How to calculate homology groups of spaces of nonsingular algebraic projective hypersurfaces

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Abstract

A general method of computing cohomology groups of the space of nonsingular algebraic hypersurfaces of degree d in $\mathbf{CP}^n$ is described. Using this method, rational cohomology groups of such spaces with $n = 2, d \leq 4$ and $n = 3 = d$ are calculated.

1 Introduction

I describe a method of calculating the cohomology groups of spaces of nonsingular algebraic hypersurfaces of fixed degree in $\mathbf{CP}^n$: a complex analog of the rigid homotopy classification of algebraic hypersurfaces in $\mathbf{RP}^n$.

This method is essentially the one described in [12], [14]; it consists of two main parts. The first one is due to Arnold, who remarked [1] that often it is convenient to replace the calculation of cohomology groups of spaces of nonsingular objects by that of the (Alexander dual) homology groups of corresponding *discriminant* spaces of singular objects; see also [2]. The second one is the techniques of conical resolutions and topological order complexes, which is a continuous analog of the combinatorial inclusion-exclusion formula and is adopted very well to the calculation of the latter homology groups. This construction was proposed in [9] and generalized in [11], [12], [14].

The version of this method, applied to spaces of nonsingular projective hypersurfaces, is described in § 3. In the preceding § 2 we demonstrate all main ingredients of it, considering the simplest problem of this series: we calculate there the (well-known) cohomology groups of the space of regular quadrics in $\mathbf{CP}^2$. In § 4 we do the same for the real cohomology groups of the space of nonsingular cubics in $\mathbf{CP}^2$ (which probably also can be calculated by more standard methods, see remark after Theorem 2). In § 5, § 6 and § 7 we present similar calculations respectively for spaces of nonsingular quartics in $\mathbf{CP}^2$, nonsingular cubics in $\mathbf{CP}^3$, and nondegenerate quadratic vector fields in $\mathbf{C}^3$.

*supported in part by INTAS (project ????) and RFBR (project ????)

This work is inspired by the Arnold's problem 1970-13 on the topology of the space of nonsingular cubics, see [3].

1.1 Stating the problem and the first reduction.

Let $\Pi_{d,n} \simeq \mathbf{C}^{D(d,n)}$ be the space of all homogeneous polynomials $\mathbf{C}^{n+1} \to \mathbf{C}^1$ of degree d, $\Sigma_{d,n} \subset \Pi_{d,n}$ the set of forms, having singular points outside the origin, and $N_{d,n} \subset \mathbf{CP}^{D(d,n)-1}$ the projectivization of $\Pi_{d,n} \setminus \Sigma_{d,n}$, i.e. the set of nonsingular algebraic hypersurfaces of degree d in $\mathbf{CP}^n$. $\Sigma_{d,n}$ is a conical nonempty algebraic hypersurface in $\Pi_{d,n}$, therefore $\Pi_{d,n} \setminus \Sigma_{d,n}$ is diffeomorphic to $N_{d,n} \times \mathbf{C}^*$, in particular the homology groups of spaces $N_{d,n}$ and $\Pi_{d,n} \setminus \Sigma_{d,n}$ are related via the Künneth formula.

By the Alexander duality theorem, for $i > 0$ we have

$$\tilde{H}^i(\Pi_{d,n} \setminus \Sigma_{d,n}) \simeq \bar{H}_{2D(d,n)-i-1}(\Sigma_{d,n}), \tag{1}$$

where $\bar{H}_*$ is the notation for the Borel–Moore homology, i.e. the homology of the one-point compactification reduced modulo the added point.

2 The trivial example: nonsingular quadrics in $\mathbf{CP}^2$

In this section we assume that $n = 2$ and denote the spaces $\Pi_{d,2}, \Sigma_{d,2}$ and $N_{d,2}$ simply by Π_d, Σ_d and N_d.

PROPOSITION 1. *Only the following groups $H^i(\Pi_2 \setminus \Sigma_2)$ are nontrivial: $H^0 \simeq H^1 \simeq H^5 \simeq H^6 \simeq \mathbf{Z}$, $H^3 \simeq H^4 \simeq \mathbf{Z}_2$. In particular, only the following groups $H^i(N_2)$ are nontrivial: $H^0 \simeq H^5 \simeq \mathbf{Z}$, $H^3 \simeq \mathbf{Z}_2$.*

This statement is not new: indeed, it is easy to see that the space $\Pi_2 \setminus \Sigma_2$ is homotopy equivalent to the Lagrangian Grassmannian manifold $U(3)/O(3)$, whose cohomology groups were studied in [4] and [6], see e.g. [10]. We give here another calculation, demonstrating our general method.

This calculation is based on the classification of subsets in $\mathbf{CP}^2$, which can be the singular sets of homogeneous polynomials of degree 2 in $\mathbf{C}^3$. There are exactly the following such sets:

A) any point $x \in \mathbf{CP}^2$,

B) any line $\mathbf{CP}^1 \subset \mathbf{CP}^2$ (such lines are parametrized by the points of the dual projective space $\mathbf{CP}^{2\vee}$);

C) the entire $\mathbf{CP}^2$.

For any set of type A), B) or C), the corresponding set of quadratic forms $\mathbf{C}^3 \to \mathbf{C}$, having singularities at all points of this set (and maybe somewhere else) is isomorphic, respectively, to $\mathbf{C}^3$, $\mathbf{C}^1$ or $\mathbf{C}^0$.

The main topological tool of the calculation is the following one.

2.1 The topological order complex Λ_2

Take the spaces $\mathbf{CP}^2, \mathbf{CP}^{2\vee}$ and $\{\bullet\}$ of sets of types A) B) and C), and consider the join $\mathbf{CP}^2 * \mathbf{CP}^{2\vee} * \{\bullet\}$ of these spaces, i.e., roughly speaking, the union of all simplices (of dimensions 0, 1 and 2), whose vertices correspond to their points.

Such a simplex is called *coherent,* if all subsets in $\mathbf{CP}^2$, corresponding to its vertices, are incident to one another.

The *topological order complex* Λ_2 is defined as the union of all coherent simplices, with topology induced from that of the join.

To any set K of the form A), B) or C) there corresponds a subset $\Lambda_2(K) \subset \Lambda_2$: the union of simplices, all whose vertices are subordinate by inclusion to the point $\{K\}$. Obviously this subset is a cone with the vertex $\{K\}$: for K of type A) (respectively, B), respectively, C)) it is equal to $\{K\}$ itself (respectively, to the cone over the topological space $K \simeq \mathbf{CP}^1$, respectively, to entire Λ_2). By $\mathring{\Lambda}_2(K)$ we denote this cone with its base removed.

2.2 Conical resolution

The *conical resolution* σ_2 of Σ_2 is defined as a subset in the direct product $\Pi_2 \times \Lambda_2$. Namely, let $K \subset \mathbf{CP}^2$ be any possible set of type A), B) or C). Let $L(K)$ be the set of all quadratic forms $\mathbf{C}^3 \to \mathbf{C}$, having singular points at all points of K (and maybe somewhere else); this always is a vector subspace in Π_2. Then the space $\sigma_2(K) \subset \Pi_2 \times \Lambda_2$ is defined as the direct product $L(K) \times \Lambda_2(K)$, and the desired conical resolution σ_2 as the union of all possible sets $\sigma_2(K)$.

PROPOSITION 2 (see [12], [14]). *The obvious projection $\sigma_2 \to \Sigma_2$ is a proper map, and the induced map of one-point compactifications of these spaces is a homotopy equivalence. In particular this projection induces the isomorphism $\bar{H}_*(\sigma_2) \simeq \bar{H}_*(\Sigma_2)$.* $\square$

The space σ_2 admits a natural filtration $F_1 \subset F_2 \subset F_3$: the subspace F_1 (respectively, F_2, respectively, F_3) of σ_2 is the union of sets $\sigma_2(K)$ such that the corresponding spaces K are of type A) only (respectively, of type A) or B), respectively, of type A) or B) or C), i.e. $F_3 = \sigma_2$).

Similarly, we consider the filtration $\{\Phi_i\}$ of the space Λ_2 : the subspace Φ_1 (respectively, Φ_2, respectively, Φ_3) of Λ_2 is the union of sets $\Lambda_2(K)$ such that the corresponding spaces K are of type A) only (respectively, of type A) or B), respectively, of type A) or B) or C), i.e. $\Phi_3 = \sigma_2$).

In particular, the space $F_i \setminus F_{i-1}$ is the space of a complex vector bundle over $\Phi_i \setminus \Phi_{i-1}$, the dimension of this bundle is equal to 3 for $i = 1$, to 1 for $i = 2$ and to 0 for $i = 3$. Thus the Borel–Moore homology groups of spaces $F_i \setminus F_{i-1}$ and $\Phi_i \setminus \Phi_{i-1}$ are related by the Thom isomorphism.

Consider the spectral sequence generated by the filtration $\{F_i\}$ and converging to the group $\bar{H}_*(\sigma_2)$. By definition its term $E^1_{p,q}$ is equal to $\bar{H}_{p+q}(F_p \setminus F_{p-1})$.

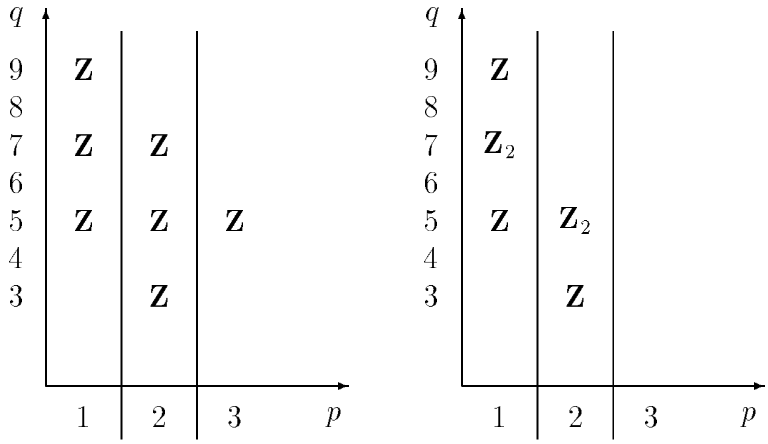


Figure 1: Terms E^1 and E^2 of the spectral sequence for $d = n = 2$.

LEMMA 1. *All nontrivial terms $E^1_{p,q}$ of this spectral sequence are as follows (see fig. 1 left): $E^1_{1,q} = \mathbf{Z}$ for $q = 5, 7, 9$; $E^1_{2,q} = \mathbf{Z}$ for $q = 3, 5, 7$; $E^1_{3,q} = \mathbf{Z}$ for $q = 5$.*

Proof. 1. F_1 is the space of a fiber bundle over $\mathbf{CP}^2$ with fiber $\mathbf{C}^3$, this proves the statement concerning $E^1_{1,q}$.

2. $F_2 \setminus F_1$ is fibered over the space $\mathbf{CP}^{2\vee}$ of all complex lines in $\mathbf{CP}^2$. The fiber over such a line K is the direct product of the space $L(K) \simeq \mathbf{C}^1$ and the space $\Lambda_2(K) \backslash \Phi_1$. But $\Lambda_2(K)$ is the cone over the set of all points of $K \sim \mathbf{CP}^1$, and the base of this cone belongs to Φ_1. Therefore $\Lambda_2(K) \setminus \Phi_1$ is an oriented 3-dimensional open disc, thus by the Thom isomorphism $\bar{H}_*(F_2 \backslash F_1) \simeq \bar{H}_{*-5}(\mathbf{CP}^{2\vee})$, and the statement concerning $E^1_{2,*}$ is proved.

3. $F_3 \setminus F_2 \equiv \Phi_3 \setminus \Phi_2$. Φ_2 is the subset of the join $\mathbf{CP}^2 * \mathbf{CP}^{2\vee}$, consisting of all segments, spanning such pairs $(x \in \mathbf{CP}^2, K \in \mathbf{CP}^{2\vee})$, that $x \in K$. In [11] it was proved that this subset is homeomorphic to S^7. Φ_3 is the cone over Φ_2, hence $F_3 \setminus F_2$ is an open 8-dimensional disc, and the statement concerning $E^1_{3,*}$ also is proved.

PROPOSITION 3. *The differential d^1 of our spectral sequence maps a generator of the group $E^1_{3,5}$ into twice a generator of $E^1_{2,5}$, maps a generator of $E^1_{2,7}$ into twice a generator of $E^1_{1,7}$, and acts trivially on all other groups $E^1_{p,q}$. In particular all nontrivial groups $E^2_{p,q}$ are as shown in fig. 1 right and our spectral sequence degenerates at this step: $E^2 \equiv E^\infty$.*

Proposition 1 follows immediately from this one, Lemma 1 and the Alexander isomorphism (1).

Proof of Proposition 3. 1. The space $F_2 \backslash F_1$ can be considered as the space of a fiber bundle with fiber $\mathbf{C}^1$, whose base $\Phi_2 \backslash \Phi_1$ is fibered over $\mathbf{CP}^{2\vee}$ with fiber equal to an open 3-dimensional disc. The image of the map $d^1 : E^1_{3,5} \to E^1_{2,5}$ is equal to the fundamental cycle (with closed supports) of the base $\Phi_2 \setminus \Phi_1$ of this bundle. The homomorphism

$$\bar{H}_7(\Phi_2 \backslash \Phi_1) \to \bar{H}_7(F_2 \backslash F_1) \to \bar{H}_5(\Phi_2 \backslash \Phi_1), \tag{2}$$

defined as the composition of the identical embedding and the Thom isomorphism, maps this fundamental cycle into a homology class, Poincaré dual to the Euler class of the complex line bundle, whose fiber over the points of the open cone $\mathring{\Lambda}_2(K)$, $K \in \mathbf{CP}^{2\vee}$, is equal to the space $L(K)$. This space consists of all quadratic functions of the form l^2, where l is an arbitrary linear form vanishing on the line K. It is easy to calculate that this Euler class is equal to twice the generator of the group $\bar{H}_5(\Phi_2 \backslash \Phi_1)$, and the first statement of Proposition 3 is proved.

2. The homomorphism $d^1 : E^1_{2,7} \to E^1_{1,7}$ maps the generator of the group $E^1_{2,7}$ into the fundamental cycle of the algebraic subset $\omega \subset F_1$, consisting of all pairs of the form (a point $u \in \mathbf{CP}^2$, a quadratic function l^2), such that l is a linear form with $l(u) = 0$. This subset is fibered over $\mathbf{CP}^2$ with a standard quadratic cone $\{\alpha\beta = \gamma^2\} \subset \mathbf{C}^3$ for a fiber. To calculate the homology class of this cycle, it is sufficient to calculate its intersection index with the generator of the group $H_2(F_1)$. This generator can be realized as follows: we take an arbitrary line $\mathbf{CP}^1 \subset \mathbf{CP}^2$, say the one given by the equation $x = 0$, and consider any section of the restriction on it of the fibration $F_1 \to \mathbf{CP}^2$. For such a section we can take the one, whose value over the point $(0:1:\lambda) \in \mathbf{CP}^1$ is the quadratic form $\frac{x(z-\lambda y)}{1+|\lambda|^2}$. This section intersects the above-described set ω at exactly one point, corresponding to $\lambda = \infty$. This intersection is of multiplicity 2, hence the fundamental cycle of this set ω is equal to twice the generator of $\bar{H}_8(F_1)$.

It is now obvious that d^1 cannot be nontrivial on any other group $E^1_{p,q}$, $(p,q) \neq (3,5), (2,7)$, and Proposition 3 is completely proved.

3 The general spectral sequence

Here we extend the above-described construction to the case of hypersurfaces of an arbitrary degree d. The spaces of sets of types A), B) and C) will be formalized in terms of the Hilbert-scheme topology.

3.1 Conical resolution of the discriminant: preliminary version.

Set $D \equiv D(d,n) = \dim \Pi_{d,n}$. Consider all Grassmann manifolds $G_i(\Pi_{d,n})$, $i = 1, \ldots, D$, whose points are the complex subspaces of *codimension* i in $\Pi_{d,n}$. Suppose that the subset $K \subset \mathbf{CP}^n$ is the set of singular points of a certain homogeneous function $\mathbf{C}^{n+1} \to \mathbf{C}^1$, and denote by $L(K)$ the subspace in $\Pi_{d,n}$ consisting of all functions having singular points everywhere in K (and maybe somewhere else). Obviously this is a vector subspace in $\Pi_{d,n}$. Let $\Omega_i \subset G_i(\Pi_{d,n})$ be the set of all subspaces of the form $L(K)$. Let $\bar{\Omega}_i$ be the closure of Ω_i in $G_i(\Pi_{d,n})$.

Consider the join $G_1(\Pi_{d,n}) * \cdots * G_{D-1}(\Pi_{d,n}) * G_D(\Pi_{d,n})$ and the subset in it, consisting of all *coherent* simplices of any dimensions, i.e. of such simplices, that a) all their vertices belong to subspaces $\bar{\Omega}_i$, and b) the collection of subspaces in $\Pi_{d,n}$, corresponding to all vertices of the simplex, forms a flag.

The union of all such simplices will be denoted by $\tilde{\Lambda}_{d,n}$. The *leading vertex* of a coherent simplex is the one lying in the space $\bar{\Omega}_i$ with the greatest value of i. For any point $\{L\} \in \bar{\Omega}_i$, the set $\tilde{\Lambda}_{d,n}(L)$ is the union of coherent simplices, whose leading vertices coincide with L. Obviously this is a cone with vertex $\{L\}$. In particular, $\tilde{\Lambda}_{d,n} = \tilde{\Lambda}_{d,n}(\{\Pi_{d,n}\})$.

The subset $\tilde{\sigma}_{d,n}(L) \subset \Pi_{d,n} \times \tilde{\Lambda}_{d,n}$ is defined as the direct product $L \times \tilde{\Lambda}_{d,n}(L)$. The *conical resolution* $\tilde{\sigma}_{d,n}$ of $\Sigma_{d,n}$ is defined as the union of all sets $\tilde{\sigma}_{d,n}(L)$ over all points $\{L\}$ of all $\bar{\Omega}_i$, $i = 1, \ldots, D$.

PROPOSITION 4. *The obvious projection $\tilde{\sigma}_{d,n} \to \Sigma_{d,n}$ is a proper map, and the induced map of one-point compactifications of these spaces is a homotopy equivalence. In particular it induces an isomorphism $\bar{H}_*(\tilde{\sigma}_{d,n}) \simeq \bar{H}_*(\Sigma_{d,n})$.*

The space $\tilde{\sigma}_{d,n}$ admits a natural increasing filtration: its term $\tilde{F}_i \subset \tilde{\sigma}_{d,n}$ is the union of sets $\tilde{\sigma}_{d,n}(L)$ over all $L \in \bar{\Omega}_j$, $j \leq i$.

Simultaneously we consider the increasing filtration $\{\tilde{\Phi}_i\}$ on the space $\tilde{\Lambda}_{d,n}$: its term $\tilde{\Phi}_i$ is the union of sets $\tilde{\Lambda}_{d,n}(L)$ such that $\text{codim}_{\mathbf{C}} L \leq i$. In particular, the space $\tilde{F}_i \setminus \tilde{F}_{i-1}$ is the space of a complex $(D-i)$-dimensional vector bundle over $\tilde{\Phi}_i \setminus \tilde{\Phi}_{i-1}$, and their Borel–Moore homology groups are related by the Thom isomorphism.

The simplicial resolution of § 2 is the special case of this one, and its filtration $F_1 \subset F_2 \subset F_3$ is obtained from our present filtration $\{\tilde{F}_i\}$ by renumbering its terms: $F_1 = \tilde{F}_3 = \tilde{F}_4$, $F_2 = \tilde{F}_5$, $F_3 = \tilde{F}_6$.

3.2 Reduced conical resolutions of discriminants.

There is another, more economical conical resolution of $\Sigma_{d,n}$, also coinciding with the previous one in the case $d = 2 = n$. Namely, we can reduce all "nongeometrical" vertices $L \in \bar{\Omega}_i \backslash \Omega_i$ and their subordinate simplices.

DEFINITION. Given a point $L \in \bar{\Omega}_i$, its *geometrization* $\bar{L}$ is the smallest plane of the form $L(K) \in \Omega_j$, $j \leq i$, containing L. In particular $\bar{L} = L$ if $L \in \Omega_i$.

EXAMPLE. Suppose that $d \geq 3$, $u \in \mathbf{CP}^n$, and $v(t)$, $t \in \mathbf{C}^1$, is an one-parametric analytic family of points in $\mathbf{CP}^n$ such that $v(t) = u \Leftrightarrow t = 0$. Then for any $t \neq 0$ the 2-point set $K(t) \equiv (u, v(t))$ defines a subspace $L(t) \equiv L(K(t))$ of codimension $2(n+1)$ in $\Pi_{d,n}$, while the limit position $L(0)$ of these subspaces when $t \to 0$ belongs to $\bar{\Omega}_{2(n+1)} \backslash \Omega_{2(n+1)}$. Then $\overline{L(0)} = L(\{u\}) \in \Omega_{n+1}$.

The segment in $\tilde{\Lambda}_{d,n}$, joining these two points $L(0)$ and $L(\{u\})$, is homotopy equivalent to its endpoint $L(\{u\})$, therefore we can contract it to this point, and to extend this retraction by linearity to all coherent simplices, containing this segment. Doing the same in all similar situations, we reduce the complex $\tilde{\Lambda}_{d,n}$ to the (properly topologized) union $\Lambda_{d,n}$ of all coherent simplices, whose vertices correspond to the points of spaces Ω_i only, but not of $\bar{\Omega}_i \backslash \Omega_i$. Then we lift this reducing map to the space $\tilde{\sigma}_{d,n}$ and get the desired space $\sigma_{d,n}$, with which we will work all the time. This general reduction map is constructed as follows.

Given a point $L \in \Omega_j$, denote by $W(L)$ the union of all coherent simplices in $\tilde{\Lambda}_{d,n}$ such that for all their vertices L' their geometrizations $\overline{L'}$ coincide with L. Obviously $W(L)$ is a compact and contractible subspace of $\tilde{\Lambda}_{d,n}$: it is a cone with vertex $\{L\}$.

The *reducing map*

$$red : \tilde{\Lambda}_{d,n} \to \Lambda_{d,n} \tag{3}$$

is defined as follows. For any i and any $L \in \bar{\Omega}_i$ we map the point L to $\bar{L}$ and extend this map by linearity to all coherent simplices; the image of a coherent simplex under this map obviously is again coherent. Then we call *equivalent* all points of $\tilde{\Lambda}_{d,n}$ sent by this extended map into one and the same point. The space $\Lambda_{d,n}$ is defined as the quotient space of $\tilde{\Lambda}_{d,n}$ by this equivalence relation.

This factorization map defines obviously a map $\tilde{\Lambda}_{d,n} \times \Pi_{d,n} \to \Lambda_{d,n} \times \Pi_{d,n}$, and hence also a map

$$Red : \tilde{\sigma}_{d,n} \to \sigma_{d,n} \tag{4}$$

of the resolved discriminant $\tilde{\sigma}_{d,n} \subset \tilde{\Lambda}_{d,n} \times \Pi_{d,n}$ onto some space $\sigma_{d,n} \subset \Lambda_{d,n} \times \Pi_{d,n}$.

PROPOSITION 5. *The map (4) induces a homotopy equivalence of one-point compactifications of spaces $\tilde{\sigma}_{d,n}$ and $\sigma_{d,n}$ and factorizes the map $\pi : \tilde{\sigma}_{d,n} \to \Sigma$ (i.e., there is a proper map $\tilde{\pi} : \sigma_{d,n} \to \Sigma$ such that $\pi = \tilde{\pi} \circ Red$).*

This follows immediately from the construction. $\square$

For any $L \in \Omega_i$ denote by $\Lambda_{d,n}(L)$ the image of $\tilde{\Lambda}_{d,n}(L)$ under the reducing map (3).

EXAMPLE. If K is a set of $r < \infty$ points, and d is sufficiently large with respect to r, then both spaces $\tilde{\Lambda}_{d,n}(L(K))$ and $\Lambda_{d,n}(L(K))$ are naturally homeomorphic to a $(r-1)$-dimensional simplex, and the restriction of the map *red* onto $\tilde{\Lambda}_{d,n}(K(L))$ is a homeomorphism between them.

DEFINITIONS AND NOTATION. For any set L we denote by $\partial\Lambda_{d,n}(L)$ the *link* of the order complex $\Lambda_{d,n}(L)$, i.e. the union of all its coherent simplices not containing its vertex $\{L\}$, and denote by $\overset{\circ}{\Lambda}_{d,n}(L)$ the *open cone* $\Lambda_{d,n}(L) \setminus \partial\Lambda_{d,n}(L)$. Their homology groups are obviously related by the boundary isomorphism

$$\bar{H}_*(\overset{\circ}{\Lambda}_{d,n}(L)) \cong H_*(\Lambda_{d,n}(L), \partial\Lambda_{d,n}(L)) \xrightarrow{\sim} H_{*-1}(\partial\Lambda_{d,n}(L), pt). \tag{5}$$

For any subset $K \subset \mathbf{CP}^n$ the complex $\Lambda_{d,n}(L(K))$ will be usually denoted simply by $\Lambda_{d,n}(K)$.

3.3 Filtrations and spectral sequences.

The space $\tilde{\Lambda}_{d,n}$ admits a natural increasing filtration $\tilde{\Phi}_1 \subset \cdots \subset \tilde{\Phi}_D \equiv \tilde{\Lambda}_{d,n}$. Namely, the filtration of any interior point of a coherent simplex $\Delta \subset \tilde{\Lambda}_{d,n}$ is equal to the codimension i of the plane $L(\Delta) \in \bar{\Omega}_i$ corresponding to the leading vertex of this simplex. Hence, a similar filtration is induced also on the quotient space $\Lambda_{d,n}$: the filtration of the equivalence class $x \in \Lambda_{d,n}$ is equal to the smallest value of filtrations of points in $\tilde{\Lambda}_{d,n}$, constituting x. The obvious projections $\tilde{\sigma}_{d,n} \to \tilde{\Lambda}_{d,n}$, $\sigma_{d,n} \to \Lambda_{d,n}$ induce also filtrations on spaces $\tilde{\sigma}_{d,n}$ and $\sigma_{d,n}$. By construction, the term $\tilde{F}_i \setminus \tilde{F}_{i-1}$ of this filtration on $\tilde{\sigma}_{d,n}$ (respectively, on $\sigma_{d,n}$) is the space of a $(D-i)$-dimensional complex vector bundle over the term $\tilde{\Phi}_i \setminus \tilde{\Phi}_{i-1}$ of the filtration on $\tilde{\Lambda}_{d,n}$ (respectively, on $\Lambda_{d,n}$).

In fact, we will use a shortened version of this filtration, removing all nonincreasing terms by the following inductive rule: its term F_1 is the first nonempty set $\tilde{F}_j$, and its term F_i is the first term $\tilde{F}_j$ of the former filtration which is *strictly* greater than F_{i-1}.

This filtration defines a spectral sequence, converging to the right-hand group in (1). Its term $E^1_{p,q}$ is isomorphic to $\bar{H}_{p+q}(F_p \setminus F_{p-1})$.

4 Non-singular forms of degree 3

THEOREM 1. *Only the following groups $H^i(\Pi_{3,2} \setminus \Sigma_{3,2}, \mathbf{R})$ are nontrivial: $H^0 \simeq H^1 \simeq H^3 \simeq H^4 \simeq H^5 \simeq H^6 \simeq H^8 \simeq H^9 \simeq \mathbf{R}$. In particular, only the following groups $H^i(N_{3,2}, \mathbf{R})$ are nontrivial: $H^0 \simeq H^3 \simeq H^5 \simeq H^8 \simeq \mathbf{R}$.*

This theorem follows immediately from the following one.

THEOREM 2. *The term E^1 of our spectral sequence (over $\mathbf{R}$), converging to the group $\bar{H}_*(\Sigma_{3,2})$, looks as in fig. 2 (where empty cells denote trivial groups $E^1_{p,q}$). $E^\infty \equiv E^1$, i.e. the unique possible differential $d_1 : E^1_{2,13} \to E^1_{1,13}$ of this sequence is trivial.*

REMARK. Theorem 1 is not surprising: it states in fact that $H^*(\Pi_{3,2} \setminus \Sigma_{3,2}, \mathbf{R}) \simeq H^*(PGL(\mathbf{CP}^2), \mathbf{R})$. There is obvious projection of the space $\Pi_{3,2} \setminus \Sigma_{3,2}$ onto the modular space of elliptic curves. The group $PGL(\mathbf{CP}^2)$ acts transitively on all fibers of this projection, and the stationary groups of this action are finite for all fibers (although they are not the same: they "jump" over the curve with additional symmetry). It seems likely that Theorem 1 could be proved by considering these subgroups and the Leray spectral sequence of this projection.

The proof of Theorem 2 occupies the rest of this section. It uses many times the following notions and facts.

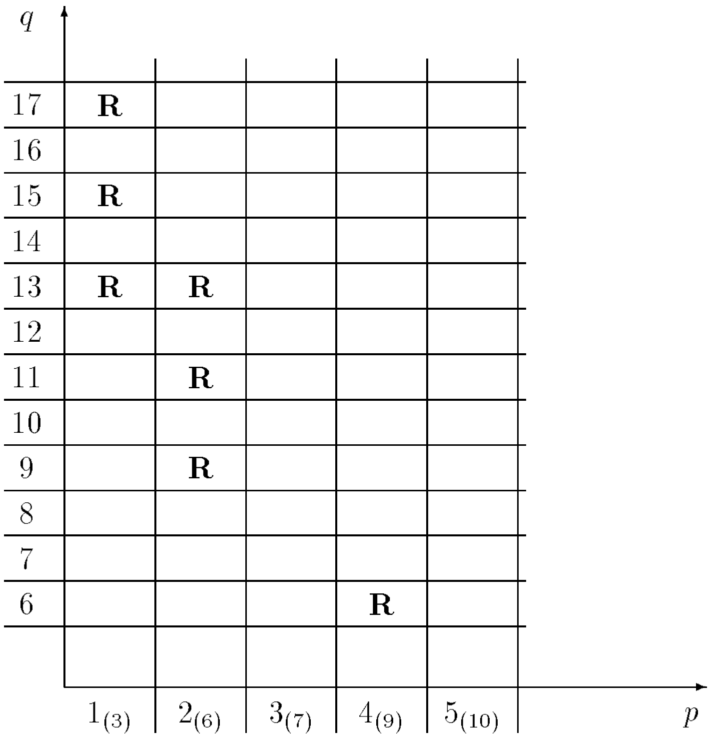


Figure 2: spectral sequence for $d = 3$.

4.1 Main homological lemmas.

DEFINITION. Given a topological space X, the *k-th configuration space $B(X, k)$ of X* is the space of all subsets of cardinality k in X, supplied with the natural topology, see e.g. [12], [14]. The *sign representation* $\pi_1(B(X, k)) \to \text{Aut}(\mathbf{Z})$ maps the paths in $B(X, k)$, defining odd (respectively, even) permutations of k points into multiplication by -1 (respectively, 1). The local system $\pm\mathbf{Z}$ over $B(X, k)$, is the one locally isomorphic to $\mathbf{Z}$, but with monodromy representation equal to the sign representation of $\pi_1(B(X, k))$. The *twisted Borel–Moore homology group* $\bar{H}_*(B(X, k), \pm\mathbf{Z})$ is defined as the homology group of the complex of *locally finite* singular chains of $B(X, k)$ with coefficients in the sheaf $\pm\mathbf{Z}$. Similarly we define local systems $\pm\mathbf{R} \equiv \pm\mathbf{Z} \otimes \mathbf{R}$ and their homology groups. For any topological space X, $\bar{X}$ denotes the one-point compactification of X.

LEMMA 2. *A. The group*

$$\bar{H}_*(B(\mathbf{C}^n, k), \pm\mathbf{R}), \tag{6}$$

$n \geq 1$, *is trivial for any $k \geq 2$.*

B. The group

$$\bar{H}_*(B(\mathbf{CP}^n, k), \pm\mathbf{R}), \tag{7}$$

$n \geq 1$, *is isomorphic to* $H_{*-k(k-1)}(G_k(\mathbf{C}^{n+1}), \mathbf{R})$, *where* $G_k(\mathbf{C}^{n+1})$ *is the Grassmann manifold of* k*-dimensional subspaces in* $\mathbf{C}^{n+1}$, *see* [8].

In particular, the group (7) is trivial if $k > n+1$.

Proof. Assertion A is well-known, see e.g. [12], [14].

Let us fix some complete flag $\bullet \subset \mathbf{CP}^1 \subset \mathbf{CP}^2 \subset \cdots \subset \mathbf{CP}^n$. To any configuration of k points in $\mathbf{CP}^n$ associate its *index* $a = (a_0 \leq a_1 \leq \cdots \leq a_n)$, $a_n = k$, where a_i is the number of its points lying in $\mathbf{CP}^i$. For any such index a denote by $B_a \subset B(\mathbf{CP}^n, k)$ the union of all configurations having this index. It follows immediately from statement A, that $\bar{H}_*(B_a, \pm\mathbf{R}) = 0$ for any index a such that $a_i - a_{i-1} > 1$ for at least one i. Thus only the indices a, which are Schubert symbols of certain Schubert cells of $G_k(\mathbf{C}^{n+1})$, can provide nontrivial groups $\bar{H}_*(B_a, \pm\mathbf{R})$. Moreover, for any Schubert symbol a the set B_a is diffeomorphic to the complex vector space of dimension $\sum_{i=1}^n i \cdot (a_i - a_{i-1})$. This implies statement B.

Definition. For any finite-dimensional topological space X, its k-th *self-join* X^{*k} is defined as follows: we embed X generically into the space $\mathbf{R}^N$ of a very large dimension and take the union of all $(k-1)$-dimensional simplices spanning all k-tuples of points in X. (The genericity of the embedding means that these simplices have only obvious intersections, namely, the intersection set of any two simplices is some their common face.)

Lemma 3. If $k \geq 2$, then $H_*((S^2)^{*k}, \mathbf{R}) \equiv 0$ in all dimensions.

Proof. The space $(S^2)^{*k}$ is obviously filtered by similar self-joins:

$$S^1 \equiv (S^2)^{*1} \subset (S^2)^{*2} \subset \cdots \subset (S^2)^{*(k-1)} \subset (S^2)^{*k}. \tag{8}$$

Consider the spectral sequence generated by this filtration. Any space $\phi_i \backslash \phi_{i-1} \equiv (S^2)^{*i} \backslash (S^2)^{*(i-1)}$ of this filtration is fibered over $B(S^2, i)$ with the fiber $\overset{\circ}{\Delta}{}^{i-1}$. This bundle is nontrivial, its fibers change their orientations exactly over the same loops, which act nontrivially on the local system $\pm\mathbf{R}$. Hence the first term of this sequence is given by the formula

$$\mathcal{E}^1_{p,q} \simeq \bar{H}_{p+q}((S^2)^{*p} \backslash (S^2)^{*(p-1)}, \mathbf{R}) \simeq \bar{H}_{q+1}(B(\mathbf{CP}^1, p), \pm\mathbf{R}).$$

By Lemma 2 there are exactly two such nontrivial cells $\mathcal{E}^1_{p,q}$, namely $\mathcal{E}^1_{2,1} \simeq \mathcal{E}^1_{1,1} \simeq \mathbf{R}$. The unique differential d^1 of our spectral sequence, which can be nontrivial, acts between these cells; it is exactly the boundary homomorphism $H_3((S^2)^{*2}, S^2; \mathbf{R}) \to H_2(S^2, \mathbf{R})$; It follows immediately from the shape of the generator of the former group (i.e., from that of the 2-dimensional Schubert cell B_a, $a = (1, 2)$ in $B(S^2, 2)$) that this differential an isomorphism. This implies our Lemma.

Denote by $\tilde{B}(\mathbf{CP}^2, k)$ the subset in $B(\mathbf{CP}^2, k)$, consisting of *generic* configurations, i.e. of such that none three points do lie in the same line.

Lemma 4. *There are isomorphisms*

$$\bar{H}_*(B(\mathbf{CP}^2, 3), \pm\mathbf{R}) \xrightarrow{\simeq} \bar{H}_*(\tilde{B}(\mathbf{CP}^2, 3), \pm\mathbf{R}), \tag{9}$$

$$\bar{H}_*(B(\mathbf{CP}^2,4),\pm\mathbf{R}) \xrightarrow{\sim} \bar{H}_*(\tilde{B}(\mathbf{CP}^2,4),\pm\mathbf{R}), \qquad (10)$$

induced by identical embeddings. Namely, both groups (10) are trivial in all dimensions, and groups (9) are trivial in all dimensions other than 6 and are isomorphic to $\mathbf{R}$ *in dimension 6.*

Proof. The space $B(\mathbf{CP}^2,3)\setminus\tilde{B}(\mathbf{CP}^2,3)$ consists of 3-configurations lying on the same line. This space is obviously fibered over $\mathbf{CP}^{2\vee}$ with fiber equal to $B(\mathbf{CP}^1,3)$. Applying Lemma 2 B) to this fiber, we get the assertion about the groups (9). Similarly, the space $B(\mathbf{CP}^2,4)\setminus\tilde{B}(\mathbf{CP}^2,4)$ is the union of two pieces, consisting of 4-configurations, some three (respectively all four) points of which lie in the same line. Both these pieces are fibered over $\mathbf{CP}^{2\vee}$ with fibers $B(\mathbf{CP}^1,3)\times\mathbf{C}^2$ (respectively, $B(\mathbf{CP}^1,4)$). By Lemma 2 B) both these pieces are $\pm\mathbf{R}$-acyclic, and Lemma 4 is proved.

4.2 Proof of Theorem 2.

There are exactly the following possible singular sets $K\subset\mathbf{CP}^2$ defined by homogeneous polynomials $\mathbf{C}^3\to\mathbf{C}^1$ of degree 3:

A) any point $u\in\mathbf{CP}^2$;

B) any pair of points in $\mathbf{CP}^2$; the corresponding space $L(K)$ consists of all 3-forms splitting into the product of two polynomials of degree 1 and 2, vanishing at these two points;

C) any line in $\mathbf{CP}^2$: the corresponding space $L(K)$ consists of 3-forms vanishing on this line with multiplicity ≥ 2;

D) any triple of points in $\mathbf{CP}^2$ not lying on the same line, i.e. defining a point of the space $\tilde{B}(\mathbf{CP}^2,3)$. The space $L(K)$ in this case is 1-dimensional and consists of forms vanishing on any of three lines containing any two of our three points;

E) the entire $\mathbf{CP}^2$.

The corresponding terms $F_i\setminus F_{i-1}$ of our filtration and the corresponding terms of the spectral sequence look as follows. Any of them is the space of a fibered product of two fiber bundles, whose base is the corresponding space of all possible sets K (i.e., respectively, $\mathbf{CP}^2$, $B(\mathbf{CP}^2,2)$, $\mathbf{CP}^{2\vee}$, $\tilde{B}(\mathbf{CP}^2,3)$ and $\{pt\}$). The fiber of the first (respectively, second) bundle over any point $\{K\}$ of such a base is the space $L(K)$ (respectively, the open cone $\overset{\circ}{\Lambda}_{3,2}(K)$). The spaces of only second fiber bundles are exactly the terms $\Phi_i\setminus\Phi_{i-1}$ of the related filtration of the continuous order complex $\Lambda_{3,2}$ of all possible sets K defined by homogeneous polynomials of order 3.

Let us describe explicitly all these terms and bundles.

A) The space $F_1\equiv\tilde{F}_3$ is the space of a fiber bundle with fiber $\mathbf{C}^7$ over $\mathbf{CP}^2$, this implies statement of Theorem 2 concerning the column $p=1$[1].

[1]the fibers of the second bundle in this case are the points

B) The space $F_2\backslash F_1 \equiv \tilde{F}_6\backslash\tilde{F}_5$. The fiber of the second bundle over a point $(u,v)\in B(\mathbf{CP}^2,2)$ is the union of two segments, joining this point (u,v) with both points u and $v\in\mathbf{CP}^2$, and not containing their boundary points u and v (which belong to F_1). Obviously such an union is homeomorphic to the open segment $(-1,1)$. Our bundle of such segments is not oriented: it changes the orientation simultaneously with the sheaf $\pm\mathbf{Z}$. Therefore the term $E^1_{2,q}$ of our spectral sequence is given by the formula

$$E^1_{2,q}\cong \bar{H}_{2+q}(F_2\backslash F_1,\mathbf{R})\simeq \bar{H}_{2+q-8}(\Phi_2\backslash\Phi_1,\mathbf{R})\simeq \bar{H}_{2+q-9}(B(\mathbf{CP}^2,2),\pm\mathbf{R}). \qquad (11)$$

Thus the assertion of Theorem 2 concerning the column $E^1_{2,q}$ of the main spectral sequence follows from Lemma 2.

C) Let $K=\mathbf{CP}^1\subset\mathbf{CP}^2$. Then the reduced order complex $\Lambda_{3,2}(K)$ is homeomorphic to the cone with vertex $\{L(K)\}\subset\Omega_7$, and base lying in Φ_2. By the construction, this base $\partial\Lambda_{3,2}(K)$ is canonically homeomorphic to the self-join $(\mathbf{CP}^1)^{*2}$, whose homology groups are trivial by Lemma 3.

The space $F_3\backslash F_2$ is fibered over $\mathbf{CP}^{2\vee}$; its fiber over a line $\{K\}\in\mathbf{CP}^{2\vee}$ is the product of the space $\mathbf{C}^3$ and the space $\Lambda_{3,2}(K)\backslash\Phi_2$, i.e. an open cone with base $\partial\Lambda_{3,2}(K)$. Thus the Borel–Moore homology group of this fiber is acyclic over $\mathbf{R}$, this implies assertion of Theorem 2 concerning the column $p=3$.

D) $\Phi_4\backslash\Phi_3$ is fibered over $\tilde{B}(\mathbf{CP}^2,3)$; its fiber over a point $(u,v,w)\subset\mathbf{CP}^2$ is the open triangle $\dot{\Delta}^2$, whose vertices are identified with these points u,v and w. In particular,

$$\bar{H}_i(F_4\backslash F_3)\equiv\bar{H}_{i-2}(\Phi_4\backslash\Phi_3)\simeq\bar{H}_{i-4}(\tilde{B}(\mathbf{CP}^2,3),\pm\mathbf{Z}). \qquad (12)$$

By Lemmas 4 and 2 B), the group $H_j(\tilde{B}(\mathbf{CP}^2,3),\pm\mathbf{R})$ is isomorphic to $\mathbf{R}$ if $j=6$ and is trivial for all $j\neq 6$. This proves the assertion of Theorem 2 concerning the column $p=4$.

E) Finally, the set $F_5\backslash F_4$ coincides with $\Phi_5\backslash\Phi_4\equiv\Lambda_{3,2}\backslash\Phi_4$; it is an open cone over the reduced topological order complex Φ_4, so that $\bar{H}_i(F_5\backslash F_4)\simeq\tilde{H}_{i-1}(\Phi_4)$. Consider the spectral sequence $e^a_{b,c}\to H_*(\Phi_4,\mathbf{R})$, generated by the filtration $\{\Phi_i\}$, $i=1,2,3,4$. Its term $e^1_{i,c}$ is isomorphic to $\bar{H}_{i+c}(\Phi_i\backslash\Phi_{i-1},\mathbf{R})$. The homology groups of spaces $(\Phi_i\backslash\Phi_{i-1})$ are related by the Thom isomorphism with these of corresponding spaces $F_i\backslash F_{i-1}$ (calculated in the previous steps A — D), hence the term e^1 of our sequence looks as is shown in fig. 3.

LEMMA 5. *The differentials $\partial^1: e^1_{2,1}\to e^1_{1,1}$ and $\partial^1: e^1_{2,3}\to e^1_{1,3}$ of our spectral sequence are isomorphisms, as well as the map $\partial^2: e^2_{4,4}\to e^2_{2,5}$. In particular, the term $e^\infty\equiv e^3$ of our spectral sequence consists of unique nontrivial cell $e^3_{1,-1}\simeq\mathbf{R}$.*

Proof. If some of these maps is not isomorphic, then there exists a nontrivial group $H_{i-1}(\Phi_4,\mathbf{R})\simeq\bar{H}_i(F_5\backslash F_4,\mathbf{R})$ with $i<9$. It follows from the shape of terms $E^1_{p,q}$ of the main spectral sequence with $p\le 4$ (determined in the previous steps) that the corresponding group $E^1_{5,i-5}$ cannot disappear in the further steps of this

q				
8				
7				
6				
5		$\mathbf{R}$		
4				$\mathbf{R}$
3	$\mathbf{R}$	$\mathbf{R}$		
2				
1	$\mathbf{R}$	$\mathbf{R}$		
0				
-1	$\mathbf{R}$			
	1	2	3	4 p

Figure 3: spectral sequence for the term $F_5 \setminus F_4$.

sequence and thus the group $\bar{H}_i(\Sigma_{3,2}) \simeq H^{19-i}(\Pi_{3,2} \backslash \Sigma_{3,2})$ also is nontrivial. This is impossible because $\Pi_{3,2} \backslash \Sigma_{3,2}$ is a 10-dimensional Stein manifold. $\Box$

Thus we have proved the assertion of Theorem 2 concerning the shape of the term E^1 of the main spectral sequence.

Lemma 6. *The homomorphism $d^1 : E^1_{2,13} \to E^1_{1,13}$ is trivial.*

Proof. It is clear from fig. 2, that the group $E_{1,15}$ cannot disappear, hence $\bar{H}_{16}(\Sigma_{3,2}, \mathbf{R}) \simeq H^3(\Pi_{3,2} \backslash \Sigma_{3,2}, \mathbf{R}) \sim \mathbf{R}$. $\Pi_{3,2} \backslash \Sigma_{3,2}$ is a direct product of something and $\mathbf{C}^*$, therefore the Poincaré polynomial of $H^*(\Pi_{3,2} \backslash \Sigma_{3,2}, \mathbf{R})$ is divisible by $(1+t)$. Since $\bar{H}_{17}(\Sigma_{3,2}, \mathbf{R}) \simeq H^2(\Pi_{3,2} \backslash \Sigma_{3,2}, \mathbf{R}) = 0$, this implies that $\bar{H}_{15}(\Sigma_{3,2}, \mathbf{R}) \neq 0$. But the unique cell $E^1_{p,q}$ with $p + q = 15$ is $E^1_{2,13}$, and our lemma is proved.

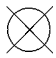
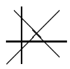
It terminates the proof of Theorems 2 and 1.

5 Non-singular quartics in $\mathbf{CP}^2$

Theorem 3. *The Poincaré polynomial of the real cohomology group of the space $N_{4,2}$ of nonsingular curves of degree 4 in $\mathbf{CP}^2$ is equal to $(1 + t^3)(1 + t^5)(1 + t^6)$.*

For the proof, consider the standard conical resolution of $\Sigma_{4,2}$ described in § 3. To describe it explicitly, we must classify all singular sets in $\mathbf{CP}^2$ of polynomials $\mathbf{C}^3 \to \mathbf{C}^1$ of degree 4.

Proposition 6. *There are exactly the following possible singular sets in $\mathbf{CP}^2$ of homogeneous polynomials of degree 4 in $\mathbf{C}^3$ (in angular parentheses we indicate* *the complex dimension of the space of all polynomials having the singularity at an arbitrary fixed set of the corresponding class):*

1. *Any point in* $\mathbf{CP}^2$ $\langle 12\rangle$
2. *Any pair of points* $\langle 9\rangle$
3. *Any three point on the same line* $\langle 7\rangle$ 
4. *Any generic triple of points (i.e. not lying on the same line)* $\langle 6\rangle$
5. *Any line* $\mathbf{CP}^1 \subset \mathbf{CP}^2$ $\langle 6\rangle$
6. *Any three points on the same line plus a point not on the line* $\langle 4\rangle$ 
7. *Any generic quadruple of points (i.e., none three of them are on the same line; the corresponding polynomial necessarily splits into the product of two quadrics)* $\langle 3\rangle$
8. *Any line* $\mathbf{CP}^1 \subset \mathbf{CP}^2$ *plus a point outside it* $\langle 3\rangle$
9. *Any five points, four of which are generic, and the fifth is the intersection point of two lines spanning some pairs of first four points (the corresponding polynomial splits into the product of two linear functions and one quadric)* $\langle 2\rangle$ 
10. *Six points, which are all the intersection points of some four lines in general position* $\langle 1\rangle$ 
11. *Any nonsingular quadric in* $\mathbf{CP}^2$ $\langle 1\rangle$
12. *Two lines in* $\mathbf{CP}^2$ $\langle 1\rangle$
13. *Entire* $\mathbf{CP}^2$*.* $\langle 0\rangle$ □

I have liberty to improve slightly the canonical filtration in the resolution space $\sigma_{4,2}$ of the discriminant and in the underlying order complex $\Lambda_{4,2}$ of singular sets. Namely, for any $i = 1, \ldots, 13$, I denote by F_i (respectively, by Φ_i) the union of all sets $\sigma_{4,2}(K) \subset \sigma_{4,2}$ (respectively, $\Lambda_{4,2}(K) \subset \Lambda_{4,2}$) where K is one of sets described in items $1, \ldots, i$ of the previous proposition.

Proposition 7. *The term* E^1 *of the spectral sequence, generated by this filtration and converging to the group* $\bar{H}_*(\sigma_{4,2}, \mathbf{R})$, *is as shown in Fig. 4. This spectral sequence degenerates at this term:* $E^\infty \equiv E^1$.

Proof. The shape of columns $p = \mathbf{1}, \mathbf{2}$ and $\mathbf{4}$ of E^1 is justified in essentially the same way as that of the same columns of the spectral sequence presented in Fig. 2.

q \ p	1	2	3	4	5	6	7	8	9	10	11	12	13
27	**R**												
26													
25	**R**												
24													
23	**R**	**R**											
22													
21		**R**											
20													
19		**R**											
18													
17													
16				**R**									
15													
14													
13										**R**			
12													
11													
10										**R**			
9													**R**
8										**R**			
7													
6													**R**
5										**R**			
4													**R**
3													
2													
1													**R**

Figure 4: Spectral sequence for the space of nonsingular homogeneous polynomials of degree 4 in $\mathbf{C}^3$

The triviality of columns $p = \mathbf{3}$ and $p = \mathbf{6}$ follows from Lemma 2 ($n = 1$, $k = 3$), and that of the column $p = \mathbf{7}$ from the same lemma and additionally Lemma 4. For any set K of type **5** (i.e., $K = \mathbf{CP}^1$) the link $\partial(\Lambda_{4,2}(K))$ of the corresponding order complex is the space $(\mathbf{CP}^1)^{*3}$, thus the triviality of the column $p = 5$ follows from Lemma 3. Similarly, for any set K of type **11** (i.e., K is a generic quadric in $\mathbf{CP}^2$) the link $\partial(\Lambda_{4,2}(K))$ is the space K^{*4}, and the triviality of the column $p = 11$ also follows from Lemma 3.

The space of configurations of type **9** is fibered over $B(\mathbf{CP}^{2\vee}, 2)$ with fiber $B(\mathbf{C}^1, 2) \times B(\mathbf{C}^1, 2)$, and everything dyes already over this fiber. Thus the column $p = 9$ also is trivial.

In the case mentioned in the column $p = \mathbf{10}$ the term $F_{10} \setminus F_9$ is the space of a fiber bundle, whose base is the configuration space $\tilde{B}(\mathbf{CP}^{2\vee}, 4)$ of all generic collections of four lines in $\mathbf{CP}^2$ (i.e., such that none three of them intersect at the same point), and the fiber is the direct product of a complex line and a 5-dimensional open simplex, whose 6 vertices correspond to all intersection points of these four lines. It is easy to check that this bundle of 5-simplices is *orientable*, thus $\bar{H}_*(F_{10} \setminus F_9, \mathbf{R}) \simeq \bar{H}_{*-7}(\tilde{B}(\mathbf{CP}^{2\vee}, 4), \mathbf{R})$. Let us calculate the latter homology group. By the projective duality, $\tilde{B}(\mathbf{CP}^{2\vee}, 4)$ is diffeomorphic to the space $\tilde{B}(\mathbf{CP}^2, 4)$ of all generic collections of 4 points, see e.g. Lemma 4. The similar space $\tilde{\mathcal{F}}(\mathbf{CP}^2, 4)$ of *ordered* generic collections of points is obviously diffeomorphic to the group $PGL(\mathbf{CP}^2)$, whose real homology groups are well-known. It is easy to check that these homology groups do not disappear when we pass to the base of the 24-fold covering $\tilde{\mathcal{F}}(\mathbf{CP}^2, 4) \xrightarrow{S(4)} \tilde{B}(\mathbf{CP}^2, 4)$, and the shape of the column $p = 10$ follows.

The triviality of the column $p = \mathbf{8}$ follows from the following lemma.

LEMMA 7. *For any set $K \subset \mathbf{CP}^2$ of type 8, $K = \mathbf{CP}^1 \sqcup *$, the link $\partial\Lambda_{4,2}(K)$ of the corresponding order complex $\Lambda_{4,2}(K)$ is homeomorphic to the suspension $\Sigma(\partial\Lambda_{4,2}(\mathbf{CP}^1))$ of the similar link for the set $\mathbf{CP}^1$ of type 5.*

Proof. This link consists of two pieces, the first of which is the order complex $\Lambda_{4,2}(\mathbf{CP}^1)$ (obviously homeomorphic to the cone over its own link), and the second is the union of all order complexes $\Lambda_{4,2}(\kappa)$, where κ is some subset of type 6 in K. All the latter order complexes are the tetrahedra, one of whose 4 vertices is the distinguished point $* = K \setminus \mathbf{CP}^1$, and another three are arbitrary different points of $\mathbf{CP}^1$. Thus the second piece also is homeomorphic to the cone over $\partial\Lambda_{4,2} \sim (S^2)^{*3}$. The intersection of these two cones is their common base, and Lemma 7 is proved.

Now let K be of type **12**, i.e. the union of two complex lines $l_1, l_2 \subset \mathbf{CP}^2$. The corresponding link $\partial\Lambda_{4,2}(K)$ is covered by two diffeomorphic sets, A_1 and A_2. Namely, A_i, $i = 1, 2$, is the union of all continuous order complexes $\Lambda_{4,2}(\kappa)$, where κ is the set of type 8 consisting of the line l_i and a point of the other line $K \setminus l_i$. The triviality of the column $p = 12$ follows now from the Mayer–Vietoris formula and the following lemma.

LEMMA 8. *All three complexes A_1, A_2 and $A_1 \cap A_2$ are acyclic (over $\mathbf{R}$) in all* *positive dimensions.*

Proof. Consider the following filtration of the set A_1:

$$\Lambda_{4,2}(l_1) \subset A_1 \cap \Phi_6 \subset A_1 \cap \Phi_8 \equiv A_1.$$

The set $A_1 \backslash (A_1 \cap \Phi_6)$ is a fiber bundle, whose base is the space $(K \backslash l_1) \sim \mathbf{C}^1$, parametrizing all possible sets of type 8 containing l_1, and fiber $\overset{\circ}{\Lambda}_{4,2}(\kappa)$, where κ is such a set. By Lemma 7 the Borel–Moore homology group of this difference (or, which is the same, the relative group $H_*(A_1, (A_1 \cap \Phi_6); \mathbf{R})$) is trivial.

The space $(A_1 \cap \Phi_6) \backslash \Lambda_{4,2}(l_1)$ also is a fiber bundle with base $(K \backslash l_1) \sim \mathbf{C}^1$; its fiber is the space $\partial \Lambda_{4,2}(l_1) \sim (S^2)^{*3}$. By Lemma 3 the Borel–Moore homology group of this fiber also is trivial. Thus $H_*(A_1)$ coincides with the homology group of the space $\Lambda_{4,2}(l_1)$, which is contractible, and acyclicity of A_1 is completely proved. The proof for A_2 is exactly the same.

Finally, $A_1 \cap A_2$ is the continuous order complex of following sets:

a) any point of K;

b) any pair of points in K lying in different lines (one of which can coincide with the intersection point $l_1 \cap l_2$);

c) any triple of points in K, the first of which is the crossing point $l_1 \cap l_2$, and two other lie in different lines l_1, l_2.

The homology groups of this complex can be easily calculated and coincide with the homology groups of a point. Lemma 8 is completely proved and the column $p = 12$ actually is empty.

As in the proof of Theorem 2, the ultimate column $p = 13$ is calculated by a spectral sequence, whose twelve columns repeat these of the main sequence, with coordinates q decreased by twice the number shown in angular parentheses in Proposition 6. The fact that its lower terms kill one another is proved in exactly the same way as it was done in the proof of Theorem 2, see Lemma 5. (The complement of the discriminant is a Stein manifold and cannot have too low-dimensional homology groups).

Finally, the fact that the term E^1, shown in Fig. 4, coincides with the stable term E^∞, also is proved similarly to the similar statement (Lemma 6) in the proof of Theorem 2. Say, the cell $E_{2,23}$ cannot disappear, because it should match the cell $E_{1,26}$, which, in its turn, cannot disappear by dimensional reasons. Cell $E_{13,9}$ cannot disappear, because it should match one of cells $E_{2,21}$ or $E_{10,13}$, etc. $\square$

6 Non-singular cubical surfaces in $\mathbf{CP}^3$

THEOREM 4. *The Poincaré polynomial of the real cohomology group of the space $N_{3,3}$ of nonsingular cubical hypersurfaces in $\mathbf{CP}^3$ is equal to $(1+t^3)(1+t^5)(1+t^7)$. In particular, again $H^*(N_{3,3}, \mathbf{R}) \simeq H^*(PGL(\mathbf{CP}^3), \mathbf{R})$.*

Proposition 8. *There are exactly the following possible singular sets in* $\mathbf{CP}^3$ *of homogeneous cubical polynomials in* $\mathbf{C}^4$*:*

1. *Any point in* $\mathbf{CP}^3$ $\langle 16 \rangle$
2. *Any pair of points* $\langle 12 \rangle$
3. *Any line* $\langle 10 \rangle$
4. *Any triple of points in general position (i.e. not on the same line)* $\langle 8 \rangle$
5. *Any generic quadric inside any plane* $\mathbf{CP}^2 \subset \mathbf{CP}^3$ $\langle 5 \rangle$
6. *Any pair of intersecting lines in* $\mathbf{CP}^3$ $\langle 5 \rangle$
7. *Any generic quadruple of points (i.e. not lying in the same two-dimensional plane)* $\langle 4 \rangle$
8. *Any plane* $\mathbf{CP}^2 \subset \mathbf{CP}^3$ $\langle 4 \rangle$
9. *Any triple of lines having one common point (such a polynomial is equal to* $F = xyz$ *in appropriate coordinates)* $\langle 1 \rangle$
10. *Any generic plane quadric (as in item 5) plus any point not in the same plane* $\langle 1 \rangle$
11. *Entire* $\mathbf{CP}^3$. $\langle 0 \rangle$

Exactly as in the proof of Theorem 3, for any $i = 1, \dots, 11$, denote by F_i the union of all sets $\sigma_{3,3}(K) \subset \sigma_{3,3}$ where K is one of sets described in items $1, \dots, i$ of the previous proposition.

Proposition 9. *The term* E^1 *of the spectral sequence, generated by this filtration and converging to the group* $\bar{H}_*(\sigma_{3,3}, \mathbf{R})$*, is as shown in Fig. 5. This spectral sequence degenerates at this term:* $E^\infty \equiv E^1$*.*

Proof. The shape of columns $p = \mathbf{1}$ and $\mathbf{2}$ follows immediately from Lemma 2B. The justification of columns $p = \mathbf{4}$ and $\mathbf{7}$ needs additionally the following lemma.

Denote by $\tilde{B}(\mathbf{CP}^3, k)$ the subset in $B(\mathbf{CP}^3, k)$, consisting of *generic* configurations, i.e. of such that none their three points lie on the same line and none four lie on the same 2-plane.

Lemma 9. *There are isomorphisms*

$$\bar{H}_*(B(\mathbf{CP}^3, 3), \pm\mathbf{R}) \xrightarrow{\sim} \bar{H}_*(\tilde{B}(\mathbf{CP}^3, 3), \pm\mathbf{R}), \tag{13}$$

$$\bar{H}_*(B(\mathbf{CP}^3, 4), \pm\mathbf{R}) \xrightarrow{\sim} \bar{H}_*(\tilde{B}(\mathbf{CP}^3, 4), \pm\mathbf{R}), \tag{14}$$

induced by identical embeddings. Namely, both groups (13) are isomorphic to $\mathbf{R}$ *in dimensions 6,8,10 and 12, and are trivial in all other dimensions, and groups (14)*

q												
37	$\mathbf{R}$											
36												
35	$\mathbf{R}$											
34												
33	$\mathbf{R}$	$\mathbf{R}$										
32												
31	$\mathbf{R}$	$\mathbf{R}$										
30												
29		$\mathbf{R}^2$										
28												
27		$\mathbf{R}$										
26				$\mathbf{R}$								
25		$\mathbf{R}$										
24				$\mathbf{R}$								
23												
22				$\mathbf{R}$								
21												
20				$\mathbf{R}$								
19												
18												
17												
16							$\mathbf{R}$					
	1	2	3	4	5	6	7	8	9	10	11	p

Figure 5: Spectral sequence for nonsingular cubical polynomials in $\mathbf{C}^4$

are trivial in all dimensions other than 12 and are isomorphic to $\mathbf{R}$ *in dimension 12.*

Proof. The isomorphism (13) follows from exactly the same arguments as (9), only with space $\mathbf{CP}^{2\vee}$ of all lines in $\mathbf{CP}^2$ replaced by the space $G_2(\mathbf{C}^4)$ of lines in $\mathbf{CP}^3$.

The space $B(\mathbf{CP}^3,4)\backslash\tilde{B}(\mathbf{CP}^3,4)$ is the union of two pieces, consisting of 4-configurations, whose projective span is a 2-plane (respectively, a line) in $\mathbf{CP}^3$. By Lemmas 2 B) and 4 the $\pm\mathbf{R}$-homology groups of both these pieces are trivial, and Lemma 9 is proved.

For K of type **3**, $K=\mathbf{CP}^1\subset\mathbf{CP}^2$, $\partial\Lambda_{3,3}(K)\simeq(S^2)^{*2}$, therefore the shape of the column $p=\mathbf{3}$ follows from Lemma 3.

Similarly, for K of type **5** $\partial\Lambda_{3,3}(K)\simeq(S^2)^{*3}$, and the shape of the column $p=5$ follows from the same lemma.

For K of type **6**, $K=l_1\cup l_2$, the link $\partial\Lambda_{3,3}(K)$ is covered by three subsets: $A_1=\Lambda_{3,3}(l_1)$, $A_2=\Lambda_{3,3}(l_2)$, and A_3 swept out by all order complexes $\Lambda_{3,3}(\kappa)$, where κ are 3-point sets, one whose point is the crossing point $l_1\cap l_2$, and two other lie in different lines l_1 and l_2. The order complexes A_1 and A_2 are contractible, and their intersection consists of one point, thus $H_*(A_1\cup A_2)\simeq H_*(pt)$. Further, $A_3\backslash(A_1\cup A_2)$ is a fiber bundle, whose base is the space $\mathbf{C}^1\times\mathbf{C}^1\equiv(K\backslash l_1)\times(K\backslash l_2)$ parametrizing all above-described sets κ, and the fiber over such a set is the triangle with two sides removed. Thus the Borel–Moore homology group of such a fiber is trivial, and by the Künneth formula

$$\bar{H}_*(\partial\Lambda_{3,3}(K)\backslash(A_1\cup A_2))\equiv H_*(\partial\Lambda_{3,3}(K),(A_1\cup A_2))\equiv 0.$$

This proves the triviality of the column $p=6$.

Similarly, if K is of type **9**, $K=l_1\cap l_2\cap l_3$, then the link $\partial\Lambda_{3,3}(K)$ is covered by four sets A_1,A_2,A_3,A_4, where A_i, $i=1,2,$and 3, are the order complexes $\Lambda_{3,3}(\mu_i)$, μ_i are unions of some two of three lines constituting K. A_4 is swept out by all order complexes $\Lambda_{3,3}(\kappa)$, where κ are 4-point sets, one whose point is the crucial point of K, and three other lie in different lines l_1,l_2 and l_3. $A_4\backslash(A_1\cup A_2\cup A_3)$ is a fiber bundle, whose base is the space $\mathbf{C}^1\times\mathbf{C}^1\times\mathbf{C}^1$ parametrizing all above-described sets κ, and the fiber over such a set is the tetrahedron with three maximal faces removed. Thus the Borel–Moore homology group of such a fiber is trivial, and by the Künneth formula

$$\bar{H}_*(\partial\Lambda_{3,3}(K)\backslash(A_1\cup A_2\cup A_3))\equiv H_*(\partial\Lambda_{3,3}(K),(A_1\cup A_2\cup A_3))\equiv 0.$$

Further, the order complexes A_1, A_2 and A_3 are acyclic, their pairwise intersections are the order complexes $\Lambda_{3,3}(l_j)$, which also are acyclic, and their total intersection is a point. This proves the triviality of the column $p=9$.

For K of type **10**, (i.e. consisting of a generic plane quadric $\tilde{K}$ and one point not in the same plane), the link $\partial\Lambda_{3,3}(K)$ is homeomorphic to the suspension $\Sigma(\partial\Lambda_{3,3}(\tilde{K}))$, cf. Lemma 7. This proves the triviality of the column $p=10$.

Column $p = \mathbf{8}$. Let $K = \mathbf{CP}^2 \subset \mathbf{CP}^3$. The link $\partial\Lambda_{3,3}(K)$ is filtered by its intersections with terms Φ_i of the filtration of $\Lambda_{3,3}$. We need only the following segment of this filtration:

$$\partial\Lambda_{3,3}(K) \cap \Phi_4 \subset \partial\Lambda_{3,3}(K) \cap \Phi_5 \subset \partial\Lambda_{3,3}(K) \cap \Phi_6 \equiv \partial\Lambda_{3,3}(K). \qquad (15)$$

The first term of this filtration, $\partial\Lambda_{3,3}(K) \cap \Phi_4$, is acyclic by Lemma 5. The set $(\partial\Lambda_{3,3}(K) \cap \Phi_5) \setminus (\partial\Lambda_{3,3}(K) \cap \Phi_4)$ is the space of a fiber bundle, whose base is the space of all generic quadrics in $\mathbf{CP}^2$ (studied in § 2), and the fiber over such a quadric κ is the open cone $\overset{\circ}{\Lambda}_{3,3}(\kappa)$. The Borel–Moore homology group of this fiber coincides (up to a shift of dimensions) with the (reduced modulo a point) homology group of the corresponding link $\partial\Lambda_{3,3}(\kappa) \sim (S^2)^{*3}$, which is trivial by Lemma 3. Finally, the set $(\partial\Lambda_{3,3}(K) \cap \Phi_6) \setminus (\partial\Lambda_{3,3}(K) \cap \Phi_5)$ is the space of a fiber bundle, whose base is the space $B(\mathbf{CP}^{2\vee}, 2)$ of pairs of complex lines in $\mathbf{CP}^2$, and the fiber over such a pair κ is the open cone $\overset{\circ}{\Lambda}_{3,3}(\kappa)$. By the boundary isomorphism (5), to find the Borel–Moore homology of such a fiber, we need only to calculate the (usual) homology of the link $\partial\Lambda_{3,3}(\kappa)$, which already was calculated (see the study of the column $p = 6$) and is trivial.

This proves the triviality of the column $p = \mathbf{8}$ and terminates the justification of columns 1 through 10 of Fig. 5. The triviality of the column $p = \mathbf{11}$ and the degeneration of the spectral sequence at the term E^1 follow from exactly the same reasons as in the proofs of previous theorems: the former follows from the fact that $\Pi_{3,3} \setminus \Sigma_{3,3}$ is a Stein manifold, and the latter from the fact that it is divisible by $\mathbf{C}^*$ (and from the explicit shape of the term E^1).

Proposition 9 and Theorem 4 are completely proved.

7 Non-degenerate quadratic vector fields in $\mathbf{C}^3$

DEFINITION. A collection of three homogeneous functions $(v_1, v_2, v_3) : \mathbf{C}^3 \to \mathbf{C}^3$ is nondegenerate, if they vanish simultaneously only at the origin of the issue space.

THEOREM 5. *The Poincaré polynomial of the rational cohomology group of the space of nondegenerate triples of homogeneous functions of degrees 2 in $\mathbf{C}^3$ is equal to $(1+t)(1+t^3)(1+t^5)$.*

Proof. Denote by $\Sigma(2,2,2)$ the discriminant subset in the space $\mathbf{C}^{18}$ of all triples of quadratic functions in $\mathbf{C}^3$, i.e. the set of all degenerate triples. The conical resolution of this subset is constructed in exactly the same way as similar resolutions of spaces $\Sigma_{d,n}$ described in § 3; instead of possible singular subsets in $\mathbf{CP}^n$ we use possible sets of common zeros of a triple of polynomials.

PROPOSITION 10. *There are exactly the following possible sets in $\mathbf{CP}^2$ of common zeros of triples of homogeneous polynomials of degree 2 in $\mathbf{C}^3$:*

1. *Any point in $\mathbf{CP}^2$* $\langle 15 \rangle$

q									
33	$\mathbf{R}$								
32									
31	$\mathbf{R}$								
30									
29	$\mathbf{R}$	$\mathbf{R}$							
28									
27		$\mathbf{R}$							
26									
25		$\mathbf{R}$							
24									
23			$\mathbf{R}$						
	1	2	3	4	5	6	7	8	9 $\quad p$

Figure 6: Spectral sequence for nondegenerate quadratic vector fields in $\mathbf{C}^3$

2. Any pair of points $\langle 12 \rangle$

3. Any generic triple of points (i.e. not lying on the same line) $\langle 9 \rangle$

4. Any line $\mathbf{CP}^1 \subset \mathbf{CP}^2$ $\langle 9 \rangle$

5. Any generic quadruple of points $\langle 6 \rangle$

6. Any line $\mathbf{CP}^1 \subset \mathbf{CP}^2$ *plus a point outside it* $\langle 6 \rangle$

7. Any nonsingular quadric in $\mathbf{CP}^2$ $\langle 3 \rangle$

8. Two lines in $\mathbf{CP}^2$ $\langle 3 \rangle$

9. Entire $\mathbf{CP}^2$. $\langle 0 \rangle$ $\square$

PROPOSITION 11. *The term E^1 of the spectral sequence, generated by the corresponding filtration and converging to the group $\bar{H}_*(\Sigma_(2,2,2)\mathbf{R})$, is as shown in Fig. 6. This spectral sequence degenerates at this term: $E^\infty \equiv E^1$.*

The proof of this proposition repeats essentially some fragments of the proof of Proposition 7 and will be omitted. The comparison of spectral sequences of Figs. 2 and 6 implies the following version of the (homological) Smale–Hirsch principle.

PROPOSITION 12. *The gradient embedding $\mathbf{C}^{10} \to \mathbf{C}^{18}$, mapping any homogeneous polynomial $\mathbf{C}^3 \to \mathbf{C}^1$ of degree 3 to the collection of its three partial derivatives,* *induces the isomorphism between rational cohomology groups of spaces of nonsingular cubical polynomials and nondegenerate quadratic vector fields in* $\mathbf{C}^3$. $\square$

8 Problems and concluding remarks

1. To explain the 6-dimensional generator of $H^*(N_{4,2})$.

2. To find general properties of the above calculations. Of course, these calculations (and the number of strata to be considered) grow at least exponentially together with n and d. Note however that all the infinite strata have contributed nothing in our spectral sequences. Is it so also for greater n and d?

Is $H^*(N_{3,n}, \mathbf{R}) \simeq H^*(PGL(\mathbf{CP}^n), \mathbf{R})$ for any n?

3. To make all the above calculations for the cohomology groups with coefficients in $\mathbf{Z}$ or at least $\mathbf{Z}_2$. The same for the real version of the problem, i.e. for spectral sequences calculating the cohomology of spaces of nonsingular algebraic hypersurfaces in $\mathbf{RP}^n$.

The involution of complex conjugation acts naturally on our spectral sequences, thus they are a good tool to check the M-property for such spaces or some strata of the discriminant.

However, the most useful for the original problem of the rigid homotopy classification should be the similar calculation of $\mathbf{R}$-valued cohomology groups of spaces of real nonsingular objects. Note that V. Kharlamov [7] considered the topology of discriminant sets in a connection with the rigid classification problem.

PROBLEM: to express the standard invariants of rigid isotopy classification of projective curves and surfaces in terms of dual homology classes of discriminants.

4. A collection of points in $\mathbf{RP}^2$ or in $\mathbf{CP}^2$ is called *d-sufficient* (respectively, *d-degenerate*), if any homogeneous polynomial of degree d with singular points in it vanishes (respectively, has singular points) at entire curve of smaller degree passing through some of them. The simplest example: any $[d/2]+1$ (respectively, d) points, lying on the same line, form a d-sufficient (respectively, d-degenerate) collection.

PROBLEM. To study the set of such configurations. For which smallest d there are *minimal* d-degenerate configurations such that the corresponding curve of singularities contains not all their points? For which i the spaces of all i-point d-nondegenerate configurations are disconnected (both in real and complex versions) and how many components do they have?

To describe the class of algebraic subsets in $\mathbf{RP}^2$ or $\mathbf{CP}^2$ which satisfy the following *d-overdeterminacy* property: any homogeneous polynomial of degree d, having singular points at this subset, is identically equal to 0.

More generally, let us call a point $y \in \mathbf{P}^2$ a *d-consequence* of points $x_1, \ldots, x_l$, if any homogeneous polynomial of degree d, having singularities at these l points, has also a singularity at the point y. E.g., if $(x_1, \ldots, x_l)$ is a d-degenerate (respectively,

d-overdetermined) set, then all points of the corresponding curve (respectively, all points at all) are its d-consequences. Are there other examples?

Consider some natural family of algebraic subsets in $\mathbf{P}^2$, e.g. the set of all collections, consisting of several curves of fixed degrees and several points not on these curves. Any point of this family defines a subspace in Π_d: the space of polynomials having singularities at all points of the corresponding algebraic set. Is it possible to describe easily the set of points of our family, for which the dimension of this subspace "jumps"?

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