ON SOME IDENTITIES FOR THE ELEMENTS OF A SYMMETRIC MATRIX

N. V. Ilyushechkin

Introduction

The following problem has its origin in the theory of continuous media (see [1]).

Let $\mathrm{Sym}(n)$ be the space of n-dimensional real symmetric matrices. Describe all indefinitely smooth maps

$$\mathcal{F} : \mathrm{Sym}(n) \longrightarrow \mathrm{Sym}(n),$$

possessing the following property: for any matrix $X \in \mathrm{Sym}(n)$ and any orthogonal matrix C, such that $C^{-1}XC$ is diagonal, the matrix $C^{-1}\mathcal{F}(X)C$ is also diagonal.

Another problem, in a sense dual to the first one, consists in describing the set of all infinitely smooth maps

$$\mathcal{G} : \mathrm{Sym}(n) \longrightarrow \mathrm{Sym}(n),$$

with the property that for any matrix $X \in \mathrm{Sym}(n)$ and any orthogonal matrix C, such that $C^{-1}XC$ is diagonal, all the diagonal elements of the matrix $C^{-1}\mathcal{G}(X)C$ are equal to zero.

In this paper, we will prove the following theorems.

Theorem 1. *If $\mathcal{F}$ is a map from the first set, then there are infinitely smooth functions*

$$a_m : \mathrm{Sym}(n) \longrightarrow R$$

for $m = 0, 1, \dots, n-1$, such that

$$\mathcal{F}(X) = a_0(X)E + a_1(X)X + \cdots + a_{n-1}(X)X^{n-1}, \tag{0.1}$$

that is $\mathcal{F}(X)$ is a matrix polynomial of degree $n-1$ with variable scalar coefficients.

Theorem 2. *For any map $\mathcal{G}$ from the second set there is an infinitely smooth skew-symmetric matrix $A(X)$ such that*

$$\mathcal{G}(X) = A(X)X - XA(X), \tag{0.2}$$

that is $\mathcal{G}(X)$ is the commutator of X with a variable skew-symmetric matrix.

The proof of these theorems relies on some matrix identities, notably, a formula for the volume of a parallelepiped in the space of normal matrices and certain formulas related to the discriminants of the characteristic polynomials. These formulas represent independent interest, and we start the paper by establishing them.

In section 1, we will discuss the volumes generated by functions of a normal matrix. Section 2 is about discriminant polynomials. Section 3 contains a proof of one of the identities stated in Section 2 as well as some useful formulas arising from the Hodge isomorphism. Finally, in Section 4 we give the proof of the two main theorems.

§1. On volumes generated by functions of a normal matrix

Although this study is primarily devoted to symmetric matrices, the formula we are going to prove in this section is valid in the more general case of normal matrices with complex elements.

Let $\mathrm{Mat}_C(n)$ be the space of square complex matrices of order n. It is a unitary space of dimension n^2 with the scalar product of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ given by

$$(A, B) = \sum_{i,j=1}^{n} a_{ij}\bar{b}_{ij}.$$

Let $Z \in \mathrm{Mat}_C(n)$ be a normal matrix, and let $f_1(\zeta), \dots, f_n(\zeta)$ be complex-valued analytic functions of a complex argument defined on the spectrum $\lambda_1, \dots, \lambda_n$ of the matrix Z. We are interested in the volume of the parallelepiped in $\mathrm{Mat}_C(n)$ built on the vectors $f_1(Z), \dots, f_n(Z)$.

We begin with the following observation. Let $\mathbf{w}_1, \dots, \mathbf{w}_l$ be vectors of a unitary space of dimension $m \geq l$. Denote by $V(\mathbf{w}_1, \dots, \mathbf{w}_l)$ the volume of the parallelepiped built on these vectors. Viewing all $\mathbf{w}_j$ as vector-columns $\mathbf{w}_j = (w_{1j}, \dots, w_{mj})^T$, we can arrange their coordinates into a matrix $W = (w_{ij})$, where $i = 1, \dots, m$ and $j = 1, \dots, l$. Denote by $\mathfrak{M}_r(W)$ the sum of squares of absolute values of all order r minors of the matrix W. It is known [2] that

$$V^2(\mathbf{w}_1, \dots, \mathbf{w}_l) = \mathfrak{M}_l(W) \tag{1.1}$$

Observe that the volume does not change under the action of a unitary transformation Q on all the vectors $\mathbf{w}_j$:

$$V(\mathbf{w}_1, \dots, \mathbf{w}_l) = V(Q\mathbf{w}_1, \dots, Q\mathbf{w}_l) \tag{1.2}$$

Indeed, the square of the volume $V^2(\mathbf{w}_1, \dots, \mathbf{w}_l)$ coincides with the Gramian generated by the vectors $\mathbf{w}_1, \dots, \mathbf{w}_l$, and scalar products $(\mathbf{w}_j, \mathbf{w}_k) = (Q\mathbf{w}_j, Q\mathbf{w}_k)$ which are the elements of the Gram matrix, are preserved under unitary transformations of the whole space.

We return to the evaluation of the volume $V(f_1(Z), \dots, f_n(Z))$. From the elements of matrices $f_1(Z), \dots, f_n(Z)$ we make up a new matrix $\Phi(Z)$ of size $n^2 \times n$, in which column number $\nu = 1, \dots, n$ consists of the elements of the matrix $f_\nu(Z)$ taken in a certain order, same for all ν. Due to (1.1), we have

$$V^2(f_1(Z), \dots, f_n(Z)) = \mathfrak{M}_n(\Phi(Z)) \tag{1.3}$$

The similarity transformation $N = U^{-1}MU$, where M, N are variable matrices, while U is a constant unitary matrix, can be viewed as a unitary transformation of the space $\mathrm{Mat}_C(n)$.

We can choose the unitary matrix U so that the matrix $L = U^{-1}ZU$ be diagonal:

$$L = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$$

Then, of course, $f_\nu(L) = U^{-1} f_\nu(Z) U$ and

$$f_\nu(L) = \mathrm{diag}(f_\nu(\lambda_1), \dots, f_\nu(\lambda_n))$$

Since the similarity transformation is unitary, (1.2) implies that

$$V(f_1(Z), \dots, f_n(Z)) = V(f_1(L), \dots, f_n(L)),$$

and owing to (1.3), to determine the unknown volume we only need to find $\mathfrak{M}_n(\Phi(L))$. The matrix $\Phi(L)$ contains only one nonzero minor of order n made of diagonal elements of matrices $f_\nu(Z)$, and $\mathfrak{M}_n(\Phi(L))$ equals the square of the absolute value of this minor. Extracting the square root, we receive the final formula.

Proposition 1. *If Z is a complex normal matrix and $f_1, \dots, f_n$ are analytic functions defined on the spectrum of Z, then the volume of the parallelepiped spanned by the vectors $f_1(Z), \dots, f_n(Z)$ in the space of matrices is given by the formula;*

$$V(f_1(Z), \dots, f_n(Z)) = \left| \begin{vmatrix} f_1(\lambda_1) & f_2(\lambda_1) & \dots & f_n(\lambda_1) \\ f_1(\lambda_2) & f_2(\lambda_2) & \dots & f_n(\lambda_2) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(\lambda_n) & f_2(\lambda_n) & \dots & f_n(\lambda_n) \end{vmatrix} \right|$$

For example, suppose that all functions $f_\nu(\zeta)$ are polynomials $p_\nu(\zeta)$ of degree $n-1$:

$$p_\nu(\zeta) = b_{\nu 0} + b_{\nu 1}\zeta + \dots + b_{\nu, n-1}\zeta^{n-1}.$$

The coefficients of these polynomials make up a square matrix of order n:

$$B = \begin{pmatrix} b_{10} & b_{11} & \dots & b_{1,n-1} \\ b_{20} & b_{21} & \dots & b_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n0} & b_{n1} & \dots & b_{n,n-1} \end{pmatrix}$$

Then, by the determinant multiplication rule,

$$V(p_1(Z), \dots, p_n(Z)) = |\det(B)||W(\lambda_1, \dots, \lambda_n)|,$$

where W is the Vandermonde determinant corresponding to numbers $\lambda_1, \dots, \lambda_n$. Raising this in the square, we get

$$V^2(p_1(Z), \dots, p_n(Z)) = |\det(B)|^2 |\mathrm{Dis}(\chi_Z)|,$$

where χ_Z is the characteristic polynomial of matrix Z, and $\mathrm{Dis}(\chi_Z)$ is its discriminant. In particular,

$$V^2(E, Z, \ldots, Z^{n-1}) = |\mathrm{Dis}(\chi_Z)|$$

Taking into account (1.3), we obtain the following algebraic analog of the previous formula.

Proposition 2. *If χ_Z is the characteristic polynomial of a complex normal matrix Z, and* Dis *its discriminant, then*

$$|\mathrm{Dis}(\chi_Z)| = \mathfrak{M}_n(Z_*), \tag{1.4}$$

where Z_* is the matrix of size $n^2 \times n$, whose column number ν contains the elements of the matrix Z^ν, arranged in a definite order, same for all $\nu = 0, 1, \ldots, n-1$.

In the case when the normal matrix $Z = X \in \mathrm{Sym}(n)$ is symmetric, formulas (1.3) and (1.4) can be slightly modified. This modification is related to the fact that the dimension of the space $\mathrm{Sym}(n)$, which we shall denote by $N = C^2_{n+1}$, is less than the dimension n^2 of the space $\mathrm{Mat}_C(n)$.

Passing to exact definitions, let us view a matrix $X = (x_{ij}) \in \mathrm{Sym}(n)$ as a column $\widehat{X}$ of dimension N, whose elements are x_{ii} and $\sqrt{2}x_{ij}$ with $i < j$, arranged in the order

$$11, 22, \ldots, nn, 12, 13, \ldots, 1n, 23, \ldots, 2n, \ldots, n-1\; n \tag{1.5}$$

The choice of the factor $\sqrt{2}$ is stipulated by the fact that the similarity transformation

$$X = C^{-1}YC, \tag{1.6}$$

where X and Y are variable symmetric matrices, and C is a constant orthogonal matrix, preserves the quadratic form

$$\sum_{i=1}^{n} x_{ii}^2 + 2\sum_{i<j} x_{ij}^2.$$

By E_{ij} we denote the square matrix of order n, which has number 1 on the intersection of i'th row and j'th column and has zeros elsewhere. Define the symmetric matrices Γ_{ii} with $i = 1, \ldots, n$ and Γ_{ij} with $i < j$ by the equalities

$$\Gamma_{ii} = E_{ii}, \qquad \Gamma_{ij} = \frac{1}{\sqrt{2}}(E_{ij} + E_{ji}). \tag{1.7}$$

Then $\mathrm{Sym}(n)$ can be considered as a Euclidean space of dimension N with orthonormal basis (1.7), arranged in the order (1.5).

Slightly modifying the argument used above, we can deduce the formulas similar to (1.3) and (1.4), in dimension N. Instead of the matrix $\Phi(X)$ from (1.3), we use the $N \times n$ matrix $\Phi_s(X)$ whose ν'th column, $\nu = 1, \ldots, n$, contains the elements of the matrix $f_\nu(X)$, arranged in the order (1.5) and multiplied, where necessary, by $\sqrt{2}$. Then formula (1.3) transforms to

$$V^2(f_1(X), \ldots, f_n(X)) = \mathfrak{M}_n(\Phi_s(X)),$$

and formula (1.4) turns into the assertion of the following proposition.

Proposition 3. *The discriminant of the characteristic polynomial of a complex symmetric matrix X is equal to the sum of squares of absolute values of all $n\times n$ minors of an auxiliary matrix $X_!$:*

$$\mathrm{Dis}(\chi_X)=\mathfrak{M}_n(X_!), \tag{1.8}$$

where the matrix $X_!$ is formed by the columns $\widehat{X^0},\widehat{X},\dots,\widehat{X^{n-1}}$, disposed in a natural order.

If $X=(x_{ij})\in\mathrm{Sym}(n)$, then we denote the elements of the matrix X^m for $m=0,1,\dots,n-1$, by $x_{ij}^{(m)}$. Thus, the row of $X_!$ with number ii contains the elements

$$1,x_{ii},x_{ii}^{(2)},\dots,x_{ii}^{(n-1)},$$

and the line number ij with $i<j$ is composed by the elements

$$0,\sqrt{2}x_{ij},\sqrt{2}x_{ij}^{(2)},\dots,\sqrt{2}x_{ij}^{(n-1)}.$$

For example, in the case $n=3$ the matrix $X_!$ looks like

$$X_!=\begin{pmatrix}1 & x_{11} & x_{11}^{(2)}\\ 1 & x_{22} & x_{22}^{(2)}\\ 1 & x_{33} & x_{33}^{(2)}\\ 0 & \sqrt{2}x_{12} & \sqrt{2}x_{12}^{(2)}\\ 0 & \sqrt{2}x_{13} & \sqrt{2}x_{13}^{(2)}\\ 0 & \sqrt{2}x_{23} & \sqrt{2}x_{23}^{(2)}\end{pmatrix}$$

§2. Discriminant polynomials

As we shall see below, the maximum minors of the matrix $X_!$ are closely related to the proof of theorem 1. We denote by $\langle i_1j_1,\dots,i_nj_n\rangle(X)$ the minor of order n of the matrix $X_!$, formed by lines with double indices $i_1j_1,\dots,i_nj_n$, This is a homogeneous polynomial of degree C_n^2 in the elements of matrix X. For example, if $n=3$, then

$$\langle 11,22,33\rangle(X)=\begin{vmatrix}1 & x_{11} & x_{11}^2+x_{12}^2+x_{13}^2\\ 1 & x_{22} & x_{12}^2+x_{22}^2+x_{23}^2\\ 1 & x_{33} & x_{13}^2+x_{23}^2+x_{33}^2\end{vmatrix}=$$

$$=(x_{22}-x_{11})(x_{33}-x_{11})(x_{33}-x_{22})-$$

$$-(x_{22}-x_{11})x_{12}^2+(x_{33}-x_{11})x_{13}^2-(x_{33}-x_{22})x_{23}^2;$$

$$\langle 33,12,23\rangle(X)=2\begin{vmatrix}x_{12} & x_{11}x_{12}+x_{12}x_{22}+x_{13}x_{23}\\ x_{23} & x_{12}x_{13}+x_{22}x_{23}+x_{23}x_{33}\end{vmatrix}=$$

$$= 2x_{12}x_{23}(x_{33} - x_{11}) + 2x_{13}(x_{12}^2 - x_{23}^2)$$

As we already have proved, the sum of squares of all these minors is equal to the discriminant $\mathrm{Dis}(\chi_X)$. In the sequel, the minors $\langle i_1j_1, \dots, i_nj_n\rangle(X)$ will be referred to as *discriminant polynomials.*

The central position in the set of all discriminant polynomials is occupied by $T(X) = \langle 11, \dots, nn\rangle(X)$.

Proposition 4. *The polynomial*

$$T(X) = \langle 11, \dots, nn\rangle(X)$$

is irreducible.

PROOF. The permutation group $S(n)$ acts on the set of square matrices. Let $A = (a_{ij})$ be a matrix of order n, and $\sigma \in S(n)$ a substitution. We set by definition $(\sigma A)_{ij} = a_{\sigma(i)\sigma(j)}$. These transformations are ring automorphisms, in particular, for any $X \in \mathrm{Sym}(n)$ we have

$$\sigma(X^m) = [\sigma(X)]^m \tag{2.1}$$

The group $S(n)$ also operates in the ring $\{P(X)\}$ of polynomials in the variables x_{ij}:

$$\sigma P(\dots, x_{ij}, \dots) = P(\dots, x_{\sigma(i)\sigma(j)}, \dots)$$

These σ's are ring automorphisms, too.

Let us now prove the irreducibility of $T(X)$. As

$$T(X) = \begin{vmatrix} 1 & x_{11} & x_{11}^{(2)} & \dots & x_{11}^{(n-1)} \\ 1 & x_{22} & x_{22}^{(2)} & \dots & x_{22}^{(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{nn} & x_{nn}^{(2)} & \dots & x_{nn}^{(n-1)} \end{vmatrix}, \tag{2.2}$$

owing to (2.1) for any substitution σ we have

$$\sigma x_{ij}^{(m)} = x_{\sigma(i)\sigma(j)}^{(m)},$$

so

$$\sigma T(X) = \pm T(X), \tag{2.3}$$

where the sign depends on the parity of a substitution.

We will refer to the elements x_{ii} as *diagonal*, and to all remaining variables as *nondiagonal.* It is easy to see that the element $x_{ii}^{(m)}$ can be represented as $x_{ii}^{(m)} = x_{ii}^m + \widetilde{x}_{ii}^{(m)}$, where each monomial of the polynomial $\widetilde{x}_{ii}^{(m)}$ contains nondiagonal variables. Therefore, we can write

$$T(X) = W(x_{11}, x_{22}, \dots, x_{nn}) + \widetilde{T}(X), \tag{2.4}$$

where W, as before, is a Vandermonde's determinant, and each monomial of the polynomial $\widetilde{T}(X)$ contains nondiagonal variables. On the other hand, it is evident from (2.2) that every monomial in $\widetilde{T}(X)$ contains diagonal variables, too.

We shall now assume that $Q(X)$ is an irreducible divisor of $T(X)$. Due to (2.4), polynomial $Q(X)$ has the form

$$Q(X) = R(x_{11}, x_{22}, \ldots, x_{nn}) + \widetilde{Q}(X), \tag{2.5}$$

where R depends only on diagonal variables and is a divisor of W, and the nondiagonal variables appear in each monomial of $\widetilde{Q}(X)$. From (2.3) it follows that for any substitution $\sigma \in S(n)$ the polynomial $\sigma(Q)$ is also an irreducible divisor of $T(X)$, and from (2.4) and (2.5) we conclude that $\sigma(R)$ is a divisor of W. As the polynomial W factors into a product of linear forms $x_{jj} - x_{ii}$, and for any two such forms there is a substitution carrying one of them into the other, any other irreducible divisor $T(X)$ is equal to $\sigma(Q)$ for some substitution σ. Now we consider three cases.

1. Q is a form of degree 1. Then $T(X)$ is a product of linear forms. Also, if $\widetilde{Q}(X) \equiv 0$, then $T(X) = W(x_{11}, \ldots, x_{nn})$, which obviously cannot be true. If $\widetilde{Q}(X) \not\equiv 0$, then also all of $\sigma(\widetilde{Q}) \not\equiv 0$ do not depend on diagonal variables. But then $T(X)$ contains monomials which do not depend on diagonal variables, which is also impossible.

2. Q is a form of degree greater than 1. Then R is a product of several factors $x_{jj} - x_{ii}$, and all of them are different. It is always possible to select such a substitution σ, that $\sigma(R) \neq R$, but R and $\sigma(R)$ have an identical divisor of the form $x_{jj} - x_{ii}$ (there is an exceptional case $n = 4$, when the degree of Q is 2; we will discuss it separately). As the forms Q and $\sigma(Q)$ are different and irreducible, they both appear in the expansion of the polynomial $T(X)$ into irreducible factors. Therefore W is divisible by the product $R\sigma(R)$ that contains a factor $(x_{jj} - x_{ii})^2$, and this is a contradiction.

3. The exceptional case $n = 4$:

$$R = (x_{jj} - x_{ii})(x_{ll} - x_{kk}),$$

where (i, j, k, l) is a permutation of numbers 1,2,3,4. In this case $T(X)$ factors into a product of three quadratic forms. Obviously, the form $\widetilde{Q}$ should have degree 1 in diagonal variables and degree 1 in nondiagonal variables. Then any monomial $T(X)$ has at least third degree in diagonal variables. Let us show that this is impossible. Assign the following values to the variables x_{ij} :

$$x_{11} = x, \qquad x_{12} = x_{23} = x_{34} = t,$$

and assume all remaining variables equal to zero. Then

$$\langle 11, 22, 33, 44 \rangle(X) = x^2 t^4,$$

whence $T(X)$ is obliged to contain monomials having degree in diagonal variables no higher than 2. This contradiction completes the proof.

It turns out that the discriminant polynomials can be obtained by a totally different construction. Let $\mathrm{so}(n)$ be the space of skew-symmetric matrices of order n. The dimension of this space is $M = C_n^2$. A matrix $A = (a_{kl}) \in \mathrm{so}(n)$ can be considered as a column $\bar{A}$ of dimension M, whose elements $\sqrt{2}a_{kl}$ with $k < l$ are arranged in the order (1.5), only without the first n elements. We shall consider $\mathrm{so}(n)$ as a Euclidean space of dimension M with orthonormal basis

$$\Delta_{ij} = \frac{1}{\sqrt{2}}(E_{ij} - E_{ji}) \tag{2.6}$$

with $i < j$.

Let $X \in \mathrm{Sym}(n)$. Consider the linear map

$$X^{\square} : \mathrm{so}(n) \longrightarrow \mathrm{Sym}(n),$$

defined by the formula

$$X^{\square}(A) = AX - XA$$

We denote by $X^!$ the matrix of this map in bases (1.7) and (2.6). In the notation introduced earlier, the following matrix-vectorial equality holds: $\widehat{H} = X^!\bar{A}$, where $H = X^{\square}(A)$. Let us describe the matrix $X^!$. First of all it has the size $N \times M$. Its lines are numbered by double indices ij with $i \le j$ and are arranged in the order (1.5). The columns are numbered by double indices kl with $k < l$, disposed in the same order.

If

$$\Psi_{kl} = X^{\square}(\Delta_{kl}),$$

then $\widehat{\Psi}_{kl}$ is the column of the matrix $X^!$ with number kl.

Therefore the row of $X^!$ with number ii looks as follows. If $k < i$, then the element in the column ki is $-\sqrt{2}x_{ki}$. If $k > i$, the element in the column ik is $\sqrt{2}x_{ik}$. All remaining elements of this line are zero.

Now we shall describe the line with number ij, where $i < j$. If $k < i$, the element in the column ki is $-x_{kj}$, and the element in the column kj is $-x_{ki}$. If $i < k < j$, in the column ik we have the number x_{kj}, and in the column kj, the number $-x_{ik}$. If $j < k$, the column ik contains the number x_{jk}, and the column jk, the number x_{ik}. Finally, in the column ij we have the number $x_{jj} - x_{ii}$. All remaining elements of this line are equal to zero.

For example, let $n = 3$. Then

$$X^! = \begin{pmatrix} \sqrt{2}x_{12} & \sqrt{2}x_{13} & 0 \\ -\sqrt{2}x_{12} & 0 & \sqrt{2}x_{23} \\ 0 & -\sqrt{2}x_{13} & -\sqrt{2}x_{23} \\ x_{22} - x_{11} & x_{23} & x_{13} \\ x_{23} & x_{33} - x_{11} & -x_{12} \\ -x_{13} & -x_{12} & x_{33} - x_{22} \end{pmatrix}$$

We are interested in the maximal minors of the matrix $X^!$. The minor containing all lines, except $i_1j_1, \dots, i_nj_n$, will be denoted by $\{i_1j_1, \dots, i_nj_n\}(X)$. Obviously, it is a homogeneous polynomial of degree M in the elements x_{ij} of the matrix X. For example, for $n = 3$ we have

$$\{11, 22, 33\}(X) = \begin{vmatrix} x_{22} - x_{11} & x_{23} & x_{13} \\ x_{23} & x_{33} - x_{11} & -x_{12} \\ -x_{13} & -x_{12} & x_{33} - x_{22} \end{vmatrix} = \langle 11, 22, 33 \rangle(X);$$

$$\{33,12,23\}(X)=\begin{vmatrix}\sqrt{2}x_{12} & \sqrt{2}x_{13} & 0\\ -\sqrt{2}x_{12} & 0 & \sqrt{2}x_{23}\\ x_{23} & x_{33}-x_{11} & -x_{12}\end{vmatrix}=-\langle 33,12,23\rangle(X);$$

In the next section, we shall prove that a similar relation is true in general.

Proposition 5. *For any set of indices the following equality holds:*

$$\{i_1j_1,\ldots,i_nj_n\}(X)=\pm\langle i_1j_1,\ldots,i_nj_n\rangle(X) \tag{2.7}$$

§3. On Hodge's isomorphism for the space of symmetric operators

To avoid tedious computations with associated matrices in the proof of equality (2.7), we shall work with linear operators rather than with matrices. In doing so, we preserve all previous notation, using bold face to indicate the invariant objects. We will rely upon a well-known property of Hodge's isomorphism [3] stating that the coordinates of skew-symmetric tensors that correspond to each other under this isomorphism, can differ only by a sign. Now we introduce the necessary notations and briefly remind some necessary properties of invariant objects.

Let $\mathcal{L}$ be a real Euclidean space of dimension n with scalar product $(\ ,\ )$. By $\mathbf{End}(n)$ we denote the space of linear operators over $\mathcal{L}$. The zero operator will be denoted by $\mathbf{O}$.

An orthonormal basis $\mathbf{e}_1,\ldots,\mathbf{e}_n$ of the space $\mathcal{L}$ gives rise, in a natural way, to an orthonormal basis in the space $\mathbf{End}(n)$. First we define the tensor product $(\mathbf{a}\otimes\mathbf{b})$ for an ordered pair of vectors $\mathbf{a},\mathbf{b}\in\mathcal{L}$. Namely, $(\mathbf{a}\otimes\mathbf{b})\in\mathbf{End}(n)$ is the linear operator defined by the formula

$$(\mathbf{a}\otimes\mathbf{b})(\mathbf{x})=(\mathbf{b},\mathbf{x})\mathbf{a}.$$

All possible products $(\mathbf{e}_i\otimes\mathbf{e}_j)$ of the elements of a basis of $\mathcal{L}$ constitute a basis of the space $\mathbf{End}(n)$, which we will call the accompanying basis of $\mathbf{e}_1,\ldots,\mathbf{e}_n$. Obviously, for any i,j,k we have

$$(\mathbf{e}_i\otimes\mathbf{e}_j)(\mathbf{e}_k)=\delta_{jk}\mathbf{e}_i, \tag{3.1}$$

where δ_{jk} is the Kronecker delta. The operator $(\mathbf{e}_i\otimes\mathbf{e}_j)$ in the basis $\mathbf{e}_1,\ldots,\mathbf{e}_n$ corresponds to the matrix E_{ij}. If $A=(a_{ij})$ is the matrix of an operator $\mathbf{A}\in\mathbf{End}(n)$ in that basis, then

$$\mathbf{A}=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}(\mathbf{e}_i\otimes\mathbf{e}_j).$$

In the space $\mathbf{End}(n)$ there is a natural scalar product $\langle\ ,\ \rangle$, arising from the product $(\ ,\ )$ in $\mathcal{L}$. For any two operators $\mathbf{A},\mathbf{B}\in\mathbf{End}(n)$ their scalar product is

$$\langle\mathbf{A},\mathbf{B}\rangle=\mathrm{tr}(\mathbf{A}^*\mathbf{B})$$

where $\mathbf{A}^*$ is the operator conjugate to $\mathbf{A}$, and tr is the trace. The scalar products of the basic elements are:

$$\langle(\mathbf{e}_i\otimes\mathbf{e}_j),(\mathbf{e}_k\otimes\mathbf{e}_l)\rangle=\delta_{ik}\delta_{jl}, \tag{3.2}$$

hence the accompanying basis is orthonormal, and the coordinates of an operator $\mathbf{A} \in \mathbf{End}(n)$ in this basis coincide with the elements of its matrix in the basis $\mathbf{e}_1, \dots, \mathbf{e}_n$.

An operator $\mathbf{A}$ is called *symmetric*, if $\mathbf{A}^* = \mathbf{A}$, and *skew-symmetric*, if $\mathbf{A}^* = -\mathbf{A}$. The subspace of symmetric operators we denote by $\mathbf{Sym}(n)$, and the subspace of skew-symmetric operators, by $\mathbf{so}(n)$.

Let $\mathbf{X} \in \mathbf{Sym}(n)$. Since the matrix $X = (x_{ij})$ of $\mathbf{X}$ in the basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ is symmetric, we have:

$$\mathbf{X} = \sum_{i=1}^{n} x_{ii}(\mathbf{e}_i \otimes \mathbf{e}_i) + \sum_{i<j} x_{ij}((\mathbf{e}_i \otimes \mathbf{e}_j) + (\mathbf{e}_j \otimes \mathbf{e}_i)).$$

The symmetric operators

$$\mathbf{\Gamma}_{ii} = (\mathbf{e}_i \otimes \mathbf{e}_i), \qquad \mathbf{\Gamma}_{ij} = \frac{1}{\sqrt{2}}(\mathbf{e}_i \otimes \mathbf{e}_j) + \frac{1}{\sqrt{2}}(\mathbf{e}_j \otimes \mathbf{e}_i), \tag{3.3}$$

where $i < j$, correspond to the matrices Γ_{ii} and Γ_{ij}, introduced above. Any operator $\mathbf{X} \in \mathbf{Sym}(n)$ is a linear combination of the basic elements (3.3):

$$\mathbf{X} = \sum_{i=1}^{n} x_{ii}\mathbf{\Gamma}_{ii} + \sqrt{2}\sum_{i<j} x_{ij}\mathbf{\Gamma}_{ij}$$

Besides, (3.2) implies that the system (3.3) is orthonormal. Therefore, this system constitutes an orthonormal basis of N-dimensional Euclidean space $\mathbf{Sym}(n)$, and the coordinates of an operator $\mathbf{X} \in \mathbf{Sym}(n)$ in this basis are the diagonal elements x_{ii} of its matrix X in the basis $\mathbf{e}_1, \dots, \mathbf{e}_n$, as well as nondiagonal elements x_{ij} multiplied on $\sqrt{2}$. We always put the elements of the basis (3.3) in the order (1.5).

Now let $\mathbf{A} \in \mathbf{so}(n)$. Since the matrix $A = (a_{ij})$ of the operator $\mathbf{A}$ in the basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ is skew-symmetric, we have

$$\mathbf{A} = \sum_{i<j} a_{ij}((\mathbf{e}_i \otimes \mathbf{e}_j) - (\mathbf{e}_j \otimes \mathbf{e}_i)),$$

therefore in the space $\mathbf{so}(n)$ there is a natural orthonormal basis

$$\mathbf{\Delta}_{ij} = \frac{1}{\sqrt{2}}(\mathbf{e}_i \otimes \mathbf{e}_j) - \frac{1}{\sqrt{2}}(\mathbf{e}_j \otimes \mathbf{e}_i) \tag{3.4}$$

with $i < j$. We arrange the elements of this basis in the usual order (1.5). Let Δ_{ij} be the corresponding matrices. The coordinates of an operator $\mathbf{A} \in \mathbf{so}(n)$ in this basis are nondiagonal elements a_{ij}, $i < j$, of its matrix A in the basis $\mathbf{e}_1, \dots, \mathbf{e}_n$, multiplied by $\sqrt{2}$.

From now on, we fix an orientation in the space $\mathcal{L}$ and consider as admissible only proper orthogonal transformations (with determinant 1). Under such transformations in $\mathcal{L}$ the accompanying bases (3.3) and (3.4) are also exposed to proper orthogonal transformations. Indeed, suppose that two bases $\mathbf{e}_1, \dots, \mathbf{e}_n$ and $\mathbf{e}'_1, \dots, \mathbf{e}'_n$ are connected by the formulas

$$\mathbf{e}'_i = \sum_{j=1}^{n} c_{ji}\mathbf{e}_j,$$

where the matrix C is proper orthogonal, $\det C = 1$. Then the accompanying transformation is given by

$$\mathbf{\Gamma}'_{ii} = \sum_{k=1}^{n} c_{ki}^2 \mathbf{\Gamma}_{kk} + \sqrt{2} \sum_{k<l} c_{ki} c_{li} \mathbf{\Gamma}_{kl}$$

$$\mathbf{\Gamma}'_{ij} = \sqrt{2} \sum_{k=1}^{n} c_{ki} c_{kj} \mathbf{\Gamma}_{kk} + \sum_{k<l} (c_{ki} c_{lj} + c_{kj} c_{li}) \mathbf{\Gamma}_{kl} \tag{3.5}$$

$$\mathbf{\Delta}'_{ij} = \sum_{k<l} (c_{ki} c_{lj} - c_{kj} c_{li}) \mathbf{\Delta}_{kl}$$

Fort an operator in $\mathcal{L}$ we denote by $C^\wedge$ and $C_\wedge$ the matrices of accompanying transformations in the spaces $\mathbf{Sym}(n)$ and $\mathbf{so}(n)$, respectively. Since all bases connected by formulas (3.5), are orthonormal, both matrices $C^\wedge$ and $C_\wedge$ are orthogonal. For example, if $n = 2$ and

$$C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \qquad \text{where} \qquad a^2 + b^2 = 1,$$

then

$$C^\wedge = \begin{pmatrix} a^2 & b^2 & -\sqrt{2}ab \\ b^2 & a^2 & \sqrt{2}ab \\ \sqrt{2}ab & -\sqrt{2}ab & a^2 - b^2 \end{pmatrix} \qquad \text{and} \qquad C_\wedge = (1).$$

Now let us prove that the matrices $C^\wedge$ and $C_\wedge$ are proper orthogonal, that is their determinants are positive. Indeed, since the matrix C is proper orthogonal, it can be continuously transformed to the unit matrix. More exactly, there are such continuous functions $c_{ij}(t)$, defined for $0 \le t \le 1$, that $c_{ij}(0) = \delta_{ij}$, $c_{ij}(1) = c_{ij}$, and the matrix $C(t) = (c_{ij}(t))$ is orthogonal for all values of t. Considering the family of transformations (3.5), where the elements c_{ij} are replaced by $c_{ij}(t)$, we obtain a continuous set of orthonormal bases $\mathbf{\Gamma}'_{ij}(t)$ and $\mathbf{\Delta}'_{ij}(t)$ that satisfy

$$\mathbf{\Gamma}'_{ij}(0) = \mathbf{\Gamma}_{ij}, \quad \mathbf{\Gamma}'_{ij}(1) = \mathbf{\Gamma}'_{ij}, \qquad \mathbf{\Delta}'_{ij}(0) = \mathbf{\Delta}_{ij}, \quad \mathbf{\Delta}'_{ij}(1) = \mathbf{\Delta}'_{ij}$$

Therefore, matrices $C^\wedge$ and $C_\wedge$ can be continuously deformed to unit matrices of appropriate dimensions and thus have positive determinant.

We have proved that an orientation in space $\mathcal{L}$ naturally induces an orientation in spaces $\mathbf{Sym}(n)$ and $\mathbf{so}(n)$. Henceforth, we will assume the bases (3.3) and (3.4) to be positively oriented.

The dimensions of spaces $\mathbf{Sym}(n)$ and $\mathbf{so}(n)$ are connected by the relation $M + n = N$. We shall study the exterior algebra over the space $\mathbf{Sym}(n)$:

$$\bigwedge(\mathbf{Sym}(n)) = \bigoplus_{p=0}^{N} \bigwedge^p (\mathbf{Sym}(n)) \tag{3.6}$$

In this algebra we are interested in two multivectors $\mathbf{X}_!$ and $\mathbf{X}^!$ of degrees n and M respectively, associated with an operator $\mathbf{X} \in \mathbf{Sym}(n)$. The first one is defined as the exterior product of powers $\mathbf{X}^q$ of the operator $\mathbf{X}$ for $q = 0, 1, \ldots, n-1$:

$$\mathbf{X}_! = \mathbf{X}^0 \wedge \mathbf{X}^1 \wedge \cdots \wedge \mathbf{X}^{n-1}$$

The definition of the second multivector is a little bit more involved. Let

$$\mathbf{X}^\Box : \mathbf{so}(n) \longrightarrow \mathbf{Sym}(n),$$

be the linear map given by

$$\mathbf{X}^\Box(\mathbf{A}) = \mathbf{AX} - \mathbf{XA}$$

Let $\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_M$ be an arbitrary positively oriented orthonormal basis of space $\mathbf{so}(n)$. Denote $\mathbf{\Psi}_r = \mathbf{X}^\Box(\mathbf{\Phi}_r)$. We set by definition

$$\mathbf{X}^! = \mathbf{\Psi}_1 \wedge \mathbf{\Psi}_2 \wedge \cdots \wedge \mathbf{\Psi}_M$$

To prove the correctness of this definition, suppose that $\mathbf{\Phi}'_1, \ldots, \mathbf{\Phi}'_M$ is another positively oriented orthonormal basis of space $\mathbf{so}(n)$, connected with the first basis by a proper orthogonal matrix G. By the linearity of operator $\mathbf{X}^\Box$, systems of vectors $\mathbf{\Psi}_1, \ldots, \mathbf{\Psi}_M$ and $\mathbf{\Psi}'_1, \ldots, \mathbf{\Psi}'_M$, where $\mathbf{\Psi}'_r = \mathbf{X}^\Box(\mathbf{\Phi}'_r)$, are related by the same matrix G. And since $\det G = 1$, we have

$$\mathbf{\Psi}'_1 \wedge \cdots \wedge \mathbf{\Psi}'_M = (\det G)\mathbf{\Psi}_1 \wedge \cdots \wedge \mathbf{\Psi}_M = \mathbf{X}^!$$

Matrices $X_!$ and $X^!$ considered above are matrices of multivectors $\mathbf{X}_!$ and $\mathbf{X}^!$, if X is the matrix of the operator $\mathbf{X}$, and operators $\mathbf{\Delta}_{ij}$ are used as basis vectors $\mathbf{\Phi}_r$ in $\mathbf{so}(n)$. The minors $\langle i_1 j_1, \ldots, i_n j_n \rangle(X)$ and $\{i_1 j_1, \ldots, i_n j_n\}(X)$ represent the coordinates of multivectors $\mathbf{X}_!$ and $\mathbf{X}^!$, respectively.

There is a very simple connection between the multivectors $\mathbf{X}_!$ and $\mathbf{X}^!$. To find it, we note that the summands of complementary degrees in formula (3.6) are related by the Hodge isomorphism

$$\mathcal{H}_p : \bigwedge^p(\mathbf{Sym}(n)) \longrightarrow \bigwedge^{N-p}(\mathbf{Sym}(n)).$$

In particular, the spaces of skew-symmetric tensors of degrees n and M are isomorphic, because $M = N - p$. With the orientation of $\mathbf{Sym}(n)$ that we have chosen the following relation holds:

$$\begin{gathered}\mathcal{H}_n(\mathbf{\Gamma}_{11} \wedge \mathbf{\Gamma}_{22} \wedge \cdots \wedge \mathbf{\Gamma}_{nn}) = \\ = \mathbf{\Gamma}_{12} \wedge \cdots \wedge \mathbf{\Gamma}_{1n} \wedge \mathbf{\Gamma}_{23} \wedge \cdots \wedge \mathbf{\Gamma}_{2n} \wedge \cdots \wedge \mathbf{\Gamma}_{n-1,n}\end{gathered} \tag{3.7}$$

We shall prove that the Hodge isomorphism $\mathcal{H}_n$ transforms the multivector $\mathbf{X}_!$ into the multivector $\mathbf{X}^!$, that is

$$\mathcal{H}_n(\mathbf{X}_!) = \mathbf{X}^! \tag{3.8}$$

First of all, relations (3.1), (3.3) and (3.4) imply that

$$\mathbf{\Gamma}^\Box_{ii}(\mathbf{\Delta}_{ij}) = -\mathbf{\Gamma}_{ij}, \quad \mathbf{\Gamma}^\Box_{jj}(\mathbf{\Delta}_{ij}) = \mathbf{\Gamma}_{ij}, \quad \mathbf{\Gamma}^\Box_{kk}(\mathbf{\Delta}_{ij}) = \mathbf{O}, \tag{3.9}$$

if $i < j$ and $k \neq i, j$.

Further, we can choose a basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of $\mathcal{L}$ in which the matrix of $\mathbf{X}$ is diagonal:

$$X = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$$

Then for any $q \geq 0$

$$\mathbf{X}^q = \lambda_1^q \mathbf{\Gamma}_{11} + \lambda_2^q \mathbf{\Gamma}_{22} + \cdots + \lambda_n^q \mathbf{\Gamma}_{nn},$$

and, therefore,

$$\mathbf{X}_! = W(\lambda_1, \ldots, \lambda_n) \mathbf{\Gamma}_{11} \wedge \mathbf{\Gamma}_{22} \wedge \cdots \wedge \mathbf{\Gamma}_{nn}$$

Finally, from (3.9) it follows that

$$\mathbf{X}^{\square}(\mathbf{\Delta}_{ij}) = (\lambda_j - \lambda_i) \mathbf{\Gamma}_{ij},$$

whence

$$\mathbf{X}^! = \prod_{i<j} (\lambda_j - \lambda_i) \times \mathbf{\Gamma}_{12} \wedge \cdots \wedge \mathbf{\Gamma}_{1n} \wedge \mathbf{\Gamma}_{23} \wedge \cdots \wedge \mathbf{\Gamma}_{2n} \wedge \cdots \wedge \mathbf{\Gamma}_{n-1,n}.$$

Here the coefficient of the exterior product is the Vandermonde determinant $W(\lambda_1, \ldots, \lambda_n)$. The equality (3.8) now follows from (3.7).

In an arbitrary coordinate system multivectors $\mathbf{X}_!$ and $\mathbf{X}^!$ have representations

$$\mathbf{X}_! = \sum \langle i_1 j_1, \ldots, i_n j_n \rangle (X) \mathbf{\Gamma}_{i_1 j_1} \wedge \cdots \wedge \mathbf{\Gamma}_{i_n j_n}$$

$$\mathbf{X}^! = \sum \{ i_1 j_1, \ldots, i_n j_n \} (X) \mathbf{\Gamma}_{k_1 l_1} \wedge \cdots \wedge \mathbf{\Gamma}_{k_M l_M},$$

where $k_1 l_1, \ldots, k_M l_M$ is the set of double indexes complementary to the set $i_1 j_1, \ldots, i_n j_n$ in the sequence (1.5). By Hodge's isomorphism we have

$$\mathcal{H}_n(\mathbf{\Gamma}_{i_1 j_1} \wedge \cdots \wedge \mathbf{\Gamma}_{i_n j_n}) = \pm \mathbf{\Gamma}_{k_1 l_1} \wedge \cdots \wedge \mathbf{\Gamma}_{k_M l_M},$$

and equality (2.7) follows from (3.8). Due to (2.7), the minors $\{ i_1 j_1, \ldots, i_n j_n \}(X)$ can also be called discriminant polynomials.

§4. Proof of the main theorems

Proof of Theorem 1. A matrix $X \in \operatorname{Sym}(n)$ will be called *regular*, if all its eigenvalues are distinct. The set of all regular matrices will be denoted by $\mathfrak{R}$. Matrices belonging to $\operatorname{Sym}(n) \setminus \mathfrak{R}$ will be referred to as *critical*.

Let X be a regular matrix. Then there is an orthogonal matrix C such that

$$C^{-1} X C = L = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n), \tag{4.1}$$

and all numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ are pairwise different. By our assumptions

$$C^{-1} \mathcal{F}(X) C = M = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_n)$$

Diagonal matrices $E, L, \dots, L^{n-1}$ are linearly independent, therefore diagonal matrix M can be (uniquely) represented as their linear combination:

$$M = a_0 E + a_1 L + \dots + a_{n-1} L^{n-1},$$

whence

$$\mathcal{F}(X) = a_0 E + a_1 X + \dots + a_{n-1} X^{n-1}.$$

If the matrix X is regular, then the coefficients $a_0, a_1, \dots, a_{n-1}$ are uniquely determined by the elements of X.

In what follows, by smoothness of a function we understand its infinite smoothness. Let us prove that functions $a_0(X), a_1(X), \dots, a_{n-1}(X)$ are smooth on the set $\mathfrak{R}$. Denote the elements of the matrix $\mathcal{F}(X)$ by $\theta_{ij}(X)$. For any matrix $X \in \mathfrak{R}$ and any collection of double indices $i_1 j_1, i_2 j_2, \dots, i_n j_n$ the equality

$$\begin{pmatrix} x^{(0)}_{i_1 j_1} & x^{(1)}_{i_1 j_1} & \dots & x^{(n-1)}_{i_1 j_1} \\ x^{(0)}_{i_2 j_2} & x^{(1)}_{i_2 j_2} & \dots & x^{(n-1)}_{i_2 j_2} \\ \vdots & \vdots & \ddots & \vdots \\ x^{(0)}_{i_n j_n} & x^{(1)}_{i_n j_n} & \dots & x^{(n-1)}_{i_n j_n} \end{pmatrix} \begin{pmatrix} a_0(X) \\ a_1(X) \\ \vdots \\ a_{n-1}(X) \end{pmatrix} = \begin{pmatrix} \theta_{i_1 j_1}(X) \\ \theta_{i_2 j_2}(X) \\ \vdots \\ \theta_{i_n j_n}(X) \end{pmatrix},$$

is fulfilled. The right hand member of this equality is defined and smooth on all of $\mathrm{Sym}(n)$. If $i_r < j_r$, we multiply line number r by $\sqrt{2}$. Multiplying after that both parts of the equality on the left by the matrix adjoined to the matrix in the left hand side, we obtain, for all $m = 0, 1, \dots, n-1$, the following relations:

$$\langle i_1 j_1, \dots, i_n j_n \rangle (X) a_m(X) = \varphi_{i_1 j_1, \dots, i_n j_n, m}(X), \tag{4.2}$$

where all functions $\varphi_{i_1 j_1, \dots, i_n j_n, m}$ are defined and smooth on the entire space $\mathrm{Sym}(n)$.

Let $X_0 \in \mathfrak{R}$. In the point X_0, matrices $E, X, \dots, X^{n-1}$ are linearly independent, that is the rank of matrix $X_!$ is exactly n. Therefore, $X_!$ contains a minor $\langle i_1 j_1, \dots, i_n j_n \rangle (X)$, nondegenerate in the point X_0. There is a neighbourhood $\Omega \ni X_0$, such that everywhere in Ω we have

$$\langle i_1 j_1, \dots, i_n j_n \rangle (X) \neq 0$$

This, together with (4.2), implies that $a_m(X) \in C^\infty(\Omega)$ for all m.

Functions $a_m(X)$ smoothly extend to the critical set. Indeed, as we saw before, for any $m = 0, 1, \dots, n-1$ and any collection of double indices $k_1 l_1, \dots, k_n l_n$ the function $f(X)$, defined on the set $\mathfrak{R}$ by the equality

$$f(X) = \langle k_1 l_1, \dots, k_n l_n \rangle (X) a_m(X),$$

can be smoothly prolonged to the entire space $\mathrm{Sym}(n)$. Therefore, for any linear combination of the minors $\langle k_1 l_1, \dots, k_n l_n \rangle (X)$

$$P(X) = \sum P_{k_1 l_1, \dots, k_n l_n} \langle k_1 l_1, \dots, k_n l_n \rangle (X),$$

where all $P_{k_1 l_1,\dots,k_n l_n}$ are constants, the function $g(X) = P(X)a_m(X)$ is smooth on $\mathrm{Sym}(n)$. Let $Q(X)$ be another linear combination and $h(X) = Q(X)a_m(X)$. Obviously,

$$h(X)P(X) = g(X)Q(X) \tag{4.3}$$

in all points of $\mathfrak{R}$. As the critical matrices make in $\mathrm{Sym}(n)$ a set of measure zero and functions $g(X)$, $h(X)$ are smooth everywhere, equality (4.3) holds on the whole set $\mathrm{Sym}(n)$. Now, if we could prove, proceeding from (4.3), that $g(X)$ is divisible by $P(X)$ in a neighbourhood of a critical point X_0 and the quotient is a smooth function, then we would be able to smoothly prolong the function $a_m(X)$ to a neighbourhood of X_0. To establish this fact, under a special choice of polynomials $P(X)$ and $Q(X)$, we will need the Malgrange theorem [4].

THE MALGRANGE THEOREM. *Let Ω be an open set in R^N and $f_1, \dots, f_r$ be analytical functions in Ω. Let $\varphi \in C^\infty(\Omega)$. Then φ is representable as*

$$\varphi = \sum_{i=1}^{r} f_i \psi_i,$$

where $\psi_i \in C^\infty(\Omega)$, if and only if for every $a \in \Omega$ the Taylor expansion $T_a\varphi$ belongs to the ideal generated by the expansions $T_a f_i$ in the ring of formal power series in the point a.

The Malgrange theorem is about the division of smooth functions by analytical functions. In the present investigation, analytical functions will always be polynomials. Therefore we shall touch upon the problem of division of formal power series by a polynomial. A polynomial of several variables can be uniquely represented as a sum of homogeneous polynomials. The one that has the smallest degree, is usually called the *initial form* of the given polynomial.

We shall use the following sufficient condition of divisibility. Let P_1 and P_2 be polynomials in several variables, while R_1 and R_2 be formal power series in the same variables, such that $R_1P_2 = R_2P_1$. Suppose that the initial forms of polynomials P_1 and P_2 are coprime. It is easy to prove that then the polynomials P_1 and P_2 must be coprime in the ring of formal power series. Since this ring is factorial, the series R_1 is divisible by P_1, and the series R_2 by P_2.

Here we encounter the problem about the degrees of the initial forms of discriminant polynomials, decomposed into a finite Taylor series in a neighbourhood of a critical point X_0. Formula (1.8) suggests that for the solution of this problem it is important to find the multiplicity of a zero of the discriminant $\mathrm{Dis}(\chi_X)$ in a critical point.

Below, we will use the following notion. An integer sequence $(m_1, m_2, \dots, m_k)$ is the *root type* of a matrix $X \in \mathrm{Sym}(n)$, if X has exactly k different eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$, and m_i is the multiplicity of λ_i. Of course, the sum $m_1 + m_2 + \dots + m_k$ is always equal to n. We say that two root types $(m_1, m_2, \dots, m_k)$ and $(r_1, r_2, \dots, r_l)$ are equivalent, if $k = l$ and the two sequences differ only by a permutation.

Further, we will say that the root type $(r_1, r_2, \dots, r_l)$ is obtained from the root type $(m_1, m_2, \dots, m_k)$ by an elementary operation, if $l = k - 1$ and there are integers α, β: $1 \le \alpha < \beta \le k$, that the root type

$$(m_1, \dots, m_{\alpha-1}, m_\alpha + m_\beta, m_{\alpha+1}, \dots, m_{\beta-1}, m_{\beta+1}, \dots, m_k)$$

is equivalent to $(r_1, r_2, \ldots, r_l)$. Finally, a root type A precedes a root type B, if B can be obtained from A by means of several elementary operations.

Evidently, the root type $(1, 1, \ldots, 1)$ of a regular matrix precedes any other, and any root type precedes (n), the root type of a scalar matrix.

Let $X_0 \in \mathrm{Sym}(n)$ be a critical matrix of root type $(m_1, m_2, \ldots, m_k)$. We are interested in the behaviour of linear combinations of discriminant polynomials in the neighbourhood of X_0 and the multiplicity of the zero of the discriminant $\mathrm{Dis}(\chi_X)$ in the point X_0. The matrix X_0 can be reduced to diagonal form by a conjugation $Y_0 = CX_0C^{-1}$. Transformation (1.6) gives a one-to-one correspondence between the neighbourhoods of points X_0 and Y_0. Viewing $\mathrm{Sym}(n)$ as a Euclidean space of dimension N this transformation can be written as $\widehat{X} = C^{\wedge}\widehat{Y}$, where $C^{\wedge}$ is an orthogonal matrix of order N. Equality (1.6) implies that for any q we have $X^q = C^{-1}Y^qC$, or in another notation, $\widehat{X^q} = C^{\wedge}\widehat{Y^q}$. Setting $q = 0, 1, \ldots, n-1$, we get the transformation rule

$$X_! = C^{\wedge}Y_!$$

This, together with the Binet–Cauchy formula, shows that polynomials $\langle i_1j_1, \ldots, i_nj_n\rangle(X)$ are constant coefficient linear combinations of polynomials $\langle i_1j_1, \ldots, i_nj_n\rangle(Y)$ and vice versa. As the discriminant is invariant with respect to similarity transformations, we can assume that X_0 is a diagonal matrix

$$X_0 = \begin{pmatrix} L_1 & & & \\ & L_2 & & \\ & & \ddots & \\ & & & L_k \end{pmatrix}, \qquad \text{where} \qquad L_\alpha = \begin{pmatrix} \lambda_\alpha & & \\ & \ddots & \\ & & \lambda_\alpha \end{pmatrix} \tag{4.4}$$

is a scalar matrix of dimension m_α. Now it is easy to prove (using, for example, Gershgorin's circles [2]), that the zero multiplicity in question is equal to 2ν, where

$$\nu = C^2_{m_1} + C^2_{m_2} + \ldots + C^2_{m_k} \tag{4.5}$$

Taking into account (1.8), we conclude that any discriminant polynomial has, in the point X_0, a zero of order greater than or equal to ν.

Return to the proof of the theorem. We have estimated from below the degrees of the initial forms of discriminant polynomials expanded in the Taylor series in a neighbourhood of any critical point. Now we can construct polynomials $P(X)$ and $Q(X)$ with relatively prime initial forms for equality (4.3).

As we already did earlier, we shall divide the set of variables x_{ij} into two groups: diagonal variables x_{ii} and nondiagonal variables x_{ij} with $i < j$. Suppose that $R(Z, W)$ is a homogeneous polynomial of degree M depending on two groups of variables $Z = (z_1, z_2, \ldots, z_k)$ and $W = (w_1, w_2, \ldots, w_l)$. Then it can be uniquely represented as

$$R(Z, W) = \sum_{s+t=M} R_{s,t}(Z, W),$$

where each polynomial $R_{s,t}(Z,W)$ is homogeneous of degree s in variables Z and homogeneous of degree t in variables W. The nonzero summand $R_{s,t}(Z,W)$ with the greatest s will be referred to as the *highest part of $R(Z,W)$ with respect to variables Z*. If two homogeneous polynomials $R_1(Z,W)$ and $R_2(Z,W)$ have coprime highest parts with respect to Z, then the polynomials themselves are coprime, too.

Denote by $\Delta K(X)$ the highest part of a homogeneous polynomial $K(X)$ with respect to diagonal variables. For any m, elements $x_{ij}^{(m)}$ of the matrix X^m are homogeneous polynomials of degree m. It is easy to see that

$$\Delta x_{ii}^{(m)} = x_{ii}^m, \qquad \Delta x_{ij}^{(m)} = x_{ij}\frac{x_{jj}^m - x_{ii}^m}{x_{jj} - x_{ii}} \tag{4.6}$$

This implies the formula

$$\Delta\langle 11, 22, \ldots, nn\rangle(X) = W(x_{11}, x_{22}, \ldots, x_{nn}), \tag{4.7}$$

where W is the Vandermonde determinant. Further, let $1 \le i < j \le n$ and denote by $P_{ij}(X)$ the determinant obtained from the determinant $\langle 11, 22, \ldots, nn\rangle(X)$, if its row jj is replaced by the row ij. Due to (4.6) we have

$$\Delta P_{ij}(X) = \frac{x_{ij}}{x_{jj} - x_{ii}} W(x_{11}, x_{22}, \ldots, x_{nn}) \tag{4.8}$$

Now let X_0 be a critical matrix. Let's prove, that the coefficients $a_m(X)$ can be continued to a neighbourhood of X_0 with smoothness preserved. The proof proceeds by induction on the root type of X_0. The induction base is the root type $(1, 1, \ldots, 1)$ of regular matrices. Suppose that such prolongation is possible for all points, whose root type precedes that of X_0.

We can assume that X_0 is a diagonal matrix (4.4). Choose a neighbourhood Ω of the point X_0 such that the root type of all its points precedes or is equivalent to the type of X_0, and set $X = X_0 + H$. The initial form in the Taylor expansion of the polynomial $K(X)$ in the point X_0 will be denoted by $K_{X_0}(H)$. Further, we will say that a pair of indices (i, j) is admissible, if there exists an integer t such that

$$m_0 + m_1 + \cdots + m_t + 1 \le i < j \le m_0 + m_1 + \cdots + m_{t+1}$$

(we assume that $m_0 = 0$, $0 \le t \le n - 1$). Setting

$$Q(X) = \langle 11, 22, \ldots, nn\rangle(X)$$

in equality (4.3), we can see that $Q(X)$ has in the point X_0 a zero of order ν, defined by (4.5). By (4.7), the highest part of the initial form $Q_{X_0}(H)$ with respect to diagonal variables is

$$\Delta Q_{X_0}(H) = B \prod (h_{jj} - h_{ii}),$$

where $B \ne 0$ is a constant depending only on X_0, and the product is taken over all admissible pairs.

In the same way, if (i,j) is an admissible pair, then by (4.8) the polynomial $P_{ij}(X)$ in a vicinity of X_0 satisfies

$$\Delta P_{ij,X_0}(H) = \frac{h_{ij}}{h_{jj}-h_{ii}} \Delta Q_{X_0}(H).$$

Therefore, if we put, in equality (4.3),

$$P(X) = \sum P_{ij}(X),$$

with the summation going over all admissible pairs, we will obtain

$$\Delta P_{X_0}(H) = \Delta Q_{X_0}(H) \sum \frac{h_{ij}}{h_{jj}-h_{ii}},$$

which shows that polynomials $\Delta P_{X_0}(H)$ and $\Delta Q_{X_0}(H)$ are coprime. Then the initial forms $P_{X_0}(H)$ and $Q_{X_0}(H)$ are coprime, too, and in the point X_0 the formal series $T_{X_0}g(X)$ is divisible by $T_{X_0}P(X)$.

Furthermore, in all points of the neighbourhood Ω, whose root type precedes the root type of X_0, by the inductive assumption the coefficient $a_m(X)$ is smooth function and hence the formal series for $g(X)$ is divisible by the expansion of $P(X)$.

Finally, we have to consider those points in the neighbourhood Ω whose root type is equivalent to that of X_0. In these points the polynomials $P(X)$ and $Q(X)$ have zero of order no smaller than ν. If the neighbourhood Ω is small enough, the forms of degree ν of polynomials $P(X)$ and $Q(X)$ expanded in any point of Ω, will be arbitrarily close to the corresponding forms of these polynomials expanded in the point X_0. Therefore, first, $P(X)$ and $Q(X)$ in the points of Ω of equivalent root type have zero of order exactly ν, and second, the initial forms of the expansions of $P(X)$ and $Q(X)$ in these points are coprime. Therefore in these points the formal expansion of $g(X)$ is divisible by the expansion of $P(X)$.

The statement that close enough forms are coprime, requires some clarification. Let $\alpha_X(H)$ and $\beta_X(H)$ be two forms of degree ν in variables H whose coefficients are continuous functions in variables X. If for some value of X these forms were not relatively prime, then we would be able to find nonzero forms $\gamma(H)$ and $\delta(H)$ of degree $\nu - 1$, such that

$$\alpha_X(H)\delta(H) = \beta_X(H)\gamma(H) \tag{4.9}$$

The last equality is a system of linear equations on the coefficients of the forms $\gamma(H)$ and $\delta(H)$. For $X = X_0$ this system has only trivial solution, that is its rank is not less than the number of unknowns. The matrix of system (4.9) continuously depends on X. Therefore for X, close enough to X_0, its rank is also not less than the number of unknowns. It follows that system (4.9) has only a trivial solution, and the forms $\alpha_X(H)$ and $\beta_X(H)$ are coprime.

So, in any point X of a certain neighbourhood $\Omega \ni X_0$ the formal series $T_X g$ belongs to the ideal generated by the series $T_X P$. By the Malgrange theorem the function $g(X)$ is divisible by $P(X)$ in the domain Ω, and the quotient is a smooth function. We have thus proved that any coefficient $a_m(X)$ can be extended to a neighbourhood of the point X_0 as a smooth function.

As X_0 is an arbitrary critical point, the functions $a_m(X)$ prolong to the entire critical set. This completes the proof of theorem 1.

Of course, Theorem 1 remains valid in the case, when we have a function $\mathcal{F}$ defined only on a subset of the space $\mathrm{Sym}(n)$.

PROOF OF THEOREM 2. Theorem 2 is proved in a similar way, using equality (2.7). The only essential difference is in the proof of the induction base, which we now shall fulfill. Let X be a regular matrix. Then there is an orthogonal matrix C, such that equality (4.1) holds. Let

$$C^{-1}\mathcal{G}(X)C = T = (t_{ij}),$$

where all $t_{ii} = 0$. We can choose a skew-symmetric matrix B, such that

$$T = BL - LB$$

Indeed, the elements b_{ij} of B are uniquely determined by relations $t_{ij} = (\lambda_j - \lambda_i)b_{ij}$.

Applying an appropriate similarity transformation, we obtain (0.2), where $A(X) = CBC^{-1}$. Thus, on the regular set $\mathfrak{R}$ the matrix $A(X)$ exists and is unique. Relation (0.2) can be written in matrix–vectorial form

$$\widehat{\mathcal{G}(X)} = X^{!}\overline{A(X)},$$

and we see that the prolongation of the matrix elements $a_{ij}(X)$ of $A(X)$ to the critical set can be constructed in the same way as above, using the discriminant polynomials $\{i_1j_1, \ldots, i_nj_n\}(X)$.

Acknowledgments. The author is grateful to his teacher A. S. Mischenko for the statement of the problems and to P. P. Mosolov, A. T. Fomenko, S. V. Duzhin and V. I. Arnold for their interest in this work.

References

[1] *A. I. Lourie.* Teorija uprugosti. Moscow, Nauka, 1970.
[2] *F. R. Gantmacher.* The Theory of Matrices Vol. 1, 2, Chelsea, New York. 1959.
[3] *M. M. Postnikov.* Lectures on geometry. Semester II. Linear algebra. M.: Nauka. 1986.
[4] *B. Malgrange.* Ideals of differentiable functions. London: Oxford University Press, 1966.
[5] *N. V. Ilyushechkin.* On one class of smooth matrix valued functions. Uspehi Mat. Nauk. 1985. v. 40. No. 1.
[6] *N. V. Ilyushechkin.* Discriminant of the characteristic polynomial of a normal matrix. Mathematical Notes. 1992. v. 51. No. 3. p. 230 – 235.
[7] *N. V. Ilyushechkin.* On certain classes of smooth transformations of the space of symmetric matrices. Matem. Sbornik, 1993, v. 184, No. 9.