SINGULARITIES OF CONVEX HULLS OF THREE-DIMENSIONAL HYPERSURFACES *

I. A. Bogaevski

Abstract

We investigate singularities of the convex hull of a generic compact smooth hypersurface in four-dimensional affine space up to diffeomorphisms. It turns out there are only two new singularities (in comparison with the previous dimension case) which appear at separate points of the boundary of the convex hull and are not removed by small perturbation of the original hypersurface. The first singularity does not contain functional, but has at least nine number moduli. A normal form which does not contain moduli at all is found for the second singularity.

Introduction

The *convex hull* of a compact subset of affine space is the intersection of all closed half-spaces which contain the subset. The boundary of the convex hull of a compact smooth hypersurface can have singularities. For example, the singularities of the boundary of the convex hull of a generic closed plane curve are discontinuities of the second derivative (Fig. 1).

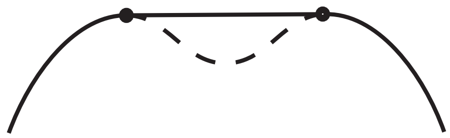

Figure 1: Singularities of the convex hull of a generic plane curve

In the present paper we investigate up to diffeomorphisms singularities of the convex hulls of generic compact C^∞-smooth hypersurfaces (without boundary) embedded into four-dimensional affine space. A *singularity* of a convex hull is called its germ at a singular point of the boundary. As usual, *generic* hypersurfaces are embeddings which form an open everywhere dense set in the C^∞-space of all considered embeddings. In other words, we are interested only in singularities which are not removed by any C^∞-small perturbation of the original hypersurface.

A tangent hyperplane is called *support* if the hypersurface lies entirely in one of two closed half-spaces defined by the hyperplane. A support hyperplane is called *non-singular* if it has with the hypersurface only one common point which is a point of non-degenerate tangency. All rest support planes are called *singular*. For example, the singular support hyperplanes to a generic plane curve are straight lines of double non-degenerate tangency. Singularities of the convex hull of a compact hypersurface can appear only in its singular support hyperplanes.

Three-dimensional space. The convex hull of a generic compact surface in three-dimensional space can have only two kinds of singularities which are called *simplest* and *angle* in our terminology. The normal form of the angle singularity contains a number *modulus* (continuous invariant) with respect to diffeomorphisms. These singularities shown in Fig. 2 are found in [1]. The results of this paper are the following.

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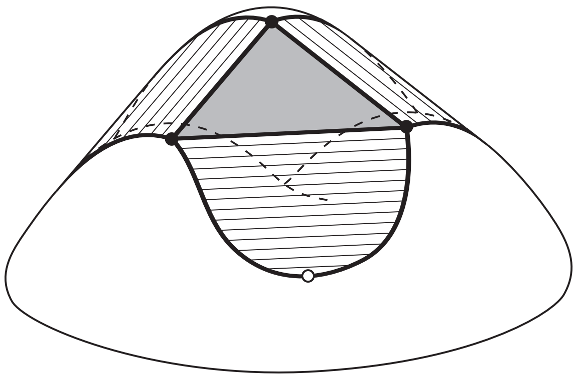

Figure 2: Singularities of the convex hull of a generic surface

Typical singular support hyperplane to a generic surface in three-dimensional space has two (and only two) points of non-degenerate tangency with the surface ($2A_1$-*plane*). Separate singular support hyperplanes can have either three points of non-degenerate tangency with the surface which form a triangle or one point of degenerate tangency A_3. Such hyperplanes are called $3A_1$- and A_3-*planes* respectively.

The segment between the points of tangency of a surface and its support $2A_1$-plane is called *support*. It lies entirely in the boundary of the convex hull of the original surface. The boundary is smooth at the interior points of the support segment. In neighborhoods of the endpoints of the support segment the boundary is diffeomorphic to the product of a line and a curve with the singularity shown in Fig. 1. Such singularities of a convex hull are called *simplest* and denoted by $\mathcal{R}_1$.

The triangle with vertices at the points of tangency of a surface and its support $3A_1$-plane is called *support*. In its neighborhood the boundary of the convex hull of the surface consists of the support triangle itself, three smooth surfaces webbed from support segments and adjoining the sides of the support triangle, and three parts of the original surface adjoining the vertices of the support triangle. The convex hull has the simplest singularities at the interior points of the sides of the support triangle. To describe singularities at its vertices let us define the under-graph as the set of points lying under the graph of a given function. It turns out that in a neighborhood of each vertex of the support triangle the convex hull is diffeomorphic to the under-graph of the square of the distance to an angle of a value β where $0 < \beta < \pi$ is a unique modulus in the normal form. Such singularities of a convex hull are called *angle*. Let us note that we can consider the epigraph of the square of the distance instead of its under-graph – they become diffeomorphic after the substitution $\beta \mapsto \pi - \beta$.

At last, the convex hull of a generic surface has the simplest singularity at any *support* A_3-point that is a point of tangency of the surface and its support A_3-plane. In a neighborhood of a such point the boundary of the convex hull consists of a part of the original surface and a surface webbed from support segments which degenerate into the support A_3-point.

Four-dimensional space. Some singularities of the convex hull of a generic hypersurface in four-dimensional space are investigated in [2] and [3]. In [2] normal forms of singularities of convex hulls are found in the case when the support hyperplane is tangent to the original hypersurface in one of the ways described above for three-dimensional space. Thus the simplest and angle singularities appear in four-dimensional space as well.

The boundary of the convex hull is smooth at the interior points of the support segments of the $2A_1$-planes and has the simplest singularities at their endpoints. The support segments themselves lie entirely in the boundary of the convex hull of the original hypersurface.

At the interior points of the support triangles of the $3A_1$-planes the boundary of the convex hull is smooth and has the simplest singularities at the interior points of their sides. The support triangles themselves lie entirely in the boundary of the convex hull of the original hypersurface. In a neighborhood of each their vertex the convex hull is diffeomorphic to the curvilinear cylinder over the above angle singularity with $\beta(z) = \beta_0 + z$, $\beta(z) = \beta_0 + z^2$, or $\beta(z) = \beta_0 - z^2$ where $0 < \beta_0 < \pi$ is a unique modulus in each of the three normal forms and z is a coordinate along the element of the cylinder. Such singularities of a convex hull are called *angle* and denoted by $\mathcal{R}_2^0$, $\mathcal{R}_2^+$, and $\mathcal{R}_2^-$ respectively.

At last, the convex hull of a generic hypersurface has the simplest singularity at any support A_3-point again.

However, among the support hyperplanes of a three-dimensional hypersurface there can be new for us $4A_1$- and A_1A_3-*planes* which are not removed by any small perturbation of the hypersurface. Each $4A_1$-plane has four points of non-degenerate tangency with the hypersurface which form a tetrahedron. Each A_1A_3-plane has one point of non-degenerate tangency and one point of tangency A_3.

The tetrahedron with vertices at the points of tangency of a hypersurface and its support $4A_1$-plane is called *support*. In its neighborhood the boundary of the convex hull of the hypersurface consists of the support tetrahedron itself, four smooth strata webbed from support triangles and adjoining the faces of the support tetrahedron, six smooth strata webbed from support segments and adjoining the edges of the support tetrahedron, and four parts of the original hypersurface adjoining the vertices of the support tetrahedron. Thus in this case the singular points of the boundary divide it into 15 $(1+4+6+4)$ strata as shown in Fig. 3, left.

To imagine a convex hull in four-dimensional space it is convenient to project it affinely into the support hyperplane to the original hypersurface. Then the boundary of the convex hull is locally the graph of a continuously differentiable function (see, e. g., [1]) whose typical singularities are discontinuities of the second derivative. In a neighborhood of a support tetrahedron of a generic three-dimensional hypersurface the points of such discontinuities form 28 smooth strata which are shown in Fig. 3, left.

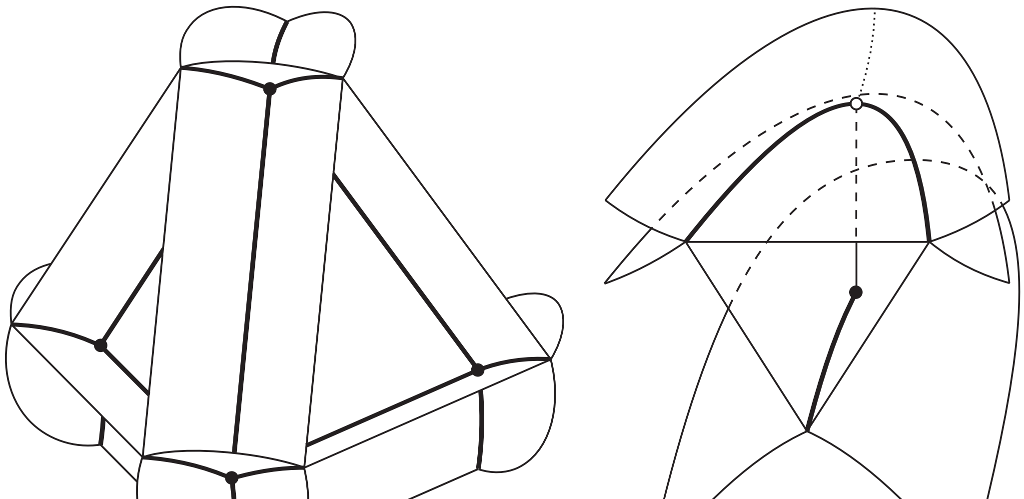

Figure 3: Singularities of the convex hull of a generic three-dimensional hypersurface in a neighborhood of a support tetrahedron and in a neighborhood of a support A_1A_3-segment

According to [2], the convex hull of a generic three-dimensional hypersurface has the simplest singularities at the interior points of the faces of a support tetrahedron and the angle singularities at the interior points of its edges. In the present paper it is proved that the germ of our convex hull at each vertex of a support tetrahedron does not contain functional moduli with respect to diffeomorphisms. This singularity is denoted by $\mathcal{R}_3$. Its normal form is not found as well as the precise quantity of number moduli. It is only proved certainly not to exceed nine but apparently is much more.

Besides, we investigate singularities in the only rest case when a generic hypersurface has a support A_1A_3-plane. The segment between the points of their tangency is called the *support A_1A_3-segment*. It lies entirely in the boundary of the convex hull of the original hypersurface. According to our results, the convex hull of a generic three-dimensional hypersurface has the simplest singularities at the interior points of its support A_1A_3-segment and the angle singularity $\mathcal{R}_2^0$ at the endpoint A_3. At the endpoint A_1 of the support A_1A_3-segment there appears one more singularity of the convex hull. This singularity is denoted by $\mathcal{V}_3$, does not contain functional moduli, and is diffeomorphic to its normal form. This normal form is the under-graph of the square of the distance to that component of the complement to the swallowtail which consists of polynomials without real roots.

In a neighborhood of a support A_1A_3-segment the boundary of the convex hull of a generic hypersurface is divided by the singular points into five strata four of which are smooth and the fifth one is non-smooth – this phenomenon has not occur before. The projection of the singular points of the boundary of the convex hull into the support A_1A_3-plane is shown in Fig. 3, right. It consists of three smooth surfaces, the cut swallowtail, and the half of the Whitney umbrella. The support A_1A_3-segment lies in the Whitney umbrella, its endpoint A_3 is left by a smooth curve which consists of support A_3-points and is shown by dots in Fig. 3. It should be noted that in general the normalizing diffeomorphisms preserve neither the support A_1A_3-segment, nor the curve from support A_3-points.

Inside of the cut swallowtail the boundary of the convex hull is the original hypersurface. A smooth stratum webbed from support triangles which degenerate into the support A_1A_3-segment is inside of the Whitney umbrella. The stratum between these two ones is non-smooth and webbed from support segments. One more part of the original hypersurface bounded by two smooth surfaces adjoins the non-singular stratum. The rest smooth stratum is bounded by two surfaces too and is webbed from support segments degenerating into support A_3-points.

The union of the cut swallowtail and the half of the Whitney umbrella whose intersection lines coincide and whose tangent cones are transversal is called a *sail-boat* in [3]. There it is proved that the projection of the singular points of the boundary of the convex hull into a support A_1A_3-plane to a generic hypersurface is a sail-boat in a neighborhood of the point of tangency A_1 and all sail-boats are diffeomorphic to each other. According to our results, in this case the convex hull itself is reduced to the above local normal form which does not contain moduli as well.

The following conjecture is formulated in [4]: the singularities of the convex hull of a generic compact hypersurface in four-dimensional affine space do not contain functional moduli. Thus this conjecture is proved in the present paper.

So in a neighborhood of a typical point of the boundary the convex hull of a generic compact three-dimensional hypersurface is diffeomorphic to the closed half-space. The points of the boundary where the convex hull has the simplest singularities $\mathcal{R}_1$ form a smooth two-dimensional surface. At last, the angle singularities $\mathcal{R}_2^0$ appear along smooth curves, the singularities $\mathcal{R}_2^+$, $\mathcal{R}_2^-$, $\mathcal{R}_3$, and $\mathcal{V}_3$ appear at separate points of the boundary of the convex hull.

The singularities $\mathcal{R}_1$, $\mathcal{R}_2^0$, $\mathcal{R}_2^\pm$, and $\mathcal{V}_3$ allow the following uniform representations: the simplest singularity $\mathcal{R}_1$ is diffeomorphic to the under-graph of the square of the distance to a three-dimensional half-space, the angle singularity $\mathcal{R}_2^0$ or $\mathcal{R}_2^\pm$ is diffeomorphic to the under-graph of the square of the distance to a dihedral angle of the variable value $\beta(z)$ where $\beta(z) = \beta_0 + z$ or $\beta(z) = \beta_0 \pm z^2$ respectively, and the singularity $\mathcal{V}_3$ is diffeomorphic to the under-graph of the square of the distance to one of the components of the complement to the swallowtail. For the singularity $\mathcal{R}_3$ there is apparently no analogous representation.

Higher dimensions. According to [5], since five-dimensional space the convex hull can have functional moduli which are not removed by small perturbation of the original hypersurface. For example, they appear as relations between several number moduli (which are already in four-dimensional space) along the lines formed by the vertices of the support tetrahedrons.

Terminology. The term "smooth" always means "infinitely smooth". The term "generic" is used for smooth mappings and means that the given proposition is only true for some open everywhere dense set in the C^∞-space of all considered mappings. The term "typical" is used for points of varieties and means that the given proposition is only true for some open everywhere dense set of points.

Distribution of material. In Section 1 we give the necessary definitions and rigorously formulate the results (Theorems 0 and 1) outlined in Introduction. In Section 2 the proof of these results is divided into three steps which are Theorems 2, 3, and 4 proved in Section 3.

Theorem 2 from Section 2 states that the boundary of the convex hull of a generic compact three-dimensional hypersurface is the front of a Legendre variety which can have only the following standard singularities: $\widetilde{R}_1$, $\widetilde{R}_2$, $\widetilde{R}_3$, and $\widetilde{V}_3$. These singularities do not have moduli and their normal forms with respect to contact diffeomorphisms are found in Section 2. The singularities $\widetilde{R}_1$ form smooth two-dimensional surfaces, the singularities $\widetilde{R}_2$ appear along smooth curves, and the singularities $\widetilde{R}_3$ and $\widetilde{V}_3$ appear at separate points of the Legendre variety.

Theorem 3 from Section 2 contains the classification of germs of a generic Legendre bundle at points of the Legendre variety up to contact diffeomorphisms preserving its local normal forms $\widetilde{R}_1$, $\widetilde{R}_2$, $\widetilde{R}_3$, and $\widetilde{V}_3$ found already. However, we consider only bundles with the following property: the front of the Legendre variety must be a continuously differentiable manifold. (The boundary of the convex hull of any compact hypersurface is continuously differentiable.) In the proof of Theorem 3 we use the techniques developed in [6].

At last, Theorem 4 from Section 2 calculates the fronts of the Legendre mappings from Theorem 3. They are the required normal forms of convex hulls: the singularities $\widetilde{R}_1$ of the Legendre variety are projected at the simplest singularities of the convex hull, the singularities $\widetilde{R}_2$ are projected at the angle ones, the singularities $\widetilde{R}_3$ are projected at the singularities $\mathcal{R}_3$, and the singularities $\widetilde{V}_3$ are projected at the singularities $\mathcal{V}_3$.

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1 Classification of Singularities of Convex Hulls

Convex Hulls and Support Simplexes

Definition. The *convex hull* of a compact subset of affine space is the intersection of all closed half-spaces which contain the subset.

Definition. A hyperplane is called *support* to a subset of affine space if the subset lies entirely in one of two half-spaces defined by the hyperplane and has at least one common point with the hyperplane.

If the subset is a manifold then so-called $A_{l_1}\dots A_{l_m}$-planes are distinguished among its support hyperplanes.

Definition. A support hyperplane to a manifold is called a *$A_{l_1}\dots A_{l_m}$-plane* (l_1, ..., l_m are positive odd numbers) if:

1) it has m points of tangency A_{l_1}, ..., A_{l_m} with the manifold;

2) these points of tangency are the vertices of a $(m-1)$-dimensional simplex which is called *a support simplex of the $A_{l_1}\dots A_{l_m}$-plane* or *a support $A_{l_1}\dots A_{l_m}$-simplex.*

A support A_1-plane is called *non-singular*. The rest support hyperplanes are called *singular*.

Remark. If $l_1 = \ldots = l_m = 1$ then we write mA_1 instead of $A_1 \dots A_1$.

Remark. A k-dimensional manifold has a point of tangency A_l (l is positive integer) with a hypersurface if the restriction of some local equation of the hypersurface has the form $\pm\xi_1^{l+1} \pm \xi_2^2 \pm \ldots \pm \xi_k^2$ in proper local coordinates on the manifold.

According to [1], if $k = 1$ or $n \leq 7$ then the boundary of the convex hull of a generic compact k-dimensional manifold in n-dimensional affine space consists of support $A_{l_1} \ldots A_{l_m}$-simplexes where $l_1 + \ldots + l_m \leq n$. In particular, in the considered case $n = 4$ it consists of support A_1-points, support $2A_1$-segments, support $3A_1$-triangles, support $4A_1$-tetrahedrons, support A_3-points, and support A_1A_3-segments.

Hierarchy of Singularities of Convex Hulls

Simplest singularity $\mathcal{R}_1$. A germ of a convex hull in four-dimensional space has the *singularity* $\mathcal{R}_1$ if it is diffeomorphic to the germ (at the origin) of the set

$$\mathcal{R}_1 = \left\{ (x, y, z, t) \in \mathbb{R}^3 \times \mathbb{R} \ : \ \min_{p \geq 0} \left(\frac{p^2}{2} + px + t \right) \leq 0 \right\}.$$

Remark. The set $\mathcal{R}_1$ is the under-graph of the function of x, y, and z which is equal to half of the square of the standard distance on straight line from the point x to the ray $x \geq 0$.

Angle singularities $\mathcal{R}_2^0$ and $\mathcal{R}_2^{\pm}$. A germ of a convex hull in four-dimensional space has the *singularity* $\mathcal{R}_2^0$, $\mathcal{R}_2^+$, or $\mathcal{R}_2^-$ if it is diffeomorphic to the germ (at the origin) of the set $\mathcal{R}_2(\alpha)$ where $\alpha(z) = a + z$, $\alpha(z) = a + z^2$, or $\alpha(z) = a - z^2$ respectively, $|a| < 1$, and

$$\mathcal{R}_2(\alpha) = \left\{ (x, y, z, t) \in \mathbb{R}^3 \times \mathbb{R} \ : \ \min_{p,q \geq 0} \left(\frac{p^2 + q^2}{2} + \alpha(z)pq + px + qy + t \right) \leq 0 \right\}.$$

Remark. The set $\mathcal{R}_2(\alpha)$ is the under-graph of the function of x, y, and z which is equal to half of the square of the distance from the point (x, y) to the coordinate angle $\{x \geq 0, y \geq 0\}$ in plane with the Euclidean metric

$$ds^2 = \frac{dx^2 - 2\alpha(z)dxdy + dy^2}{1 - \alpha^2(z)}.$$

The value of the angle is $\beta(z) = \pi - \arccos \alpha(z)$.

Singularity $\mathcal{R}_3$. A germ of a convex hull in four-dimensional space has the *singularity* $\mathcal{R}_3$ if it is diffeomorphic to the germ (at the origin) of the set

$$\mathcal{R}_3(F) = \left\{ (x, y, z, t) \in \mathbb{R}^3 \times \mathbb{R} \ : \ \min_{p,q,r \geq 0} F(p, q, r; x, y, z, t) \leq 0 \right\}$$

where F is a polynomial whose quasihomogeneous expansion has the form $F = F_2 + F_3 + \ldots$ if $\deg p = \deg q = \deg r = 1$, $\deg x = \deg y = \deg z = 1$, and $\deg t = 2$,

$$F_2(p, q, r; x, y, z, t) = \frac{p^2 + q^2 + r^2}{2} + apq + bpr + cqr + px + qy + rz + t,$$

and the quadratic form $(p^2 + q^2 + r^2)/2 + apq + bpr + cqr$ is positive definite.

For example, the set $\mathcal{R}_3(F_2)$ is the under-graph of half of the square op the distance from the point (x, y, z) to the coordinate angle $\{x \geq 0, y \geq 0, z \geq 0\}$ in space with the Euclidean metric defined by the matrix

$$\left\| \begin{matrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{matrix} \right\|^{-1}.$$

REMARK. The numbers a, b, and c are moduli of the singularity $\mathcal{R}_3$ with respect to diffeomorphisms. These moduli are described in [4]. Moreover, Theorems 3 and 4 imply that six moduli are among the coefficients of the quasihomogeneous component F_3 of the polynomial F.

Singularity $\mathcal{V}_3$. A germ of a convex hull in four-dimensional space has the *singularity* $\mathcal{V}_3$ if it is diffeomorphic to the germ (at the origin) of the under-graph $\mathcal{V}_3$ of half of the square of the standard distance from the point (x, y, z) to the body

$$V_3 = \left\{(x, y, z) \in \mathbb{R}^3 \ : \ \forall\, \tau \in \mathbb{R} \quad \tau^4 + x\tau^2 + y\tau + x^2/4 + z \geq 0\right\}$$

which is bounded by the cut swallowtail and consists of the non-negative reduced polynomials of degree four.

Adjacencies of singularities. The singularities of convex hulls adjoin each other in the following way:

$$\begin{array}{ccccccc} & & & & \mathcal{R}_2^{\pm} & & \\ & & & & \downarrow & & \\ \mathcal{R}_0 & \leftarrow & \mathcal{R}_1 & \leftarrow & \mathcal{R}_2^0 & \leftarrow & \mathcal{R}_3 \\ & & & & \uparrow & & \\ & & & & \mathcal{V}_3 & & \end{array}$$

where $\mathcal{R}_0$ denotes a germ of a closed half-space at a point of its boundary.

Formulation of Results

The following theorem is proved in [2].

THEOREM 0. *The convex hull of a generic compact hypersurface lying in four-dimensional affine space has the singularities*

$\mathcal{R}_0$ *at the support A_1-points and at the interior points of the support simplexes of the $2A_1$-, $3A_1$-, and $4A_1$-planes;*

$\mathcal{R}_1$ *at the endpoints of the support $2A_1$-segments, at the interior points of the sides of the support $3A_1$-triangles, and at the interior points of the faces of the support $4A_1$-tetrahedrons;*

$\mathcal{R}_2^0$ *at typical vertices of the support $3A_1$-triangles and at typical interior points of the edges of the support $4A_1$-tetrahedrons;*

$\mathcal{R}_2^{\pm}$ *at the rest finite number of vertices of the support $3A_1$-triangles and at the rest finite number of interior points of the edges of the support $4A_1$-tetrahedrons.*

Main Theorem 1 of the present paper completes this classification.

THEOREM 1. *The convex hull of a generic compact hypersurface lying in four-dimensional affine space has the singularities*

$\mathcal{R}_3$ *at the vertices of the support $4A_1$-tetrahedrons;*

$\mathcal{R}_1$ *at the interior points of the support A_1A_3-segments;*

$\mathcal{R}_2^0$ *at the points of tangency A_3 of the support A_1A_3-planes;*

$\mathcal{V}_3$ *at the points of tangency A_1 of the support A_1A_3-planes;*

and the quasidegree of the polynomial F from the definition of the singularity $\mathcal{R}_3$ is bounded by a number $d \geq 3$ which does not depend on the original hypersurface.

Theorems 0 and 1 follow from Theorems 2, 3, and 4, formulated in Section 2 and proved in Section 3 of the present paper.

2 Reducing to Normal Forms

Singularities of Legendre Varieties

Let B be a manifold and $\xi \in B$ be any its points. A *cooriented contact element* to B is any closed linear half-space in the tangent space $T_\xi B$. The point ξ is called the *point of applying* of the contact element.

It is well known that in the space ST^*B of all cooriented contact elements to B there is a natural *contact structure* (hyperplane distribution satisfied the condition of maximal non-integrability): a contact element is allowed to move so that its boundary contains the velocity of its point of applying. Varieties which are tangent to the distribution and whose dimension is the most $(\dim B - 1)$ is called *Legendre.*

The hyperplanes of the contact structure in the space ST^*B is naturally *cooriented* outside: a contact element move in positive direction if it does not contain the velocity of the point of applying.

Later on we realize n-dimensional affine space as an open half-sphere in $n+1$-dimensional Euclidean space and work with all sphere which has a natural projective structure: subspaces passing through the center cut planes of various dimensions in the sphere. Each hyperplane divides the sphere into two half-spaces, and the above definitions of a convex hull and a support hyperplane are suitable for its subsets.

A cooriented contact element to affine space or sphere is naturally identified with the pair consisting of the closed half-space and the point of applying which lies in the boundary of the half-space. Two cooriented contact elements are called *complementary* to each other if they consist of different closed half-spaces having common boundary and the same point of applying.

Definition. A cooriented contact element is called *support* to a subset C of affine space or sphere if it consists of a closed half-space containing C and a point from C. (The point lies in the boundary of the half-space.) A cooriented contact element which is complementary to a support one is called *antisupport.* All support to C elements form a subset $C^\perp$, all antisupport to C elements form a subset $C^\top$.

Definition. A cooriented contact element is called *infinitesimally support* to a subset C of a manifold if it is applied at a point $\xi \in C$ and contains the cone which is tangent to C at the point ξ. A cooriented contact element which is complementary to an infinitesimally support one is called *infinitesimally antisupport.* All infinitesimally antisupport to C elements form a subset $\widetilde{C}$.

Remark. The sets of the infinitesimally support and antisupport (to a subset of a manifold) elements are functorial with respect to diffeomorphisms of the manifold. In general, this is not true for the sets of the support and antisupport elements.

Remark. Let $\xi \in C \subset B$ be a point of a subset C of a manifold B. Let us consider any Riemannian metric on B and some curve beginning at the point ξ and possessing the following property: the distance from a point of the curve to the subset C is an infinitesimal whose degree is more than one if the considered point approaches to ξ. The cone lying in the tangent space $T_\xi B$ and consisting of the rays which are tangent to all such curves, is called *tangent* to the subset C at the point ξ.

Let R_0 denote the hyperplane $s = 0$ in the affine space $\mathbb{R}^3 \times \mathbb{R}$ with coordinates (u, v, w, s). Let us consider the following subsets of R_0:

1) the hyperplane R_0 itself;

2) the half-space $R_1 = \{(u, v, w) \in R_0 \ : \ u \geq 0\}$;

3) the dihedral angle $R_2 = \{(u, v, w) \in R_0 \ : \ u \geq 0,\ v \geq 0\}$;

4) the octant $R_3 = \{(u, v, w) \in R_0 \ : \ u \geq 0,\ v \geq 0,\ w \geq 0\}$;

5) the body $V_3 = \{(u, v, w) \in R_0 \ : \ \forall\, \tau \in \mathbb{R} \quad \tau^4 + u\tau^2 + v\tau + u^2/4 + w \geq 0\}$ bounded by the cut swallowtail and consisting of the non-negative reduced polynomials of degree four.

Let $(p, q, r; u, v, w, s)$ be local coordinates on $ST^*(\mathbb{R}^3 \times \mathbb{R})$ such that a cooriented contact element applied at the point $(u, v, w, s) \in \mathbb{R}^3 \times \mathbb{R}$ has the form $pdu + qdv + rdw + ds \leq 0$, and let $E_0 \subset ST^*(\mathbb{R}^3 \times \mathbb{R})$ denote the space of all such elements. Let $\widetilde{R_0}$, $\widetilde{R_1}$, $\widetilde{R_2}$, $\widetilde{R_3}$, and $\widetilde{V_3} \subset E_0$ be the Legendre varieties consisting of the cooriented contact elements which are infinitesimally antisupport to the subsets R_0, R_1, R_2, R_3, and $V_3 \subset \mathbb{R}^3 \times \mathbb{R}$ respectively. The variety $\widetilde{R_0}$ is smooth, the varieties $\widetilde{R_1}$, $\widetilde{R_2}$, $\widetilde{R_3}$, and $\widetilde{V_3}$ have singularities, for example, at the origin.

Definition. We say that a three-dimensional Legendre variety has the singularity $\widetilde{R_1}$, $\widetilde{R_2}$, $\widetilde{R_3}$, or $\widetilde{V_3}$ at some point if its germ at this point is (up to a local diffeomorphism respecting the contact structures and their coorientations) respectively the germ of the Legendre variety $\widetilde{R_1}$, $\widetilde{R_2}$, $\widetilde{R_3}$, or $\widetilde{V_3} \subset E_0$ at the origin.

These singularities of Legendre varieties adjoin each other in the following way:

$$\begin{array}{ccccccc} \widetilde{R_0} & \leftarrow & \widetilde{R_1} & \leftarrow & \widetilde{R_2} & \leftarrow & \widetilde{R_3} \\ & & & & \uparrow & & \\ & & & & \widetilde{V_3} & & \end{array}$$

where the indices are equal to the codimensions of the strata of a Legendre variety which consist of the corresponding singular points.

Theorem 2. *Let $M \subset S^4$ be a compact smooth hypersurface, $[M] \subset S^4$ be its convex hull, $[M]^\perp \subset ST^*S^4$ be the set of all support to $[M]$ cooriented contact elements, and $\pi : ST^*S^4 \to S^4$ be the natural projection.*

Then $[M]^\perp$ is a Legendre variety which is uniquely projected onto its front $\pi([M]^\perp)$ being the boundary of the convex hull of the original hypersurface M. If the hypersurface M is generic, the Legendre variety $[M]^\perp$ can have only the above singularities $\widetilde{R_1}$, $\widetilde{R_2}$, $\widetilde{R_3}$, and $\widetilde{V_3}$. Moreover:

1) the Legendre variety $[M]^\perp$ is smooth above the support A_1-points and above the interior points of the support simplexes of the $2A_1$-, $3A_1$-, and $4A_1$-planes of the hypersurface M;

2) the singularities $\widetilde{R_1}$ appear above the endpoints of the support $2A_1$-segments, above the interior points of the sides of the support $3A_1$-triangles, above the interior points of the faces of the support $4A_1$-tetrahedrons, and above the interior points of the support A_1A_3-segments of the hypersurface M;

3) the singularities $\widetilde{R_2}$ appear above the vertices of the support $3A_1$-triangles, above the interior points of the edges of the support $4A_1$-tetrahedrons, and above the points of tangency A_3 of the support A_1A_3-planes of the hypersurface M;

4) the singularities $\widetilde{R_3}$ appear above the vertices of the support $4A_1$-tetrahedrons of the hypersurface M;

5) the singularities $\widetilde{V_3}$ appear above the points of tangency A_1 of the support A_1A_3-planes of the hypersurface M.

Normal Forms of Legendre Bundles

A *Legendre* mapping is the diagram $L^{n-1} \hookrightarrow E^{2n-1} \to B^n$ consisting of an embedding of a Legendre variety L^{n-1} into the space E^{2n-1} with a cooriented contact structure and a Legendre bundle $E^{2n-1} \to B^n$. (A smooth bundle whose fibers are Legendre manifolds is called *Legendre*.) *Equivalence* of Legendre mappings is the commutative diagram

$$\begin{array}{ccccc} L & \hookrightarrow & E & \to & B \\ \updownarrow & & \updownarrow & & \updownarrow \\ L' & \hookrightarrow & E' & \to & B' \end{array}$$

where the middle vertical arrow is a diffeomorphism sending the cooriented contact structure to each other.

Legendre bundles are locally given with the help of the generating families defined below.

Let us consider again the space E_0 of the cooriented contact elements $\varkappa d\lambda + ds \le 0$ applied at the points $(\lambda, s) \in \mathbb{R}^3 \times \mathbb{R}$ where $\varkappa = (p, q, r)$, $\lambda = (u, v, w)$. The cooriented contact structure on E_0 is given by the zero subspaces of the form $\varkappa d\lambda + ds$.

Let $F : (\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a germ of a family of smooth functions of $\varkappa$ which depend smoothly on the parameters $\mu = (x, y, z) \in \mathbb{R}^3$ and $t \in \mathbb{R}$. Let $F_\varkappa(0) = 0$ and F satisfy the condition of non-degeneracy at the origin:

$$\det \begin{Vmatrix} F_{\varkappa\mu} & F_{\varkappa t} \\ F_\mu & F_t \end{Vmatrix} \neq 0.$$

Then F is called the *generating family* of the germ

$$\pi : (E_0, 0) \to (\mathbb{R}^3 \times \mathbb{R}, 0), \quad \pi(\varkappa, \lambda, s) = (\mu, t)$$

of the Legendre bundle whose fibers are given by the formula

$$\pi^{-1}(\mu, t) = \{(\varkappa, \lambda, s) \in E_0 : \lambda = F_\varkappa(\varkappa, \mu, t), s = F(\varkappa, \mu, t) - \varkappa\lambda\}.$$

This bundle is correctly defined in a neighborhood of the origin in consequence of non-degeneracy of F.

Thus $\varkappa$ are local coordinates on the fibers of the germ π, (μ, t) are local coordinates on its base, and the cooriented contact structure on E_0 is defined by the form $F_\mu d\mu + F_t dt$.

For example, the natural Legendre bundle $(\varkappa, \lambda, s) \mapsto (\lambda, s)$ is given by the generating family $\varkappa\mu + t = px + qy + rz + t$.

Theorem 3. *Let $L \subset ST^*S^4$ be a Legendre variety with the singularities $\widetilde{R_1}$, $\widetilde{R_2}$, $\widetilde{R_3}$, or $\widetilde{V_3}$ and $\pi : ST^*S^4 \to S^4$ be a generic Legendre bundle. If $\pi(L)$ is a continuously differentiable manifold then the germ of the Legendre mapping $L \hookrightarrow ST^*S^4 \xrightarrow{\pi} S^4$*

1) at each smooth point of the variety L is equivalent to the germ $(\widetilde{R_0}, 0) \hookrightarrow (E_0, 0) \to (\mathbb{R}^3 \times \mathbb{R}, 0)$ of a Legendre mapping such that its second arrow is given by the generating family

$$px + qy + rz + t;$$

2) at each singular point $\widetilde{R_1}$ of the variety L is equivalent to the germ $(\widetilde{R_1}, 0) \hookrightarrow (E_0, 0) \to (\mathbb{R}^3 \times \mathbb{R}, 0)$ of a Legendre mapping such that its second arrow is given by the generating family

$$\frac{p^2}{2} + px + qy + rz + t;$$

3) at each singular point $\widetilde{R_2}$ of the variety L is equivalent to the germ $(\widetilde{R_2}, 0) \hookrightarrow (E_0, 0) \to (\mathbb{R}^3 \times \mathbb{R}, 0)$ of a Legendre mapping such that its second arrow is given by one of the generating families having the form

$$\frac{p^2 + q^2}{2} + \alpha(z)pq + px + qy + rz + t$$

where $\alpha(z) = a + z$ for a typical singular point $\widetilde{R_2}$, $\alpha(z) = a \pm z^2$ for the rest finite number of singular points $\widetilde{R_2}$, and $|a| < 1$ is a continuous invariant;

4) at each singular point $\widetilde{R_3}$ of the variety L is equivalent to the germ $(\widetilde{R_3}, 0) \hookrightarrow (E_0, 0) \to (\mathbb{R}^3 \times \mathbb{R}, 0)$ of a Legendre mapping such that its second arrow is given by a polynomial generating family $F(p, q, r; x, y, z, t)$ whose quasihomogeneous expansion has the form $F = F_2 + \ldots + F_d$ if $\deg p = \deg q = \deg r = 1$, $\deg x = \deg y = \deg z = 1$, and $\deg t = 2$, where

$$F_2 = \frac{p^2 + q^2 + r^2}{2} + apq + bpr + cqr + px + qy + rz + t,$$

a, b, and c are continuous invariants, the quadratic form $(p^2+q^2+r^2)/2+apq+bpr+cqr$ is positive definite, among the coefficients of F_3 there are six continuous invariants, and the number $d \geq 3$ depends on neither L nor π;

5) at each singular point $\widetilde{V_3}$ of the variety L is equivalent to the germ $(\widetilde{V_3},0) \hookrightarrow (E_0,0) \to (\mathbb{R}^3 \times \mathbb{R},0)$ of a Legendre mapping such that its second arrow is given by the generating family

$$\frac{p^2+q^2+r^2}{2}+px+qy+rz+t.$$

REMARK. In the case 5) the generating family can be reduced to the simpler form

$$\frac{q^2+r^2}{2}+px+qy+rz+t.$$

Normal Forms of Fronts

The *front* of a Legendre mapping $L^{n-1} \hookrightarrow E^{2n-1} \to B^n$ is the image of L^{n-1} in B^n. The coorientation of the contact structure on E induces a coorientation of the front that is, in general, a hypersurface with singularities.

THEOREM 4. *The cooriented fronts of the germs of Legendre mappings from the items 1)–5) of Theorem 3 are the germs (at the origin) of the boundaries of the sets $\mathcal{R}_0$, $\mathcal{R}_1$, $\mathcal{R}_2(\alpha)$, $\mathcal{R}_3(F)$, and $\mathcal{V}_3$ which are cooriented in the direction of the axis t.*

3 Proofs

Duality

All closed half-spaces of the sphere S^n form the *dual* sphere $\widehat{S^n}$. Any point of the sphere S^n is canonically identified with the closure of an open half-space in the dual sphere $\widehat{S^n}$. This open half-space in $\widehat{S^n}$ is formed by all closed half-spaces in S^n which do not contain the given point of the sphere S^n. In other words, the sphere S^n itself is dual to the sphere $\widehat{S^n}$. Moreover, the spaces of all cooriented contact elements to the dual spheres S^n and $\widehat{S^n}$ are canonically identified with each other. The natural cooriented contact structures coincide after this identification, so later on we do not distinguish these spaces and denote them uniformly: $E^{2n-1} = ST^*S^n \cong ST^*\widehat{S^n}$.

LEMMA 1. *Let $M \subset S^n$ be a compact subset, $[M] \subset S^n$ be its convex hull, and $[M]^\perp \subset E^{2n-1}$ be the set of all cooriented contact elements which are support to $[M]$. Let $M^* \subset \widehat{S^n}$ denote the set of all closed half-spaces containing M and $\widetilde{M^*} \subset E^{2n-1}$ denote the set of all cooriented contact elements which are infinitesimally antisupport to M^*. Then $[M]^\perp = \widetilde{M^*}$.*

PROOF. Let $M^{*\top} \subset E^{2n-1}$ denote the set of all cooriented contact elements which are antisupport to M^*. Then $[M]^\perp = M^{*\top}$. Now let us prove that $M^{*\top} = \widetilde{M^*}$.

The inclusion $M^{*\top} \subset \widetilde{M^*}$ is obvious. Let $\mathbf{l} \notin M^{*\top}$ and ξ be the point of applying of the element $\mathbf{l}$. If $\xi \notin M^*$ then $\mathbf{l} \notin \widetilde{M^*}$. If $\xi \in M^*$ then in the interiority of the element $\mathbf{l}$ there exists a point $\eta \in M^*$. But the segment between the points ξ and η lies entirely in M^* and, consequently, in the cone which is tangent to M^* at the point ξ. Again $\mathbf{l} \notin \widetilde{M^*}$ that proves the inclusion $M^{*\top} \supset \widetilde{M^*}$. $\square$

Legendre Varieties $\widetilde{R_0}$, $\widetilde{R_1}$, $\widetilde{R_2}$, $\widetilde{R_3}$, and $\widetilde{V_3}$ in Coordinates

In the coordinates $(p, q, r; u, v, w, s)$ on E_0 the normal forms of the above singularities of Legendre varieties are the following:

$$\widetilde{R_0} = \{p = q = r = s = 0\},$$

$$\widetilde{R_1} = \{pu = q = r = s = 0, p \geq 0, u \geq 0\},$$

$$\widetilde{R_2} = \{pu = qv = r = s = 0, p \geq 0, u \geq 0, q \geq 0, v \geq 0\},$$

$$\widetilde{R_3} = \{pu = qv = rw = s = 0, p \geq 0, u \geq 0, q \geq 0, v \geq 0, r \geq 0, w \geq 0\}.$$

The Legendre variety $\widetilde{V_3}$ consists of the following three strata (τ is a real parameter):

$$\{p = q = r = 0, u \geq -2\tau^2, v = -4\tau^3 - 2u\tau, w \geq 3\tau^4 + u\tau^2 - u^2/4, s = 0\},$$

$$\{p = r\tau^2 + ru/2, q = r\tau, r \geq 0, u \geq -2\tau^2, v = -4\tau^3 - 2u\tau, w = 3\tau^4 + u\tau^2 - u^2/4, s = 0\},$$

$$\{p = 0, |q| \leq r|\tau|, r \geq 0, u = -2\tau^2, v = 0, w = 0, s = 0\}.$$

The first and third strata can be extended up to manifolds, the second stratum can be extended up to an irreducible algebraic variety. In implicit form this variety is given by the following polynomials: $32u^3v^2+64u^2w^2+144uv^2w-27v^4+256w^3$, $2pu+3qv+4rw$, $3pv+4qw-2ruv$, $16pw-8quv-8ruw-3rv^2$, $p^2 + qrv + r^2w$, $4pq + r^2v$, $2pr - 2q^2 - r^2u$, $prv - 4q^2v - 4qrw$, $p^2r - 4pq^2 + r^3w$, and s.

Proof of Theorem 2

Theorem 2 is partially proved in [2]. It only remains to investigate the singularities of the Legendre variety $[M]^\perp$ which appear above the vertices of the support $4A_1$-tetrahedrons and above the support A_1A_3-segments of the hypersurface M.

Case $4A_1$**.** Let $O \subset S^4$ be a closed half-space containing a generic hypersurface $M \subset S^4$ and bounded by a support to M $4A_1$-plane and A_1 be any point of tangency of the plane and M.

According to Lemma 1, $[M]^\perp = \widetilde{M^*}$. Let us choose local smooth coordinates (u, v, w, s) on $\widehat{S^4}$ with an origin O such that the set M^* is locally given by the inequalities $u \geq 0$, $v \geq 0$, $w \geq 0$, and $s \geq 0$, and the cooriented contact element $ds \leq 0$ applied at O is the pair (O, A_1). To investigate the singularity of the Legendre variety $\widetilde{M^*}$ above the point A_1 let us consider the cooriented contact elements from $\widetilde{M^*}$ which have the form $pdu + qdv + rdw + ds \leq 0$. The variety of all such elements is the Legendre variety $\widetilde{R_3}$.

Case A_1A_3**.** Let $O \subset S^4$ be a closed half-space containing a generic hypersurface $M \subset S^4$ and bounded by a support to M A_1A_3-plane, A_1 and A_3 be the points of tangency (A_1 and A_3 respectively) of the plane and M.

According to Lemma 1, $[M]^\perp = \widetilde{M^*}$. Let us choose local smooth coordinates (u, v, w, s) on $\widehat{S^4}$ with an origin O such that the set M^* is locally given by the condition $(u, v, w) \in V_3$ and the inequality $s \geq 0$. Then the cooriented contact element $ds \leq 0$ applied at O is the pair (O, A_1), the element $dw \leq 0$ is the pair (O, A_3), and the elements $dw + r'ds \leq 0$ where $r' \geq 0$ consist of the half-space O and interior points of the support A_1A_3-segment of the hypersurface M.

To investigate the singularity of the Legendre variety $\widetilde{M^*}$ above the point A_1 let us consider the cooriented contact elements from $\widetilde{M^*}$ which have the form $pdu + qdv + rdw + ds \leq 0$. The variety of all such elements is the Legendre variety $\widetilde{V_3}$.

To investigate the singularities of the Legendre variety $\widetilde{M^*}$ above the point A_3 and above the interior points of the support A_1A_3-segment let us consider the Legendre variety $\widetilde{M_0^*}$ of the cooriented contact elements from $\widetilde{M^*}$ which have the form $p'du + q'dv + dw + r'ds \leq 0$. All such elements are infinitesimally antisupport to the set $\partial V_3 \times \mathbb{R}^+$ where ∂V_3 is the boundary of the body V_3 (the cut swallowtail) and $\mathbb{R}^+$ is the real ray $s \geq 0$. The set $\partial V_3 \times \mathbb{R}^+$ has the form $u \geq -2\tau^2$, $v = -4\tau^3 - 2u\tau$,

$w = 3\tau^4 + u\tau^2 - u^2/4$, $s \geq 0$ where τ is a real parameter. The Legendre variety $\widetilde{M_0^*}$ consists of the following four strata:

1) $p' \geq 0$, $u - 2p' + 2q'^2 = 0$, $s \geq 0$, $r' = 0$, $v + 4p'q' = 0$, $w + p'^2 - 4p'q'^2 = 0$;
2) $p' \geq 0$, $u - 2p' + 2q'^2 = 0$, $s = 0$, $r' \geq 0$, $v + 4p'q' = 0$, $w + p'^2 - 4p'q'^2 = 0$;
3) $p' = 0$, $u + 2q'^2 \leq 0$, $s \geq 0$, $r' = 0$, $v = 0$, $w = 0$;
4) $p' = 0$, $u + 2q'^2 \leq 0$, $s = 0$, $r' \geq 0$, $v = 0$, $w = 0$.

After the substitution $p' = U$, $q' = W$, $r' = Q$, $u = -P + 2U - 2W^2$, $v = -R - 4UW$, $w = S + PU + RW - U^2 + 4UW^2$ the cooriented contact elements have the form $PdU + QdV + RdW + dS \leq 0$ and the Legendre variety $\widetilde{M_0^*}$ coincides with $\widetilde{R_2}$ which consists of the following strata:

1) $U \geq 0$, $P = 0$, $V \geq 0$, $Q = 0$, $R = 0$, $S = 0$;
2) $U \geq 0$, $P = 0$, $V = 0$, $Q \geq 0$, $R = 0$, $S = 0$;
3) $U = 0$, $P \geq 0$, $V \geq 0$, $Q = 0$, $R = 0$, $S = 0$;
4) $U = 0$, $P \geq 0$, $V = 0$, $Q \geq 0$, $R = 0$, $S = 0$.

Contact Vector Fields

A vector field preserving a contact structure on a manifold is called *contact*. If the contact structure is given by the zero subspaces of a 1-form then any contact vector field is uniquely defined by its *generating function* that is the substitution of the field into the 1-form.

For example, on the space E_0 with the contact structure given by the form $\varkappa d\lambda + ds$ the contact vector field defined by a generating function $K(\varkappa, \lambda, s)$ has the form

$$\dot{\varkappa} = \varkappa K_s - K_\lambda, \quad \dot{\lambda} = K_\varkappa, \quad \dot{s} = K - \varkappa K_\varkappa. \tag{1}$$

Let us consider the Legendre bundle $(\varkappa, \lambda, s) \mapsto (\mu, t)$ locally given by a generating family $F(\varkappa, \mu, t)$. Let $K(F)$ denote the derivative of the generating family when the Legendre bundle is acted by the contact vector field with the generating function K.

LEMMA 2. $K(F) = -K(\varkappa, F_\varkappa, F - \varkappa F_\varkappa)$.

PROOF. Explicit calculation of the action of the vector field (1) on the fibers $\lambda = F_\varkappa$, $s = F - \varkappa F_\varkappa$. □

REMARK. The associative algebra of generating functions has a natural structure of a Lie algebra. Namely, the bracket of two generating functions is equal to the generating function of the commutator of the corresponding contact vector fields: $\{K, L\} = \varkappa K_s L_\varkappa - K_\lambda L_\varkappa + K_\varkappa L_\lambda + K L_s - \varkappa K_\varkappa L_s - K_s L$.

Proof of Theorem 3

In the present paper this theorem is proved only for the new singularities $\widetilde{R_3}$ and $\widetilde{V_3}$. The rest cases 1)–3) are actually examined in [2].

Case $\widetilde{R_3}$. In a neighborhood of any singular point $\widetilde{R_3}$ the Legendre variety L consists of eight strata which can be extended up to manifolds. Because π is generic its restriction on any of the strata is a mapping of maximal rank.

So in a singular point $\widetilde{R_3}$ of the Legendre variety L the germ $\pi : (\varkappa, \lambda, s) \mapsto (\mu, t)$, where $\varkappa = (p, q, r)$, $\lambda = (u, v, w)$, and $\mu = (x, y, z)$, is given by a generating family $F(\varkappa, \mu, t)$. It follows from the condition of maximal rank of the restriction of π on the stratum $p = q = r = s = 0$. The conditions of maximal rank of the restriction of π on the rest seven strata of $\widetilde{R_3}$ imply that each diagonal minor of the matrix $\|F_{\varkappa\varkappa}\|$ is not equal to zero at the considered singular point $\widetilde{R_3}$.

Moreover, the matrix $\|F_{\varkappa\varkappa}\|$ is positive definite at the origin. Indeed, $F_{pp} > 0$, otherwise the subset $q = r = s = 0$ of $\widetilde{R_3}$ consisting of two its strata is folded. Hence, and $F_{pp}F_{qq} - F_{pq}^2 > 0$, otherwise the pair of strata $u = r = s = 0$ is folded. At last, $\det \|F_{\varkappa\varkappa}\| > 0$, otherwise the pair of strata $u = v = s = 0$ is folded.

Let us consider in F the terms of degree one and two with respect to $\varkappa$, μ, and t. Let us reduce them to the form F_2 acting by means of the generating functions pu, qv, and rw which preserve $\widetilde{R}_3$, and choosing coordinates (μ, t) properly. Now let us fix the degrees of quasihomogeneity indicated in the formulation of the theorem.

Let $\mathcal{I}(\widetilde{R}_3)$ be the Lie algebra of the germs (at the origin) of generating functions of contact vector fields on E_0 preserving the origin and the Legendre variety $\widetilde{R}_3$, and $\mathcal{D}_{\mu,t}$ be the Lie algebra of the germs (at the origin) of vector fields on the base of the Legendre bundle π which preserve the origin. Let us consider the Lie algebra $\mathcal{G} = \mathcal{I}(\widetilde{R}_3) \oplus \mathcal{D}_{\mu,t}$ of the group of the equivalence of Legendre mappings $(\widetilde{R}_3, 0) \hookrightarrow (E_0, 0) \to (\mathbb{R}^3 \times \mathbb{R}, 0)$. Generating families of such germs are acted by the first term of the Lie algebra $\mathcal{G}$ by means of the formula from Lemma 2 and are acted by the second term by means of derivations. The results of this action of the Lie algebra $\mathcal{G}$ are tangent vectors to the space of generating families. These tangent vectors are elements of the ideal

$$\mathfrak{p}_{\varkappa,\mu,t} = \{f \in \mathcal{E}_{\varkappa,\mu,t} : f|_{\varkappa,\mu,t=0} = f_\varkappa|_{\varkappa,\mu,t=0} = 0\}$$

lying in the associative algebra $\mathcal{E}_{\varkappa,\mu,t}$ of the germs (at the origin) of smooth functions of $(\varkappa, \mu, t)$.

Lemma 3. *Let $\mathcal{S}(F_2) = \{g \in \mathcal{G} : gF_2 = 0\}$ be the stabilizer of F_2. Then there exists a number N with the following property: if the coefficients a, b, c and the cubic quasihomogeneous polynomial F_3 are typical then* $\dim_{\mathbb{R}} \mathfrak{p}_{\varkappa,\mu,t}/(\mathcal{G}F_2 + \mathcal{S}(F_2)F_3) \le N$.

The number N from Lemma 3 bounds the codimension of the orbit of the germ π in the space of jets of Legendre mappings with respect to the good geometrical group ([9], Chapter 3, § 2) of the Legendre equivalence preserving the embedding $\widetilde{R}_3 \hookrightarrow E_0$. According to the general finite-definiteness theorem ([9], Chapter 3, § 2), the generating family of the germ π can be reduced to a polynomial of quasidegree d depending only on N.

The numbers of moduli among the coefficients of the polynomials F_2 and F_3 are equal $\dim_{\mathbb{R}} \mathfrak{p}_2/\mathcal{G}_0 F_2 = 3$ and $\dim_{\mathbb{R}} \mathfrak{p}_3/(\mathcal{G}_1 F_2 + \mathcal{S}_0(F_2)F_3) = 6$ respectively because $\mathcal{S}_i(F_2) = 0$ if $i < 0$. Here $\mathfrak{p}_i$, $\mathcal{G}_i$, and $\mathcal{S}_i(F_2)$ are quasihomogeneous components of the corresponding algebras of quasidegree i. (See, e. g., [7], § 14.)

Thus the considered case $\widetilde{R}_3$ of Theorem 3 follows from Lemma 3 which is remained to prove.

Proof of Lemma 3. Let $\mathcal{E}_{\mu,t}$ be the associative algebra of the germs (at the origin) of smooth functions of (μ, t), $\mathfrak{m}_{\mu,t} \subset \mathcal{E}_{\mu,t}$ be the maximal ideal, and

$$\mathfrak{M} = \mathcal{E}_{\varkappa,\mu,t}\langle p\partial_p F_2, q\partial_q F_2, r\partial_r F_2, F_2\rangle + \mathfrak{m}_{\mu,t}\langle p, q, r, 1\rangle + \mathcal{E}_{\mu,t}\langle F_3\rangle \subset \mathfrak{p}_{\varkappa,\mu,t}$$

which is a $\mathcal{E}_{\mu,t}$-submodule where ∂ means partial derivation of the indicated variable. Let us prove that $\mathcal{G}F_2 + \mathcal{S}(F_2)F_3 \supset \mathfrak{M}$.

Indeed, the Lie algebra $\mathcal{I}(\widetilde{R}_3)$ contains the germs pu, qv, rw, and s which generate an ideal in the associative algebra of the germs of generating functions. Acting on F_2 this ideal which lies entirely in $\mathcal{I}(\widetilde{R}_3)$ gives us the first term of $\mathfrak{M}$. The Lie algebra $\mathcal{D}_{\mu,t} = \mathfrak{m}_{\mu,t}\langle \partial_x, \partial_y, \partial_z, \partial_t\rangle$ applied to F_2 gives the second term of $\mathfrak{M}$. At last, $\mathcal{S}(F_2)$ is a $\mathcal{E}_{\mu,t}$-module and $F_3 \in \mathcal{S}(F_2)F_3$ because $gF_2 = 0$ and $gF_3 = F_3$ in consequence of Lemma 2 where $g = (pu + qv + rw + 2s, x\partial_x + y\partial_y + z\partial_z + 2t\partial_t) \in \mathcal{I}(\widetilde{R}_3) \oplus \mathcal{D}_{\mu,t}$.

Now it remains to prove that $\dim_{\mathbb{R}} \mathfrak{p}_{\varkappa,\mu,t}/\mathfrak{M} \le N < \infty$ if a, b, c, and F_3 are typical.

Let us consider $\mathcal{E}_{\mu,t}$-submodule $\mathfrak{M}' = \mathcal{E}_{\varkappa,\mu,t}\langle p\partial_p F_2, q\partial_q F_2, r\partial_r F_2\rangle + \mathfrak{m}_{\mu,t}\langle p, q, r, 1\rangle \subset \mathfrak{M}$. If $1 - a^2 \ne 0$, $1 - b^2 \ne 0$, $1 - c^2 \ne 0$, and $\Delta \ne 0$ where $\Delta = 1 + 2abc - a^2 - b^2 - c^2$ then $\mathcal{E}_{\mu,t}$-module $\mathfrak{p}_{\varkappa,\mu,t}/\mathfrak{M}'$ is generated by the cosets $[pqr]$, $[pq]$, $[pr]$, and $[qr]$ because $\mathcal{E}_{\mu,t}$-module $\mathcal{E}_{\varkappa,\mu,t}/\mathcal{E}_{\varkappa,\mu,t}\langle p\partial_p F_2, q\partial_q F_2, r\partial_r F_2\rangle$ is generated by the monomials pqr, pq, pr, qr, p, q, r, and 1. The last is implied by the Weierstrass–Malgrange preparation theorem (see, e. g., [7], Chapter 1, 4.4 and 6.6) applied to the mapping $(\varkappa, \mu, t) \mapsto (\mu, t)$

and the following fact: if the above inequalities are correct then the monomials generate the linear space $\mathcal{E}_\varkappa/(p^2+apq+bpr, apq+q^2+cqr, bpr+cqr+r^2)$.

In particular, the following relations are true in $\mathfrak{p}_{\varkappa,\mu,t}/\mathfrak{M}'$:

$$[p^2]=[-apq-bpr],\quad [q^2]=[-apq-cqr],\quad [r^2]=[-bpr-cqr],$$

$$[p^3]=[-ap^2q-bp^2r-xp^2],\quad [q^3]=[-apq^2-cq^2r-yq^2],\quad [r^3]=[-bpr^2-cqr^2-zr^2],$$

$$[p^2q]=(1-a^2)^{-1}[(ac-b)pqr+(ay-x)pq],\quad [pq^2]=(1-a^2)^{-1}[(ab-c)pqr+(ax-y)pq],$$

$$[p^2r]=(1-b^2)^{-1}[(bc-a)pqr+(bz-x)pr],\quad [pr^2]=(1-b^2)^{-1}[(ab-c)pqr+(bx-z)pr],$$

$$[q^2r]=(1-c^2)^{-1}[(bc-a)pqr+(cz-y)qr],\quad [qr^2]=(1-c^2)^{-1}[(ac-b)pqr+(cy-z)qr],$$

$$[p^2qr]=\Delta^{-1}[(-x+c^2x-bcy+ay-acz+bz)pqr],$$

$$[pq^2r]=\Delta^{-1}[(-bcx+ax-y+b^2y-abz+cz)pqr],$$

$$[pqr^2]=\Delta^{-1}[(-acx+bx-aby+cy-z+a^2z)pqr].$$

Indeed, the expressions for $[p^2]$, $[q^2]$, and $[r^2]$ follow from $[p\partial_pF_2]=[q\partial_qF_2]=[r\partial_rF_2]=0$, the expressions for $[p^3]$, $[q^3]$, and $[r^3]$ follow from $[p^2\partial_pF_2]=[q^2\partial_qF_2]=[r^2\partial_rF_2]=0$, the expressions for $[p^2q]$ and $[pq^2]$ follow from $[pq\partial_pF_2]=[pq\partial_qF_2]=0$, the expressions for $[p^2r]$ and $[pr^2]$ follow from $[pr\partial_pF_2]=[pr\partial_rF_2]=0$, the expressions for $[q^2r]$ and $[qr^2]$ follow from $[qr\partial_qF_2]=[qr\partial_rF_2]=0$, and, at last, the expressions for $[p^2qr]$, $[pq^2r]$, and $[pqr^2]$ follow from $[pqr\partial_pF_2]=[pqr\partial_qF_2]=[pqr\partial_rF_2]=0$.

Let us consider in $\mathcal{E}_{\mu,t}\langle pqr,pq,pr,qr\rangle$ the $\mathcal{E}_{\mu,t}$-submodule $\mathfrak{N}$ generated by the elements f_2^p, f_2^q, f_2^r, f_2^{pq}, f_2^{pr}, f_2^{qr}, f_2^{pqr}, and $f_3\in\mathcal{E}_{\mu,t}\langle pqr,pq,pr,qr\rangle$ defined by the following conditions in $\mathfrak{p}_{\varkappa,\mu,t}/\mathfrak{M}'$: $[2pF_2]=[f_2^p]$, $[2qF_2]=[f_2^q]$, $[2rF_2]=[f_2^r]$, $[2pqF_2]=[f_2^{pq}]$, $[2prF_2]=[f_2^{pr}]$, $[2qrF_2]=[f_2^{qr}]$, $[2pqrF_2]=[f_2^{pqr}]$, and $[F_3]=[f_3]$. The expressions of the previous indentation imply that the coordinates of the generators of the module $\mathfrak{N}$ in the basis $\langle pqr,pq,pr,qr\rangle$ depend rationally on the numbers a, b, c and the coefficients of the polynomial F_3. For example, if $a=b=c=0$ we get $f_2^p=ypq+zpr$, $f_2^q=xpq+zqr$, $f_2^r=xpr+yqr$, $f_2^{pq}=zpqr+(2t-x^2-y^2)pq$, $f_2^{pr}=ypqr+(2t-x^2-z^2)pr$, $f_2^{qr}=xpqr+(2t-y^2-z^2)qr$, $f_2^{pqr}=(2t-x^2-y^2-z^2)pqr$.

Let $a=b=c=0$, $F_3=zpq+2ypr+3xqr$, and $N=\dim_{\mathbb{R}}\mathcal{E}_{\mu,t}\langle pqr,pq,pr,qr\rangle/\mathfrak{N}$. Then $N<\infty$. Indeed, let us consider the matrix

$$\Omega=\left\|\begin{matrix} 0 & y & z & 0\\ 0 & x & 0 & z\\ 0 & 0 & x & y\\ z & 2t-x^2-y^2 & 0 & 0\\ y & 0 & 2t-x^2-z^2 & 0\\ x & 0 & 0 & 2t-y^2-z^2\\ 2t-x^2-y^2-z^2 & 0 & 0 & 0\\ 0 & z & 2y & 3x\end{matrix}\right\|$$

consisting of the coordinates of the generators of the module $\mathfrak{N}$ in the basis $\langle pqr,pq,pr,qr\rangle$. It is directly verified that the condition $\operatorname{rank}\Omega<4$ implies the relations $x=y=z=t=0$ in $\mathbb{C}^4$. So $\dim_{\mathbb{R}}\mathcal{E}_{\mu,t}/(M_1,\dots,M_{70})<\infty$ where $(M_1,\dots,M_{70})$ is the ideal generated by all 4×4-minors of the matrix Ω. Because all elements M_ipqr, M_ipq, M_ipr, or M_iqr where $i=1,\dots,70$ lie in $\mathfrak{N}$ we get the required finite dimensionality.

Now let a, b, c, and F_3 be typical. Then $\dim_{\mathbb{R}}\mathcal{E}_{\mu,t}\langle pqr,pq,pr,qr\rangle/\mathfrak{N}\le N$. Indeed, the set of the coefficients satisfying the inequality is not empty in consequence of the definition of N and open in the sense of Zariski (see [8], 12.7.2).

But $\dim_{\mathbb{R}} \mathfrak{p}_{\varkappa,\mu,t}/\mathfrak{M} \leq \dim_{\mathbb{R}} \mathcal{E}_{\mu,t}\langle pqr, pq, pr, qr\rangle/\mathfrak{N}$ because the natural homomorphism $\mathcal{E}_{\mu,t}\langle pqr, pq, pr, qr\rangle \to \mathfrak{p}_{\varkappa,\mu,t}/\mathfrak{M}$ is surjective if a, b, c, and F_3 are typical and the kernel of this homomorphism contains $\mathfrak{N}$. $\square$

Case $\widetilde{V_3}$. In a neighborhood of any singular point $\widetilde{V_3}$ the Legendre variety L consists of three strata two of which can be extended up to manifolds. The tangent cone to the third stratum consists of the two Legendre planes $p = q = r = s = 0$ and $p = q = w = s = 0$. These equations are obtained from the quadratic parts $2pu + 3qv + 4rw$, $3pv + 4qw$, $16pw$, p^2, $4pq$, and $2pr - 2q^2$ of the polynomials vanishing on the stratum (see p. 12). Because π is generic its restriction on any of the two smooth strata is a mapping of maximal rank. Besides, the differential of π does not vanish on the tangent cone to the non-smooth stratum.

So in a singular point $\widetilde{V_3}$ of the Legendre variety L the germ $\pi : (\varkappa, \lambda, s) \mapsto (\mu, t)$, where $\varkappa = (p, q, r)$, $\lambda = (u, v, w)$, and $\mu = (x, y, z)$, is given by a generating family $F(\varkappa, \mu, t)$. It follows from the condition of maximal rank of the restriction of π on the stratum $p = q = r = s = 0$. The condition of maximal rank of the restriction of π on the stratum $p = v = w = s = 0$ implies that $F_{qq}F_{rr} - F_{qr}^2 \neq 0$ at the origin. At last, the Legendre plane $p = q = w = s = 0$ from the tangent cone to the non-smooth stratum gives the inequality $F_{rr} \neq 0$.

Lemma 4. *Let the germ at the origin of a Legendre bundle $\pi : (\varkappa, \lambda, s) \mapsto (\mu, t)$ where $\varkappa = (p, q, r)$, $\lambda = (u, v, w)$, and $\mu = (x, y, z)$ be given by a generating family $F(\varkappa, \mu, t)$ which satisfies the following conditions: $F_{qq}F_{rr} - F_{qr}^2 \neq 0$ and $F_{rr} \neq 0$, and $\pi' : (E_0, 0) \to (\mathbb{R}^3 \times \mathbb{R}, 0)$ be the germ at the origin of a Legendre bundle which is sufficiently close to π. Then the following equivalence of germs of Legendre mappings takes place:*

$$\begin{array}{ccccc} (\widetilde{V_3}, 0) & \hookrightarrow & (E_0, 0) & \xrightarrow{\pi'} & (\mathbb{R}^3 \times \mathbb{R}, 0) \\ \updownarrow & & \updownarrow & & \updownarrow \\ (\widetilde{V_3}, 0) & \hookrightarrow & (E_0, 0) & \xrightarrow{\pi} & (\mathbb{R}^3 \times \mathbb{R}, 0) \end{array}$$

where the vertical arrows are close to the identity mappings.

The condition of non-degeneracy divides the space of the generating families into two connected components transferred to each other by the substitution $t \mapsto -t$. Each of them is divided into four components by the conditions $F_{qq}F_{rr} - F_{qr}^2 \neq 0$ and $F_{rr} \neq 0$. According to Lemma 4, the germs of Legendre mappings are equivalent if they are given by generating families from the same component. Hence, each of them can be given by the generating family $(p^2 \pm q^2 \pm r^2)/2 + px + qy + rz + t$ or even by $(\pm q^2 \pm r^2)/2 + px + qy + rz + t$ (see Note to Theorem 3). It remains to check that the image $\pi(L)$ is not a continuously differentiable manifold if there is at least one minus in these families.

Thus the considered case $\widetilde{V_3}$ of Theorem 3 follows from Lemma 4 which is remained to prove.

Proof of Lemma 4. The variety $\widetilde{V_3}$ can be replaced by its extension up to the irreducible algebraic Legendre variety $\overline{V_3}$ which consists of the following three strata: $\{p = q = r = s = 0\}$, $\{p = r\tau^2 + ru/2, q = r\tau, v = -4\tau^3 - 2u\tau, w = 3\tau^4 + u\tau^2 - u^2/4, s = 0\}$ (τ is a parameter), and $\{p = v = w = s = 0\}$. Indeed, any diffeomorphism being sufficiently close to identity one and preserving $\overline{V_3}$ preserves $\widetilde{V_3}$ as well.

Let us consider the algebra $\mathcal{E}_\varkappa$ of the germs at the origin of smooth functions on the fiber $\pi^{-1}(0)$ and the algebra $\mathcal{Q}$ of the germs at the origin of smooth functions on the intersection $\overline{V_3} \cap \pi^{-1}(0)$. There is a natural restriction homomorphism $i : \mathcal{E}_\varkappa \to \mathcal{Q}$. Let us show that $\mathfrak{m}_\varkappa^2 \subset \ker i$ where $\mathfrak{m}_\varkappa \subset \mathcal{E}_\varkappa$ is the maximal ideal. Indeed, $\pi^{-1}(0) = \{(\varkappa, \lambda, s) \in E_0 : \lambda = f_\varkappa(\varkappa), s = f(\varkappa) - \varkappa f_\varkappa(\varkappa)\}$ where $f(\varkappa) = F(\varkappa, 0, 0)$ and the functions $2pu + 3qv + 4rw$, $3pv + 4qw - 2ruv$, $16pw - 8quv - 8ruw - 3rv^2$, $p^2 + qrv + r^2w$, $4pq + r^2v$, and $-2s$ vanish on $\overline{V_3}$. So the germs $2pf_p + 3qf_q + 4rf_r$, $3pf_q + 4qf_r - 2rf_pf_q$, $16pf_r - 8qf_pf_q - 8rf_pf_r - 3rf_q^2$, $p^2 + qrf_q + r^2f_r$, $4pq + r^2f_q$, and $2pf_p + 2qf_q + 2rf_r - 2f$ lie in the kernel of i. All of them are in $\mathfrak{m}_\varkappa^2$ and their expansions up to $\mathfrak{m}_\varkappa^3$ in the basis $\langle p^2, pq, pr, q^2, qr, r^2\rangle$ form the matrix

$$\left\| \begin{array}{cccccc} * & * & * & 3f_{qq}(0) & 7f_{qr}(0) & 4f_{rr}(0) \\ * & * & * & 4f_{qr}(0) & 4f_{rr}(0) & 0 \\ * & * & 16f_{rr}(0) & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ * & * & * & f_{qq}(0) & 2f_{qr}(0) & f_{rr}(0) \end{array} \right\|$$

which is non-degenerate in accordance with the condition of the lemma. Hence, the germs generate $\mathfrak{m}_\varkappa^2/\mathfrak{m}_\varkappa^3$ as linear space over $\mathbb{R}$. According to the Nakayama lemma, they generate $\mathfrak{m}_\varkappa^2$ as a $\mathcal{E}_\varkappa$-module. Therefore $\mathfrak{m}_\varkappa^2 \subset \ker i$.

So the germ $(\overline{V_3}, 0) \hookrightarrow (E_0, 0) \xrightarrow{\pi} (\mathbb{R}^3 \times \mathbb{R}, 0)$ of a Legendre mapping is versal in the sense of 3.3 in [6] because $\mathfrak{m}_\varkappa^2 \subset \ker i$. According to Theorem $3'$ from [6], the germ is stable with respect to perturbations of π. It remains to note that the point of applying of the germ is preserved by its equivalence. $\square$

Proof of Theorem 4

According to the definition, if a germ of a Legendre bundle is given by a generating family $F(p,q,r;u,v,w,s)$ then its fibers are defined by the equations $u = F_p$, $v = F_q$, $w = F_r$, $s = F - pF_p - qF_q - rF_r$ and the cooriented contact structure is defined by the form $F_x dx + F_y dy + F_z dz + F_t dt$.

The case 1) of the variety $\widetilde{R_0}$ is trivially checked explicitly.

In the cases 2)–4) substituting the above equations of the fibers into the explicit coordinate expressions for the varieties $\widetilde{R_1}$, $\widetilde{R_2}$, and $\widetilde{R_3}$ we get that the minimum of F is equal to zero as is required.

In the case 5) the equations of the fibers have the form $u = p + x$, $v = q + y$, $w = r + z$, $s = -(p^2 + q^2 + r^2)/2 + t$ and for the Legendre bundle itself we get the formula $(p,q,r;u,v,w,s) \mapsto (u-p, v-q, w-r, s+(p^2+q^2+r^2)/2)$. If $(p,q,r;u,v,w,s) \in \widetilde{V_3}$ then $s = 0$. If in addition $(p,q,r) = 0$ then $(u,v,w) \in V_3$. If $(p,q,r) \neq 0$ then the cooriented contact element $pdu + qdv + rdw \leq 0$ applied at the point (u,v,w) is infinitesimally antisupport to $V_3 \subset \mathbb{R}^3$. In other words, $(u,v,w) \in \partial V_3$ and the vector $(-p,-q,-r)$ is perpendicular to the boundary ∂V_3 of the body V_3 and is directed outside.

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