n^2-i-j-1)$-dimensional homology class in $\beta_A(n)$ and does not depend on the choice of liftings $\stackrel{\triangle}{\rightarrow}$ of these cycles to corresponding spaces $\beta!_\cdot(n)$.

Problem. If the multiplication conjecture is true, how is the actual multiplication in the spectral sequence related with the one just described? What are its multiplicative generators?

Besides our increasing filtration (by orders) the ring (4) admits a decreasing filtration: its element is of *confinement* d if for any $n < d$ it lies in the kernels of all surjections onto the ring (3), described in § 2.

Is not it possible to characterize the primitive (i.e., indecomposable) elements of our spectral sequence in the terms of these two filtrations?

Problem 4. To find the compact general formula for the homology of the space $\partial\Xi(n)$ for any n.

This space is not a topological manifold (note that the right-hand columns of tables in Fig. 1 present the *reduced* homology of these spaces).

Therefore also their intersection homology groups are very interesting.

The last remark

All results of this work can be immediately carried over to the theory of *hyperhermitian* forms (see [4]), i.e. of quadratic forms on the realification $\mathbf{R}^{4n}$ of the quaternionic space $\mathcal{H}^n$ invariant under the left multiplications by quaternions of length 1.

In particular, the main spectral sequence calculating the rational Borel–Moore homology groups of the space of such forms with $< n$ eigenvalues degenerates at the term E^1, and the reduced homology groups of the quaternionic analog of the order complex $\partial\Xi(n)$ are trivial in dimensions not comparable with $2n^2 - n + 1$ modulo 4.

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