FIRST ORDER LOCAL INVARIANTS OF APPARENT CONTOURS
PRELIMINARY VERSION

FRANCESCA AICARDI AND TORU OHMOTO

ABSTRACT. For a smooth map from a closed surface to the plane, its critical value set is called *the apparent contour* of the map. In this paper we study *first order local* Vassiliev-type invariants of generic apparent contours.

1. INTRODUCTION

Let M be a connected compact 2-dimensional manifold, and ϕ a smooth map from M to the plane. Singular points of ϕ are the points at which the rank of the differential $d\phi$ is less than 2, and there are only two generic local types of ϕ at singular points : folds and cusps. The image Φ of the singular point set of ϕ is called the *apparent contour* of ϕ, and Φ is called *generic* if ϕ is a C^∞-stable map, that is, the map ϕ has only generic singular points and the self-intersection of Φ are transverse. Namely, a generic apparent contour is a collection of closed curves with cusp-points and transverse (self-)intersections.

The apparent contour is oriented according to the following rule: going along the curve according to its orientation, the number of preimages of the mapping is higher at the left side.

Remark 1.1. *A generic collection of wave fronts is not necessarily an apparent contour. Indeed, each wave front of the collection has the property that each cusp is of type $\vee$ (in the direction of the orientation). However, the $\vee$-property of the cusps is not sufficient to guarantee that a wave-fronts collection is an apparent contour.*

Let $\mathcal{M}$ be the mapping space $C^\infty(M, \mathbb{R}^2)$ and Γ the *discriminant*, i.e., the set of all C^∞ unstable mappings in $\mathcal{M}$. We are interested in the numerical invariants of path-connected components of the complement $\mathcal{M} - \Gamma$.

Remark 1.2. *We can assign to an apparent contour an index, i.e., the sum of numbers of projective rotation numbers of each components of the apparent contour (rotations of the normal (non oriented) direction to components). This number is constant in $\mathcal{M}$. In fact, it is equal to the Euler characteristic of the closed surface M (cf.* [9] *).*

In the following sections, we will divide Γ into several strata accoding to local types of singularities, and determine certain first order invariants (cf. [2], [1], [6], [11]). Our main result is the following theorem :

Theorem 1.3. *For generic apparent contours of maps from a closed surface to the plane, there are exactly three linearly independent first order local invariants I_1, I_2, I_3 of apparent contours, modulo constant functions. These three invariants are expressed by the number*

Date: October 19, 2002.

of cusps, the number of double points and a kind of the Bennequin invariant associated to the apparent contours.

We will also consider the special cases : the case that the surface and the plane are oriented and the case of the apparent contours of the generic projection of immersed surfaces in $\mathbb{R}^3$.

2. The codimension one strata of the discriminant

Here we denote each local stratum by a capital letter and we cooriente it by a local rule. Then we denote by the corresponding small letter a function, which jumps by 1 at every positive crossing of this stratum (i.e. a crossing in the direction of the coorientation), and is constant at the crossing of any other stratum. These jumps are called *elementary jumps*.

One can associate to each region of the complement to the plane of an apparent contour the number of the preimages of f. In the regions involved in a codimension one degenerate point the preimages consist of a set of generic regular sheets plus a set of sheets necessary for the occurrence of that event. We shall refer to such preimages as *necessary* preimages.

There are 10 codimension one local strata of the discriminant (see figure 1).

1) Stratum L (from lips). The coorientation is toward the region where the number of cusps increases: $\Delta l = 1$.

2) Stratum B (from beak-to-beak). The coorientation is toward the region where the number of cusps increases: $\Delta b = 1$.

3) Stratum S (from swallow-tail). The coorientation toward the region where the number of cusps increases: $\Delta s = 1$.

4) Stratum K_0 (from kasanie=tangency). The orientations of the two tangent branches are opposite. The new-born region between the two curves after the tangency has 0 necessary preimages. The coorientation of this stratum is toward the region where the number of double points increases: $\Delta k_0 = 1$.

5) Stratum K_1. The orientations of the two branches are concordant. The new-born region between the two curves after the tangency has 2 necessary preimages. The coorientation of this stratum is toward the region where the number of double points increases: $\Delta k_1 = 1$.

6) Stratum K_2. The orientations of the two tangent branches are opposite. The new-born region between the two curves after the tangency has 4 necessary preimages. The coorientation of this stratum is toward the region where the number of double points increases: $\Delta k_2 = 1$.

7) Stratum T_0 (from triple point). The vanishing triangle has 0 necessary preimages (The three orienting arrows exiting from the triple point are directed toward three alternating vertices of a hexagon). The coorientation of this stratum is toward the region where the number of preimages of the new-born triangle is higher: $\Delta t_0 = 1$.

8) Stratum T_1. The vanishing triangle has 2 necessary preimages (The three orienting arrows exiting from the triple point are directed toward three adjacent vertices of a hexagon). The coorientation of this stratum is toward the region where the number of preimages of the new-born triangle is higher: $\Delta t_1 = 1$.

9) Stratum C_0 (from cusp-crossing). The new born triangular region with a cuspidal vertex has three necessary preimages. The coorientation of this stratum is toward the region where the number of double points increases: $\Delta c_1 = 1$.

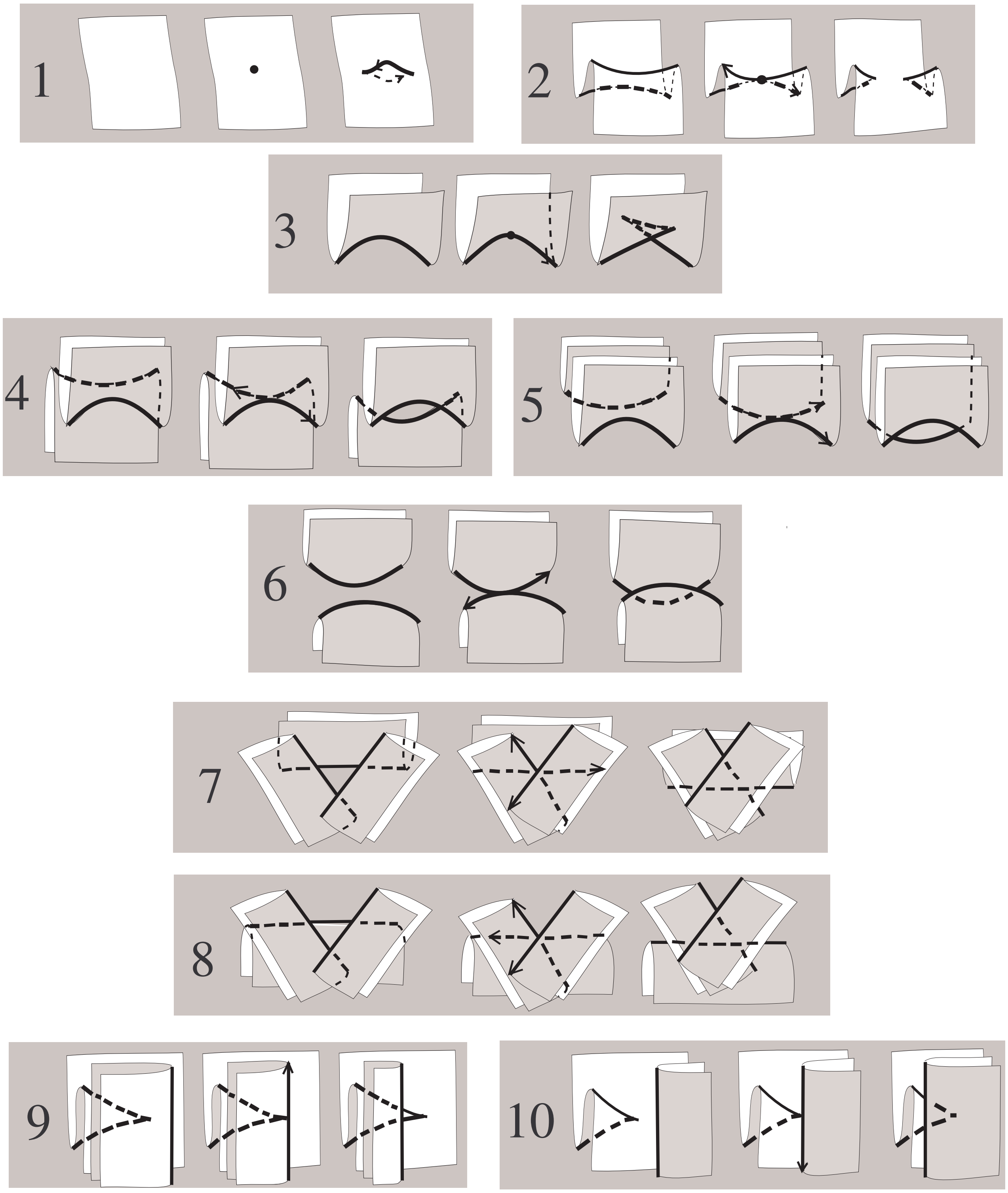

Figure 1

10) Stratum C_1. The new born triangular region with a cuspidal vertex has five necessary preimages. The coorientation of this stratum is toward the region where the number of double points increases: $\Delta c_2 = 1$.

3. The codimension two strata

Codimension two strata, which consist of transverse intersections of two local strata of codimension one are called trivial. The coorientability condition at these strata is trivially fulfilled.

Non trivial codimension two strata correspond to apparent contours with a single degenerate point of codimension two. Such a point is the image under ϕ of one up to four different points of M. The germ of ϕ at these degenerate points is thus called mono, bi, tri, and quadrigerm.

Following [5] (cf. [10]), there are two additional singularities of codimension 2 (with monogerms) besides the 18 degenerate points here considered. They correspond to the *deltoid* and the *sharksfin* with versal unfolding $(x^2 - y^2 + x^3 + ax,\ xy + bx)$ and $(x^2 + y^3 + ay,\ y^2 + x^3 + bx)$ respectively. These two singularities, which do not appear in the case of projections on a plane of a surface imbedded in $\mathbb{R}^3$, is realized by the projections on the plane of the two images of the standard Whitney umbrella $zy^2 = x^2$, when the line $(x = 0, z = 0)$ is replaced by a suitably generic curve tangent to the line (e.g., $(x = y^3, z = \pm y^2)$) respectively. Gibson and Hobbs claim that the analysis of the bifurcations in the (a, b) plane is very complicated. It however turns out that these strata do not contribute to the coherence system (see the following section). Indeed, for instance, the deltoid singularity is the appearance of a new triangular component (with three cusps) in the apparent contour : this component exists in any path linked with the codimension two stratum of this singularity, and vanishes-reappears on a non generic path crossing this stratum.

The codimension two strata are indicated by the roman numbers I-XVIII (see figure 2). The strata I, II and III correspond to the singularities 5, 8 and 7 in the Goryunov-Arnold classification in [3], [4]. The strata IV and V correspond to the cod. 1 singularities 2 and 1 in the previous section, where the apparent contour is transvesely crossed by a smooth fold-branch. The strata II, VI-VIII were considered in the study of planar wave fronts [2] together with their versal unfolding.

Here we analyze all the cases distinguished by the possible local coorientations of the codimension one strata defined in the above section.

4. The coherence system

For each codimension 2 stratum we write the condition that the variation of an invariant I, whose jump is a linear combination of elementary jumps, vanishes in a small closed path C linked with that stratum. The generic linear combination is

$$\Delta I = \lambda \Delta l + \beta \Delta b + \sigma \Delta s + \gamma_0 \Delta k_0 + \gamma_1 \Delta k_1 + \gamma_2 \Delta k_2 + \tau_0 \Delta t_0 + \tau_1 \Delta t_1 + \chi_0 \Delta c_0 + \chi_1 \Delta c_1.$$

The conditions for all codimension two strata produce a system of linear equations of valuables $\lambda, \gamma_0, \cdots, \chi_1$, which we call *the coherence system.*

Lemma 4.1. *The codimension two strata corresponding to corank 2 singularities of type sharksfin and deltoid do not contribute to the coherent system.*

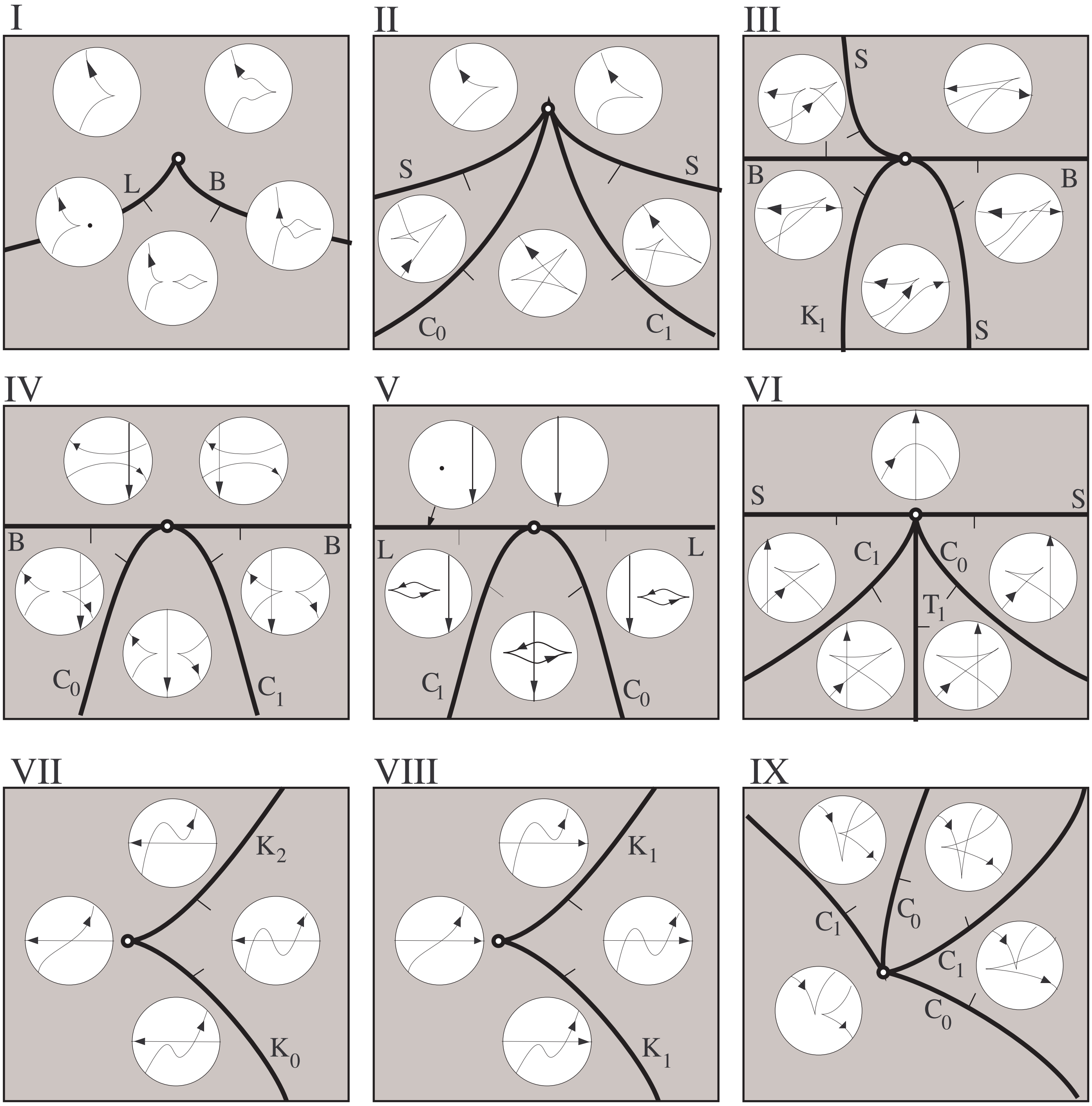

FIGURE 2. A

Proof. At first let us consider codimension two stratum corresponding to the sharksfin, i.e., the mono-germ of corank 2, whose minversal deformation is given as $G(x, y, a, b) = (x^2 + y^3 + ay, y^2 + x^3 + bx)$. We take a representative of the deformation $G : U \times W \to V$ where U and V are neighborhood of the origin of source and target, W a nighborhood of the origin in the parameter space (a, b). Set $G_{(a,b)} = G(*, a, b) : U \to V$. Note that there

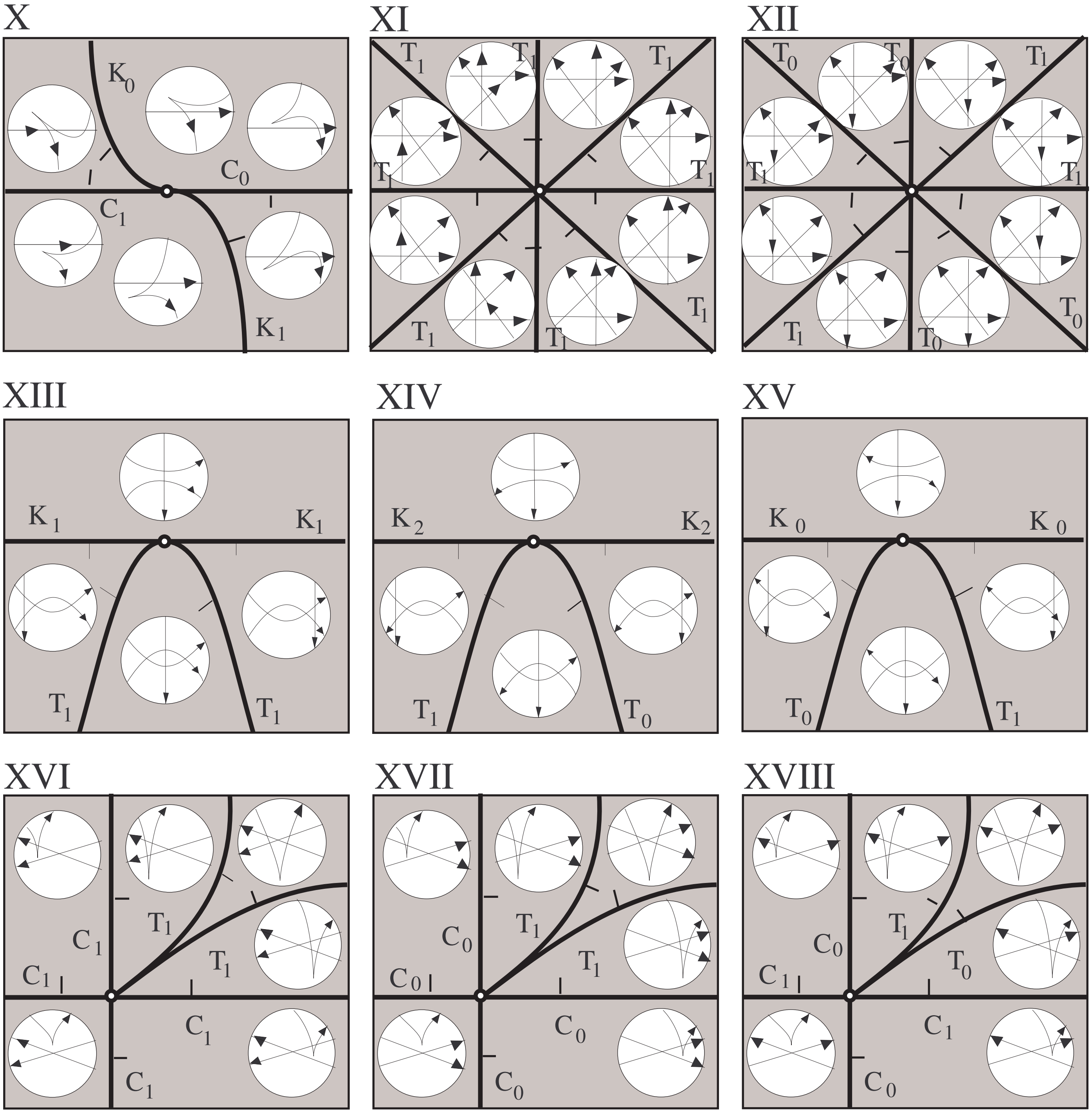

FIGURE 2. B

is $\mathbb{Z}_2$-symmetry of G caused by the parameter change $h : (a, b) \to (b, a)$: namely, the map $G_{(a,b)}$ is equivalent to $G_{h(a,b)}$: $G_{h(a,b)} = \tau \circ G_{(a,b)} \circ \phi$ where $\phi(x, y) = (y, x), \tau(u, v) = (v, u)$. Over the diagonal (a, a) off the origin the map G is regular. Take a small circle C in the parameter space W going anticlockwise around the origin. Then, for each stratum X of codimension one ($X = L, B, \cdots$), its jump Δx along the arc c of C in the area $b > a$ is the same as the jump along the arc $h \circ c$ in the area $b < a$. Hence the total jump Δx along C vanishes.

For the deltoid singularity, whose miniversal deformation is given by $G'(x, y, a, b) = (x^2 - y^2 + x^3 + ay, xy + bx)$, we can also show the same thing as above. (Besides, there are no codimension one strata adjacent to this stratum, cf. [5], [10] .) $\square$

Remark 4.2. *If we fix the orientation of source and target of the map G, we can consider positive and nagative cusps according to the mapping degree. In the proof of Lemma, the coordinate change (ϕ, τ) do not change the degree of cusps. Thus, even if we consider the case that the source manifold M is orientable, that is done in the section 8 , the codimension two strata corresponding to corank 2 singularities do not effect to the coherent system.*

¿From the above lemma, we only check codimension two strata (I) -(XVIII) corresponding to the multi-singlarities of corank less than 2. The coherence system consists of the following equations in the ten variables λ, β, σ, γ_0, γ_1, γ_2, τ_0,τ_1, χ_0, χ_1:
(I) $\lambda - \beta = 0$
(II) $\sigma + \chi_0 - \chi_1 - \sigma = 0$
(III) $\sigma - \beta - \gamma_1 + \sigma - \beta = 0$
(IV) $\beta - \chi_0 + \chi_1 - \beta = 0$
(V) $\lambda + \chi_1 - \chi_0 - \lambda = 0$
(VI) $\sigma + \chi_1 + \tau_1 - \chi_0 - \sigma = 0$
(VII) $\gamma_2 - \gamma_0 = 0$
(VIII) $\gamma_1 - \gamma_1 = 0$
(IX) $\chi_0 + \chi_1 - \chi_0 - \chi_1 = 0$
(X) $\gamma_0 - \chi_1 + \gamma_1 - \chi_0 = 0$
(XI) $\tau_1 + \tau_1 + \tau_1 + \tau_1 - \tau_1 - \tau_1 - \tau_1 - \tau_1 = 0$
(XII) $\tau_1 + \tau_0 + \tau_0 + \tau_1 - \tau_1 - \tau_0 - \tau_0 - \tau_1 = 0$
(XIII) $\gamma_1 - \tau_1 - \tau_1 - \gamma_1 = 0$
(XIV) $\gamma_2 + \tau_1 - \tau_0 - \gamma_2 = 0$
(XV) $\gamma_0 - \tau_0 + \tau_1 - \gamma_0 = 0$
(XVI) $\chi_1 + \chi_1 + \tau_1 - \tau_1 - \chi_1 - \chi_1 = 0$
(XVII) $\chi_0 + \chi_0 + \tau_1 - \tau_1 - \chi_0 - \chi_0 = 0$
(XVIII) $\chi_1 + \chi_0 - \tau_1 - \tau_0 - \chi_1 - \chi_0 = 0$

Theorem 4.3. *Any one-cocycle, i.e. any function I satisfying $\Delta I = 0$ on every generic contractile closed path in $\mathcal{M}$ is defined, up an additive constant, by a linear combination of the following independent basic jumps:*

1. $\Delta I_1 = \Delta l + \Delta b + \Delta s$
2. $\Delta I_2 = \Delta s + 2\Delta k_1 + \Delta c_0 + \Delta c_1$
3. $\Delta I_3 = 2\Delta k_0 + 2\Delta k_2 + \Delta c_0 + \Delta c_1$

Proof. The dimension of the solution of the coherence system is three. The three basic jumps above fulfill the system and are independent.

Remark 4.4. *These cocycles are cohomologous to zero. This is proved by finding explicit integral formulae for three independent linear combinations of I_1, I_2 and I_3.*

Remark 4.5. *In the same way as above, we may consider strata of higher codimension, and built up corresponding coherent systems, that makes a cochain complex (the so-called Vassiliev complex , cf. Kazarian, [8]). Instead of first order local invariants of apparent contours of generic maps $M^2 \to \mathbb{R}^2$, the higher codimensional cocycle represent some certain characteristic classes of surface bundles. Let $\eta = (E, p, B)$ be an oriented surface bundle, that is, an C^∞ fibre bundle $p : E \to B$ where B is an oriented C^∞ manifold B and the fibre is diffeomorphic to an oriented closed surface M^2. Consider a suitably generic map f of the total space E to $\mathbb{R}^2$. As a simple example, we can see the following : Let $\Sigma_2(f)$ be the bifurcation set consisting of points x in B at which the restriction of f to the fibre $E_x(= p^{-1}(x))$ has corank 2 singularities (the top strata of $\Sigma_2(f)$ corresponds to the singularity types of the deltoid and the sharksfin of $f|_{E_x}$). Then, it turns out that $\Sigma_2(f)$ forms a cocycle of codimension 2 in B which represents $e_1(\eta) \in H^2(B;\mathbb{Z})$, that is the first tautological class (the first Morita-Mumford class) of the surface bundle η.*

5. The invariants

Theorem 5.1. *There are exactly three linearly independent first order local invariants of apparent contours. They are*

1. *ν, the number of cusps, whose jump in terms of basic jumps is:*
$$\Delta\nu = 2\Delta I_1 = 2\Delta l + 2\Delta b + 2\Delta s;$$
2. *d, the number of double points, whose jump in terms of basic jumps is: $\Delta d = \Delta I_2 + \Delta I_3$, i.e.*
$$\Delta d = \Delta s + 2\Delta k_0 + 2\Delta k_1 + 2\Delta k_2 + 2\Delta c_0 + 2\Delta c_1;$$
3. *ℓ, whose interpretation is explained in term of the Legendrian link, and whose jump in terms of basic jumps is $\Delta\ell = \Delta I_1 - \Delta I_2 + \Delta I_3$, i.e.*
$$\Delta\ell = \Delta l + \Delta b + 2\Delta k_0 - 2\Delta k_1 + 2\Delta k_2.$$

Proof. The jumps of ν and d are evidently the jumps of the number of cusp points and of the number of double points respectively. As for ℓ, the geometrical interpretation needs some definitions.

Let us associate to each point of the apparent contour in the (x, y) plane the nonoriented straight line orthogonal to it, i.e. we lift the apparent contour in $PT^*\mathbb{R}^2$ (with contact form $\cos\theta dx + \sin\theta dy$), obtaining a Legendrian link F. This Legendrian link is the union of several Legendrian smooth knots F_i: $F = F_1 \cup F_2 \cup \ldots \cup F_m$. Each Legendrian knot F_i is projected to the base onto a connected component Φ_i of the apparent contour Φ.

Each Legendrian knot F_i is provided with the contact framing. We consider each framed knot F_i as a knotted strip. The boundary of the strip consists of a closed curve γ_i obtained from F_i displacing each point of F_i by ϵ in the two opposite directions along the normal to the contact plane at that point (i.e. of the straight line, lifting of the normal to the apparent contour at the corresponding point of the plane). The curve γ_i has either two components (i.e. the strip is a ring), if index(Φ_i) is integer, or one component (i.e. the strip is a Möbius band), if index(Φ_i) is semiinteger.

Let us forget the contact structure of the space $PT^*\mathbb{R}^2$, and consider each F_i as a framed knot in the solid torus. To such a knotted strip we associate a topological invariant, by means of the following definitions. We shall consider later the relation of this invariant with the invariant ℓ.

5.1. **The self-linking number.** The generic projection to a plane of a knotted strip $(F_i \cup \gamma_i)$ consists of a curves f_i, and a curve g_i, projections of F_i and γ_i respectively.

Let p be a self-intersection point of f_i. Let q be a self-intersection point of g_i, such that the branches of γ_i projecting to the branches of g_i intersecting at q are the two parallel boundary curves of the same piece of strip above the point q. Let z_a and z_b be the points of F_i or (of γ_i) which projects onto p (onto q). We assign to each point p (q) a sign σ with the rule: $\sigma(p) = 1$ $(\sigma(q) = 1)$ if the unit tangent vectors t_a and t_b of F_i above p (of γ_i above q) satisfy: $t_a \wedge t_b \cdot (z_a - z_b) > 0$. Otherwise $\sigma(p) = -1$ $(\sigma(q) = -1)$. Notice that in generic position $t_a \wedge t_b \cdot (z_a - z_b) \neq 0$.

Lemma 5.2. *The integer*

$$B_i = \sum_k \sigma(p_k) + \frac{1}{2}\sum_j \sigma(q_j)$$

(the indices k and j run over all the double points of type p and of type q of the projection respectively) is called the self-linking number of the framed knot F_i.

5.2. **The linking number.** Let us consider now our collection of Legendrian curves F_i as a link in the solid torus (i.e. we forget not only the contact structure but also the framing). Let f_i's be a generic projections of the F_i's to a plane.

Let $L_{i,j}$ be twice the linking number of F_i with F_j. Notice that

$$L_{i,j} = \sum_k \sigma(r_k^{ij}),$$

where the index k runs over all the double points of the projection where one branch belongs to f_i and the other one to f_j.

Lemma 5.3. *The integer $BL = \sum_i B_i + \sum_{i>j} L_{i,j}$ is a topological invariant of the Legendrian knot $F = \bigcup_i F_i$.*

Proof. In fact each B_i and each $L_{i,j}$ are topological invariants of the framed link. One checks that under every Reidemeister moves for the links and their generalizations for the framed knots B_i and $L_{i,j}$ are constant.

Lemma 5.4. *The invariant ℓ of the apparent contour satisfies (for a suitable choice of the additive constant) $\ell = BL$.*

Proof. We have to prove that at every crossing of the discriminant Γ, where ℓ varies, the Legendrian link undergoes a topological changing, implying an equal variation of BL. In the figures, F_i is black and γ_i is gray.

1) Positive crossing of the stratum L: $\Delta l = 1$, i.e. $\Delta\ell = 1$. The lifting of the new born lips curve is a new component F_i of the Legendrian link, unlinked with the other ones. A generic projection of this Legendrian link is an eight curve (see figure 3). There is a points of type p with positive sign and two points of type q with opposite signs. Hence $B_i = 1$, and $L_{i,j} = 0$ $\forall j$. BL increases by 1, as ℓ.

2) Positive crossing of the stratum B: $\Delta b = 1$, i.e. $\Delta\ell = 1$ (see figure 4). There are two cases: a) the two branches before the crossing belong to two different components or b) to the same component. Case a). Let F_i and F_j be such components: a double point r_{ij} of the projections f_i and f_j disappears. Its sign is equal to -1 (see figure 4). After the crossing, F_i and F_j form an unique component, say F_n, whose projection is unchanged with respect to the $f_i \cup f_j$ except for the death of the double point r_{ij} and the birth of two

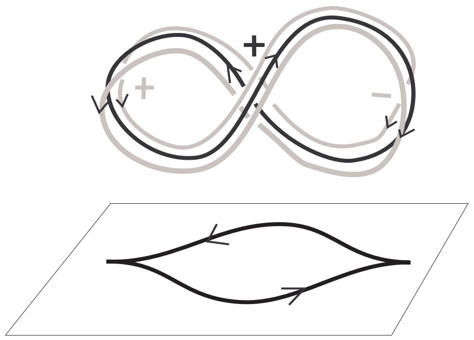

FIGURE 3

points of type q with opposite sign in the new component. Thus we have $L_{k,n}=L_{k,i}+L_{k,j}$ for every $k \neq i,j$; $B_n = B_i + B_j + L_{i,j} + 1$. Therefore BL increases by 1, as ℓ. Case b) a self-intersection point of the projection of F_i disappears, i.e. a self-intersection point of type p of the projection of f_i with sign -1 disappear. The component F_i splits into two components F_m and F_n and we have: $B_i = B_m + B_n + L_{m,n} + 1$. The two new born points of type q with opposite sign belong now one to F_m and the other one to F_n and they contribute to the B_m and B_n by $+1/2$ and $-1/2$ respectively. Therefore in total BL increases by 1, as ℓ.

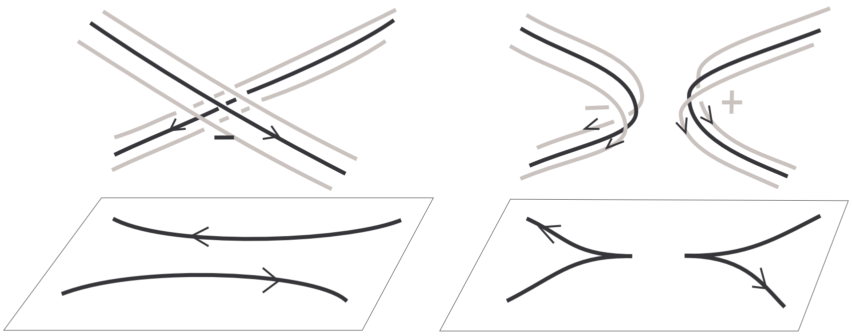

FIGURE 4

3) Positive crossing of the stratum S: $\Delta\ell = 0$. As we show in figure 5, after the swallow-tail crossing in a component F_i, there are only two new points of type q in the projection of g_i. They have opposite sign, hence BL remains unchanged, as ℓ.

4) Positive crossing of the stratum K_0: $\Delta k_0 = 1$ and $\Delta\ell = 2$. During the crossing a double point of the projection of F_i and F_j (see figure 6) changes sign from -1 to $+1$ (so $L_{i,j}$ increases by 2), or a double points of type p of the projections of a sole component F_i changes sign from -1 to $+1$ (so B_i increases by 2). In both cases BL increases by 2, as ℓ.

5) Positive crossing of the stratum K_1. $\Delta k_1 = 1$ and $\Delta\ell = 2$. During the crossing a double point of the projection of F_i and F_j (see figure 7) changes sign from $+1$ to -1 (so $L_{i,j}$ decreases by 2), or a double points of type p of the projections of a sole component F_i changes sign from $+1$ to -1 (so, B_i decreases by 2). In both cases BL decreases by 2, as ℓ.

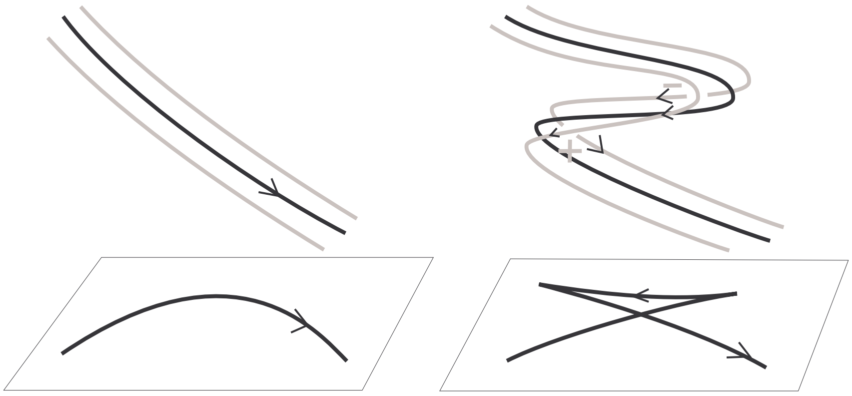

FIGURE 5

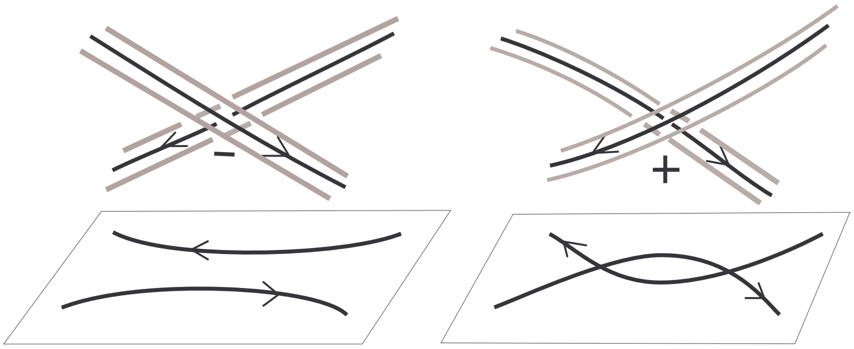

FIGURE 6

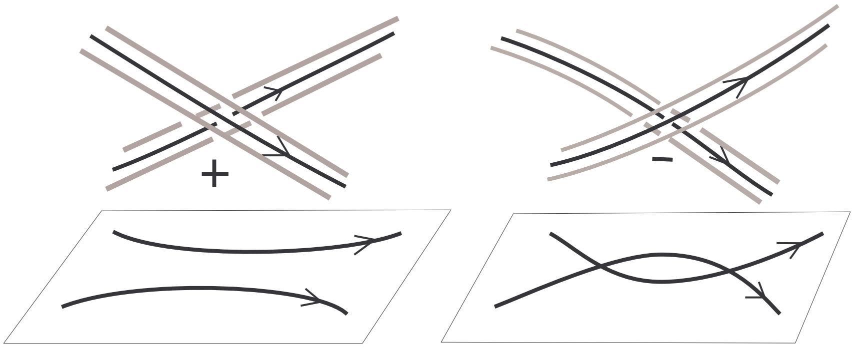

FIGURE 7

6) Positive crossing of the stratum K_2. $\Delta k_2 = 1$ and $\Delta \ell = 2$. During the crossing a double point of the projection of F_i and F_j (see figure 8) changes sign from -1 to $+1$ (so $L_{i,j}$ increases by 2), or a double point of type p of the projections of a sole component F_i changes sign from -1 to $+1$ (so, B_i increases by 2). In both cases BL increases by 2, as ℓ.

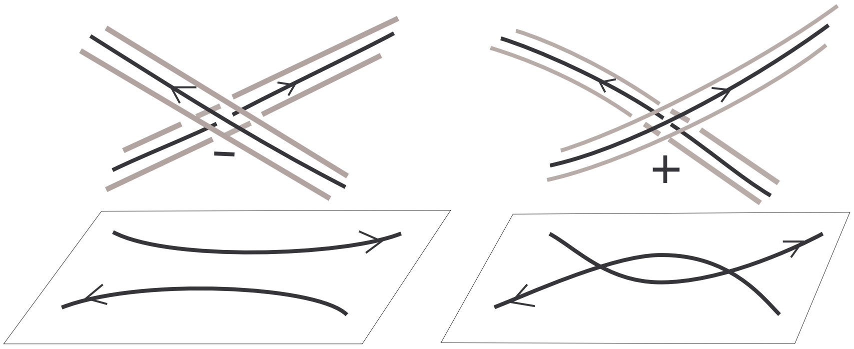

FIGURE 8

8) Positive crossing of the strata T_0 and T_1: $\Delta\ell = 0$. Since the liftings of the three moving branches have different coordinate θ, nothing happens to the isotopy class of the link F, and $\Delta\ell = 0$.

9)Positive crossing of the strata C_0 and C_1: $\Delta\ell = 0$. Like in the case 8) nothing happens to the isotopy class of the link F, and $\Delta\ell = 0$.

5.3. **BL as the Bennequin invariant.** One may consider the double covering $\tilde{F} = \cup_i \tilde{F}_i$ in $ST^*\mathbb{R}^2$ of the Legendrian link $F = \cup_i F_i$ in $PT^*\mathbb{R}^2$.

Let $\tilde{F}_\epsilon$ be a shift of $\tilde{F}$ along the positive normal direction to the contact plane. Let $Bq(\tilde{F})$ the Bennequin invariant of $\tilde{F}$, i.e. the linking number of the two links $\tilde{F}$ and $\tilde{F}_\epsilon$:

$$\begin{aligned} Bq(\tilde{F}) &= \sum_{i,j} lk(\tilde{F}_i, \tilde{F}_{j_\epsilon}) \\ &= \sum_i lk(\tilde{F}_i, \tilde{F}_{i\epsilon}) + \sum_{i\neq j} lk(\tilde{F}_i, \tilde{F}_{j_\epsilon}) \\ &= \sum_i lk(\tilde{F}_i, \tilde{F}_{i\epsilon}) + 2\sum_{i>j} lk(\tilde{F}_i, \tilde{F}_{j_\epsilon}). \end{aligned}$$

It is easily seen that $lk(\tilde{F}_i, \tilde{F}_{i\epsilon}) = 2B_i$ and $lk(\tilde{F}_i, \tilde{F}_{j_\epsilon}) = 2lk(F_i, F_j) = L_{i,j}$ for $i \neq j$, and hence $Bq(\tilde{F}) = 2BL(F)$. Hence the invariant BL is nothing but one half of the Bennequin number of $\tilde{F}$.

5.4. **The additive constant.** Let Φ and Ψ be the apparent contours of the mappings $\phi : M \to \mathbb{R}^2$ and $\psi : N \to \mathbb{R}^2$ respectively, such that $\Phi \cap \Psi = \emptyset$ and they are separeted by a line in the plane. Now we consider the additivity of invariants under any connected sum of apparent contours Φ and Ψ defined as the apparent contour of the connected sum of maps ϕ and ψ defined over the connected sum $M\sharp N$. The invariants ν and d, i.e., the number of cusps and of double points, are obviously additive under the connected sum. To reguire ℓ to be additive, we have to think of constants which normalizes the value of ℓ.

It follows from the previous argument that for each surface M there is a constant c_M such that $\ell = BL + c_M$. At first we have to normalize it in such a way that its value is zero on the apparent contour of the simplest mapping of the sphere into the plane, i. e. on a positively oriented circle. Let us calculate BL of a circle c (see figure 9). We draw a generic projection to a plane of the corresponding knotted strip in the solid torus: there are two points of type q with positive sign and a points of type p, with positive sign. Hence $BL(c) = 2$. So c_{S^2} should be 2.

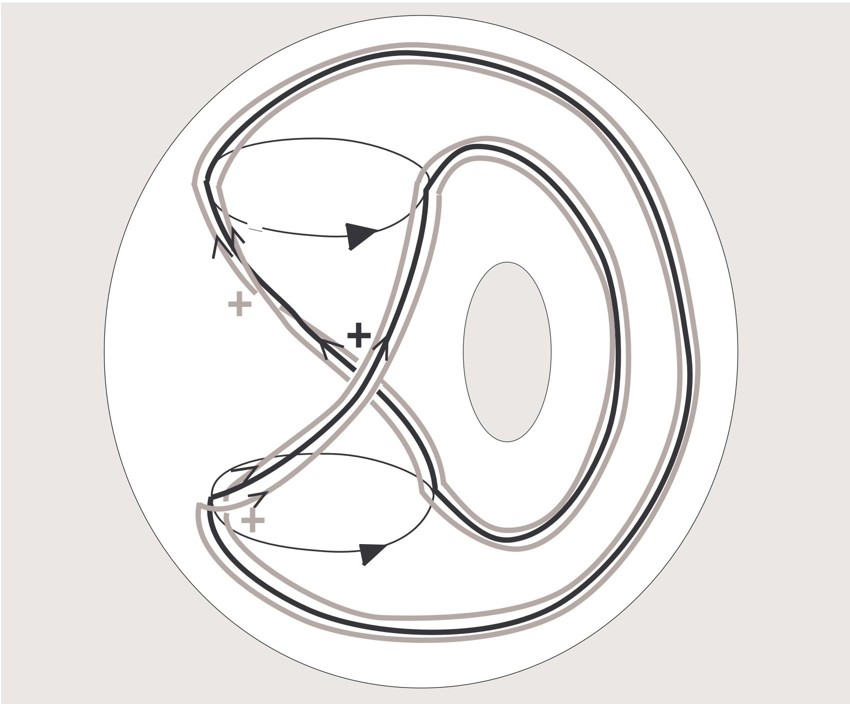

FIGURE 9

Moreover, it generally holds that

$$BL(F \sharp F') = BL(F) + BL(F') - 2.$$

Thus, we have

$$\ell(\Phi \sharp \Psi) = \ell(\Phi) + \ell(\Psi) - 2 + c_{M\sharp N} - c_M - c_N.$$

So it is reasonable to set all $c_M = 2$, otherwise, $c_M = -\chi(M)$, where $\chi(M)$ is the Euler characteristic of M. The difference of $\ell(\Phi)$ which two choice of constants makes is equal to the projective rotation number of Φ minus 2 (Remark 1.2), which can be regared as a 0-th order invariant.

Example

In figure 10 we show an example of the calculation of the ℓ invariant for an apparent contour using its differential definition; i.e. by the calculation of its jumps on the codimension one strata along an arbitrary generic path connecting it to an apparent contour with known ℓ.

The knotted strip in $PT^*\mathbb{R}^{\nvDash}$ associated to the curve Φ_f of figure 10f (the apparent contour of the Boy surface, image of an immersion in $\mathbb{R}^{\nvDash}$ of the projective plane) is rather complicated. So, we start by the calculation of ℓ for the apparent contour Φ_a of figure 10a: Let Φ_C and Φ_T be the two components of Φ_a corresponding to the circle and to the triangle, and F_C, F_T their lifting in $PT^*\mathbb{R}^{\nvDash}$. We have $\ell(\Phi_a) = \ell(\Phi_C) + \ell(\Phi_T) + lk(F_C, F_T)$.

Since $\ell(\Phi_C) = 2$, $\ell(\Phi_T) = 1/2$ and $lk(F_C, F_T) = -2$, we obtain $\ell(\Phi_a) = 1/2$. Connecting Φ_a to Φ_f along the path shown in figure 10, we cross 3 times, from (a) to (b), the stratum S, where ℓ does not jump. From (c) to (d) we cross 3 times the stratum B, and at each crossing ℓ jumps by -1. So the value of the invariant ℓ on the curve 10f is $\ell(\Phi_f) = 1/2 - 3 = -5/2$.

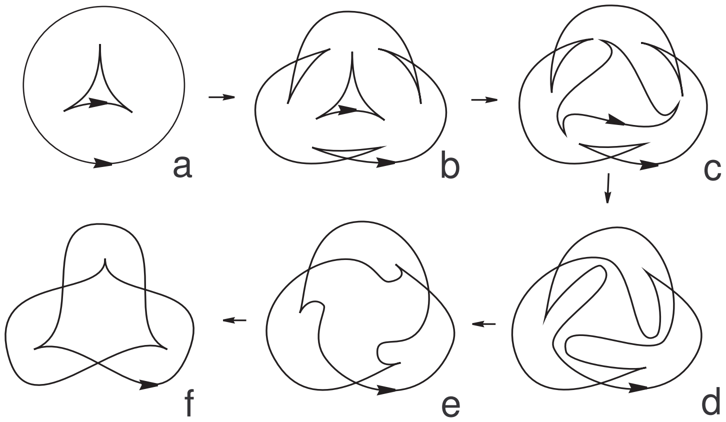

FIGURE 10. Figure 10

6. A NON-LOCAL FIRST-ORDER INVARIANT

The codimension one stratum B of the discriminant can be subdivided into two disjoint strata: B_1, if the new-born cusps belong to the same component of the apparent contour; B_2, if the new-born cusps belong to two different components.

The coherence system, written now with the new variables β_1 and β_2 replacing β, has to be modified only because of codimension two strata I, III and IV.

Stratum I. The stratum B can be only of type B_2. Hence the equation (I) becomes
(I') $\lambda - \beta_2 = 0$

Stratum III. If the stratum B at left is of type B_1, the stratum at right is of type B_2 and vice-versa. The equation (III) becomes in both cases:
(III') $\sigma - \beta_1 - \gamma_1 + \sigma - \beta_2 = 0$

Stratum IV. The strata B at left and at right are both of type B_1, or both of type B_2. So equation (IV) becomes
(IV') $\beta_1 - \chi_0 + \chi_1 - \beta_1 = 0$,
(IV") $\beta_2 - \chi_0 + \chi_1 - \beta_2 = 0$,
These two equations are trivially fulfilled by β_1 and β_2. The number of independent equations of the new coherence system is the same as before the splitting of the stratum B, hence the dimension of the solution increases by one. A new independent solution is in fact given by $\{\beta_2 = \lambda,\ \beta_1 = -\beta_2, \sigma = \gamma_0 = \gamma_1 = \gamma_2 = \tau_0 = \tau_1 = \chi_0 = \chi_1 = 0\}$, corresponding to the jump

$$\Delta I_4 = \Delta l - \Delta b_1 + \Delta b_2.$$

This is exactly the jump of the *number of components* of the apparent contour.

Remark 6.1. *This invariant is non local in the sense that to know the type (B_1 or B_2) of the local stratum B one needs to know the global shape of the apparent contour.*

7. The splitting of the invariants ν and d

A cusp point is said of type n if in its neighborhood the minimal number of preimages is n. We denote $\nu(n)$ the number of cusp points of type n. For compact M, $n \geq 2$

A double point is said of type n if in its neighborhood the minimal number of preimages is $n-2$. We denote $d(n)$ the number of double points of type n. $n \geq 2$.

We denote by:

$B[n]$ and $L[n]$ the strata B and L where the two cusps are of type n.

$S[n]$ the stratum S where the double point is of type n. (Notice that the new born cusps are of type n too).

$K_i[n]$ $(i = 0, 1, 2)$ the stratum K_i where new born double points are of type n.

$T_0[n]$ the stratum T_0 where three points of type n disappear (and three points of type $n+2$ do appear) during the positive crossing.

$T_1[n]$ the stratum T_1 where a point of type n disappear (and a point of type $n+2$ appears) during the positive crossing.

$C_0[n]$ the stratum C_0 where the new born double points are of type n.

$C_1[n]$ the stratum C_1 where the new born double points are of type n.

Proposition 7.1. *$d[n]$ and $\nu[n]$ $(n \geq 2)$ are the only order one independent additive invariants of the apparent contours obtained by the above splitting of the discriminant, with jumps:*

$$\begin{aligned}\Delta d[n] &= \Delta s[n] + 2\sum_{i=0}^{3} \Delta k_i[n] + 2\sum_{j=0}^{2} \Delta c_j[n] \\ &\qquad -3\Delta t_0[n] + 3\Delta t_0[n-2] - \Delta t_1[n] + \Delta t_1[n-2],\end{aligned}$$

and

$$\begin{aligned}\Delta \nu[n] &= +2\Delta L[n] + 2\Delta B[n] + 2\Delta s[n] \\ &\qquad -\Delta c_0[n] + \Delta c_0[n+2] + \Delta c_1[n] - \Delta c_1[n+2].\end{aligned}$$

Notice that in the case $n = 2$ both $\Delta t_0[n-2]$ and $\Delta t_1[n-2]$ are equal to zero, being the strata $T_i(0)$ undefined. Of course

$$d = \sum_{k \geq 1} d[2k], \qquad \nu = \sum_{k \geq 1} \nu[2k].$$

8. The special case of orientable surfaces

Theorem 8.1. *There are four linearly independent local first order invariants of mappings of an oriented compact surface to the plane.*

Proof. A cusp point V in the oriented plane of an apparent contour Φ is the image of a pleat point C of M. If M is oriented, we call the cusp V positive (negative) if the mapping degree at C is equal to 1 (-1). The cusp points of an apparent contour are thus subdivided into positive and negative ones. We prove that the the positive and negative cusps are in fact local first order invariants.

The codimension one strata involving cusp points are also distinguished, namely:

L^+ and L^- will denote the birth of lips with two positive and two negative cusps respectively (at the lips event the cusps have the same sign): $\Delta l = \Delta l^+ + \Delta l^-$.

B^+ and B^- will denote the beak-to-beak with two positive and two negative cusps respectively (at the beak-to-beak event the cusps have the same sign): $\Delta b = \Delta b^+ + \Delta b^-$.

S^+ (S^-) will denote the swallow-tail where the second cusp in the order given by the orientation is positive (respectively, negative). The two cusps born in the swallow-tail event have opposite signs: $\Delta s = \Delta s^+ + \Delta s^-$.

C_i^+ and C_i^- $(i = 0, 1)$ will denote the cusp crossing when the cusp is positive (respectively, negative). $\Delta c_i = \Delta c_i^+ + \Delta c_i^-$.

The coherence system is now a system of linear equations with 14 variables whose rank is equal to ten (see Remark 4.2). In this case four independent solutions for the basic jumps are for example:

1. $\Delta I_1' = \Delta l^+ + \Delta b^+ + \Delta s$
2. $\Delta I_2' = \Delta l^- + \Delta b^- + \Delta s$
3. $\Delta I_3' = \Delta s + 2\Delta k_1 + \Delta c_0^+ + \Delta c_1^- + \Delta c_1^+ + \Delta c_1^-$
4. $\Delta I_4' = 2\Delta k_0 + 2\Delta k_2 + \Delta c_0^+ + \Delta c_1^- + \Delta c_1^+ + \Delta c_1^-$

We have: $\Delta I_1 = \Delta I_1' + \Delta I_2'$, whereas $\Delta I_2 = \Delta I_3'$ and $\Delta I_3 = \Delta I_4'$. The only new invariant is the number of positive cusps ν^+, $(\nu^- = \nu - \nu^+)$, whose jump is equal to $\Delta I_1'$.

9. Generic projections of an oriented immersed surface in $\mathbb{R}^3$

Every generic apparent contour can be realized by the apparent contour of a certain composite map of a generic map $\alpha : M \to \mathbb{R}^3$ and the fixed coordinate projection $p : \mathbb{R}^3 \to \mathbb{R}^2$. A theorem of Haefliger [7] implies that for an orientable surface M, a generic apparent contour Φ is realized by the projection of immersed surface $\alpha(M)$ in $\mathbb{R}^3$ if and only if there is an even number of cusps in each component of Φ. In this case the immersed surface $\alpha(M)$ in $\mathbb{R}^3$ does not contain Whitney umbrellas, and the orientation of M induces a coorientation of the apparent contour.

In this case the apparent contour is a collection of cooriented and oriented wave fronts. One can apply to this situation the results on the local first order invariants of wave fronts [1].

In order to avoid the long classification of all local strata of codimension one and two, we try to obtain some results by useful arguments.

Fist of all, as in the non-coorientable case, the generalization to the wave fronts of the Arnold strangeness is not defined in the case of apparent contours because of the impossibility of coorientate the stratum of triple points according the order of visit of the branches (the order among components being non defined).

The second important remark, is that in this case an invariant of wave fronts, with zero jump on triple points, will be generalized to an invariant of apparent contour if its behavior at the crossing of the new strata of type B and L is well defined.

Moreover, the cusps of the apparent contour have a sign, which is not defined for wave fronts, whereas the apparent contour has no cusps of type $\wedge$.

Add statements on p-invariants of wavefronts.

References

[1] Aicardi, F., *Dicsriminants and Local Invariants of Planar Fronts,* The Arnold-Gelfand math. seminors : geometry and singularity theory, edited by V.I.Arnold etc, Birkhäuser, 1997, 1–76

[2] Arnol'd, V. I. *Invariants and Perestroikas of Plane Fronts,* Proceedings of the Steklov Institute of Mathematics, Vol. 209, 1995, p. 11-56 (Translated from Trudy Matematicheskogo Instituta imeni V.V. Steklova, 209 (1995), 14-64).

[3] Arnol'd, V. I. *Singularities of caustics and wave fronts,* Mathematics and its applications, vol. 62, Kluwer (1991).

[4] Arnol'd, V. I. , Vassiliev, V., A., Goryunov, V. V. and Lyashko, O. V. *Dynamical Systems, IV,* Encyclopaedia of Mathematical Sciences, EMS, Springer-Verlag, Vol 39, (1988)

[5] Gibson, C. G. and Hobbs, C. A., *Singularities of general two-dimensional planer motions,* New Zealand Jour. Math., vol. 25 (1996), 141–163

[6] Goryunov, V. V., *Local invariants of mappings of surfaces into three-space,* The Arnold-Gelfand math. seminors : geometry and singularity theory, edited by V.I.Arnold etc, Birkhäuser (1997), 223–255

[7] Haefliger, A., *Quelques remarques sur les applications différentibles d'une surface dans le plan,* Ann. Inst. Fourier (Grenoble) 10 (1960), 47–60

[8] Kazarian, M. É., *Singularities of functions on the circle and relative Morse theory,* Reports, Dept. Math., University of Stockholm, No.9 (1995)

[9] Levine, H., *Elimination of cusps,* Topology, vol. 3 (1965), pp. 263–296

[10] Rieger, J. H. and Ruas, M. A. S., *Classification of $\mathcal{A}$-simple germs from k^n to k^2,* Compositio Math., vol. 79 (1991), pp.99–108

[11] Vassiliev, V. A., *Cohomology of knot spaces,* Adv. Soviet. Math., vol. 21, AMS, Providence, RI. (1990), pp. 23–69