Algebras of Curvature Forms on Homogeneous Manifolds

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March 4, 1998

Abstract

Let $\mathrm{C}(X)$ be the algebra generated by the curvature two-forms of standard holomorphic hermitian line bundles over the complex homogeneous manifold $X = G/B$. The cohomology ring of X is a quotient of $\mathrm{C}(X)$. We calculate the Hilbert polynomial of this algebra. In particular, we show that the dimension of $\mathrm{C}(X)$ is equal to the number of independent subsets of roots in the corresponding root system.

1 Homogeneous Manifolds

In this section we remind the reader the basic notions and notation related to homogeneous manifolds G/B and root systems, as well as fix our terminology.

Let G be a connected complex semisimple Lie group and B its Borel subgroup. The quotient space $X = G/B$ is then a compact homogeneous complex manifold. We choose a maximal compact subgroup K of G and denote by $T = K \cap B$ its maximal torus. The group K acts transitively on X. Thus X can be identified with the quotient space K/T.

By $\mathfrak{g}$ we denote the Lie algebra of G and by $\mathfrak{h} \subset \mathfrak{g}$ its Cartan subalgebra. Also denote by $\mathfrak{g}_{\mathbb{R}} \subset \mathfrak{g}$ the real form of $\mathfrak{g}$ such that $i\,\mathfrak{g}_{\mathbb{R}}$ is the Lie algebra of K. Analogously, $\mathfrak{h}_{\mathbb{R}} = \mathfrak{h} \cap \mathfrak{g}_{\mathbb{R}}$ and $i\,\mathfrak{h}_{\mathbb{R}}$ is the Lie algebra of the maximal torus T. The *root system* associated with $\mathfrak{g}$ is the set Δ of nonzero vectors (roots) $\alpha \in \mathfrak{h}^*$ for which the root spaces

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$$

are nontrivial. Then $\mathfrak{g}$ decomposes into the direct sum of subspaces

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

For $\alpha \in \Delta$, the spaces $\mathfrak{g}_\alpha$ and $\mathfrak{h}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ are one-dimensional and there exists a unique element $h_\alpha \in \mathfrak{h}_\alpha$ such that $\alpha(h_\alpha) = 2$. The elements $h_\alpha \in \mathfrak{h}$ are called *coroots*. Actually, $\alpha \in \mathfrak{h}_{\mathbb{R}}^*$ and $h_\alpha \in \mathfrak{h}_{\mathbb{R}}$, for $\alpha \in \Delta$. Let us choose generators $e_\alpha \in \mathfrak{g}_{\mathbb{R}}$ of the root spaces $\mathfrak{g}_\alpha$ such that $[e_\alpha, e_{-\alpha}] = h_\alpha$ for any root α. Then $[h_\alpha, e_\alpha] = 2e_\alpha$ and $[h_\alpha, e_{-\alpha}] = -2e_{-\alpha}$.

The root system Δ is subdivided into a disjoint union of sets of positive roots Δ_+ and negative roots $\Delta_- = -\Delta_+$ such that the direct sum $\mathfrak{b} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$ is the Lie algebra of Borel subgroup B. The Weyl group W is the group generated by the reflections $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$, $\alpha \in \Delta_+$, given by

$$s_\alpha : \lambda \mapsto \lambda - \lambda(h_\alpha)\, \alpha.$$

The lattice $\hat{T} = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_\alpha) \in \mathbb{Z} \text{ for all } \alpha \in \Delta\}$ is called the *weight lattice*. Every weight $\lambda \in \hat{T}$ determines an irreducible unitary representations $\pi_\lambda : T \to \mathbb{C}^\times$ of the maximal torus T given by $\pi_\lambda(\exp(x)) = e^{\lambda(x)}$, for $x \in i\,\mathfrak{h}_{\mathbb{R}}$, and every irreducible unitary representation of T is of this form.

For a weight $\lambda \in \hat{T}$, the homomorphism $\pi_\lambda : T \to \mathbb{C}^\times$ extends uniquely to a holomorphic homomorphism $\bar{\pi}_\lambda : B \to \mathbb{C}^\times$. Thus any $\lambda \in \hat{T}$ determines a holomorphic line bundle $L_\lambda = G \times_B \mathbb{C} = K \times_T \mathbb{C}$ over $X = G/B = K/T$. The line bundle L_λ has a canonical K-invariant hermitian metric.

The classical Borel's theorem [1] describes the cohomology ring $\mathrm{H}^*(X, \mathbb{C})$ of the homogeneous manifold X in terms of generators and relations:

$$\mathrm{H}^*(X, \mathbb{C}) \cong \mathrm{Sym}(\mathfrak{h}^*)/I_W,$$

where I_W is the ideal in the symmetric algebra $\mathrm{Sym}(\mathfrak{h}^*)$ generated by the W-invariant elements without constant term. The natural projection from $\mathrm{Sym}(\mathfrak{h}^*)$ to $\mathrm{H}^*(X, \mathbb{C})$ is the homomorphism that sends a weight $\lambda \in \hat{T}$ to the first Chern class $c_1(L_\lambda)$ of the line bundle L_λ.

The purpose of this article is to extend the cohomology ring $\mathrm{H}^*(X, \mathbb{C})$ to the level of differential forms on X. It is possible to exhibit differential two-forms that represent the Chern classes $c_1(L_\lambda)$ in the de Rham cohomology of the homogeneous manifold X. Recall that for a holomorphic hermitian line bundle $L : E \to X$ there is a canonically associated connection on E. Denote by $\Theta(L)$ the *curvature form* of this connection, which is a differential two-form on X. Then the form $i\,\Theta(L)/2\pi$ represents $c_1(L)$.

In order to construct the curvature forms $\Theta(L_\lambda)$ explicitly, we define the elements $e^\alpha \in \mathfrak{g}_{\mathbb{R}}^*$, $\alpha \in \Delta$, by $e^\alpha(\mathfrak{h}) = 0$ and $e^\alpha(e_\beta) = \delta_{\alpha\beta}$, for any $\beta \in \Delta$. (Here $\delta_{\alpha\beta}$ is Kroneker's delta.) The space of left K-invariant differential one-forms on K can be identified with the dual to its Lie algebra, t.e., with $i\,\mathfrak{g}_{\mathbb{R}}^*$. Thus the elements $i\,e^\alpha$ can be regarded as one-forms on K. The differential two-form on K given by

$$\phi_\alpha = e^\alpha \wedge e^{-\alpha}, \quad \alpha \in \Delta,$$

is invariant with respect to the right translation action of the torus T. Thus ϕ_α produces a two-form on the manifold X, for which we will use the same notation ϕ_α. It is clear from the definition that $\phi_{-\alpha} = -\phi_\alpha$.

The following statement is implicit in [2].

Proposition 1 *For $\lambda \in \hat{T}$, the curvature form of the holomorphic hermitian line bundle L_λ is given by*

$$\Theta(L_\lambda) = \sum_{\alpha \in \Delta_+} \lambda(h_\alpha)\, \phi_\alpha.$$

Let Φ be the algebra generated by the two-forms ϕ_α, $\alpha \in \Delta_+$. The relations in Φ are easy:

$$\phi_\alpha \phi_\beta = \phi_\beta \phi_\alpha, \qquad \phi_\alpha^2 = 0.$$

Thus Φ is a 2^N-dimensional algebra, where $N = |\Delta_+|$. The main object in this paper is the subalgebra of Φ generated by the curvature forms $\Theta(L_\lambda)$.

2 Main Results

Denote by $\mathrm{C}(X)$ the subalgebra in the algebra of differential forms on X that is generated by the curvature forms $\Theta(L_\lambda)$ of line bundles. Obviously, $\mathrm{C}(X)$ has the structure of a graded ring: $\mathrm{C}(X) = \mathrm{C}^0(X) \oplus \mathrm{C}^1(X) \oplus \mathrm{C}^2(X) \oplus \cdots$, where $\mathrm{C}^k(X)$ is the subspace of $2k$-forms in $\mathrm{C}(X)$. In order to formulate our main results about $\mathrm{C}(X)$ we need some extra notation.

Let V be a collection of vectors $v_1, v_2, \ldots, v_N$ in a vector space E. A subset of vectors in V is called *independent* if they are linearly independent in E. By convention an empty subset is independent. Let $\mathrm{ind}(V)$ be number of all independent subsets in V.

A *cycle* is a minimal by inclusion not independent subset. For a cycle $C = \{v_{c_1}, \ldots, v_{c_l}\}$, there is a unique, up to a factor, linear dependence $\lambda_1 v_{c_1} + \ldots + \lambda_l v_{c_l} = 0$ with non-zero λ_i's. Let us fix a linear order $v_1 < v_2 < \cdots < v_N$ of all elements of V. For an independent subset S, a vector $v \in V \setminus S$, is called *externally active* if $S \cup \{v\}$ contains a cycle C and v is the minimal element of C. Let $\mathrm{act}(S)$ be the number of externally active vectors.

roots.

Theorem 1 *The dimension of the algebra* $\mathrm{C}(X)$ *is equal to the number* $\mathrm{ind}(\Delta_+)$ *of independent subsets in the set of positive roots* Δ_+*. Moreover, the dimension of the* k*-th component* $\mathrm{C}^k(X)$ *is equal to the number of independent subsets* $S \subset \Delta_+$ *such that* $k = N - |S| - \mathrm{act}(S)$*, where* $N = |\Delta_+|$*.*

We remark here that, although the number $\mathrm{act}(S)$ of externally active vectors depends upon a particular order of elements in V, the total number of subsets S with fixed $|S| + \mathrm{act}(S)$ does not depend upon a choice of ordering.

We will actually prove a more general result about an arbitrary collection of vectors V. Let us choose an inner product $\langle \cdot, \cdot \rangle$ and an orthogonal basis $\epsilon_1, \ldots, \epsilon_n$ in E. Denote by Φ_N the algebra generated by N commuting generators $\phi_1, \phi_2, \ldots, \phi_N$ with relations $\phi_i^2 = 0$. Let $\mathcal{C}(V)$ be the subalgebra in Φ_N generated by the elements

$$\theta_i = \sum_{j=1}^{N} \langle \epsilon_i, v_j \rangle \ \phi_j, \qquad i = 1, \ldots, n.$$

The algebra $\mathcal{C}(V)$ has an obvious grading $\mathcal{C}(V) = \mathcal{C}^0(V) \oplus \mathcal{C}^1(V) \oplus \mathcal{C}^2(V) \oplus \cdots$ by degree of elements.

If $E = \mathfrak{h}$ and V is the collections of coroots h_α, $\alpha \in \Delta_+$, then $\mathcal{C}(V) = \mathrm{C}(X)$. In general, we have the following result.

Theorem 2 *The dimension of the algebra* $\mathcal{C}(V)$ *is equal to the number* $\mathrm{ind}(V)$ *of independent subsets in* V*. Moreover, the dimension of the* k*-th component* $\mathcal{C}^k(V)$ *is equal to the number of independent subsets* $S \subset V$ *such that* $k = N - |S| - \mathrm{act}(S)$*.*

3 Dual Statement and Proof

Next we give a dual version of Theorem 2. Let $\mathrm{Sym}(E)$ be the symmetric algebra of the vector space E. For a subset $S \subset V$, denote $\langle S \rangle = \prod_{v \in S} v \in \mathrm{Sym}(E)$, a square-free monomial in v_i's. Let us denote by $\mathcal{S}(V)$ a subspace in $\mathrm{Sym}(E)$ spanned by 2^N square-free monomials $\langle S \rangle$, where S ranges over all subsets in V. Let $\mathcal{S}^k(V)$ be the k-th graded component of $\mathcal{S}(V)$.

Theorem 3 *The dimension of the subspace* $\mathcal{S}(V)$ *in* $\mathrm{Sym}(E)$ *is equal to the number* $\mathrm{ind}(V)$ *of independent subsets in* V*. Moreover, the dimension of* $\mathcal{S}^k(V)$ *is equal to the number of independent subsets* $S \subset V$ *such that* $k = N - |S| - \mathrm{act}(S)$*.*

For $I = (1 \le i_1 < i_2 < \ldots < i_k \le N)$ and $J = (1 \le j_1 \le j_2 \le \cdots \le j_k \le n)$, we denote by a_{IJ} the coefficient of the square-free monomial $\phi_{i_1} \cdots \phi_{i_k} \in \Phi_N$ in the product of generators

$\theta_{j_1} \cdots \theta_{j_k} \in \mathcal{C}(V)$. Let $A_k = (a_{IJ})$ be the $\binom{N}{k} \times \binom{n+k-1}{k}$-matrix formed by the a_{IJ}. Clearly, $\dim \mathcal{C}^k(V)$ is equal to the rank of the matrix A_k.

On the other hand, a_{IJ} is also the coefficient of the monomial $\epsilon_{j_1} \cdots \epsilon_{j_k} \in \mathrm{Sym}(E)$ in the square-free monomial $v_{i_1} \cdots v_{i_k} \in \mathcal{S}(V)$. Thus $\dim \mathcal{S}^k(V)$ is equal to the rank of the transposed matrix A_k^T, which is the same as the rank of A_k. This shows that $\dim \mathcal{C}^k(V) = \dim \mathcal{S}^k(V)$ and Theorem 2 is equivalent to Theorem 3.

We say that a subset $S \subset V$ is *robust* if there is no cycle $C \subset V$ with minimal element v such that $S \cap C = \{v\}$.

Lemma 1 *The number of robust subsets in V is equal to the number of independent subsets. Moreover, the number of k-element robust subsets is equal to the number of independent subsets S with $k = N - |S| - \mathrm{act}(S)$.*

Proof — We present an explicit bijection between robust subsets and independent subsets. A subset A is the complement to a robust subset in V if and only if for any cycle C with minimal element v, inclusion $C \setminus \{v\} \subset A$ implies $C \subset A$. We will call such subsets *antirobust.*

For a independent subset S, let M be the collection of all externally active $v \in V \setminus S$. Then $S \cup M$ is antirobust. Conversely, for an antirobust subset A, let M be the collection of all $v \in V$ such that v is a minimal element in some cycle $C \subset A$. Then $A \setminus M$ is an independent subset.

Clearly, both these mapping are inverse to each other and the statement of lemma follows. □

Theorem 4 *The set of square-free monomials $\langle S \rangle$, where S ranges over robust subsets, forms a basis of the subspace $\mathcal{S}(V)$.*

First, we prove a weaker version of Theorem 4.

Lemma 2 *The square-free monomials $\langle S \rangle$, where S ranges over robust subsets, span $\mathcal{S}(V)$.*

Proof — Suppose not. Let $\langle R \rangle$ be the maximal in the lexicographical order square-free monomial which cannot be expressed linearly via the monomials $\langle S \rangle$ with robust S. Then there is a cycle $C = \{v_{c_1}, v_{c_2}, \ldots, v_{c_l}\}$ with the minimal element $v = v_{c_1}$ such that $R \cap C = \{v\}$. We can replace v in the monomial $\langle R \rangle$ by a linear combination of $v_{c_2}, v_{c_3}, \ldots, v_{c_l}$. Thus $\langle R \rangle$ is a linear combination of square-free monomials which are greater than $\langle R \rangle$ in the lexicographical order. By assumption each of these monomials can be expressed via the monomials $\langle S \rangle$ with robust S. Contradiction. □

We can now conclude the proof.

Proof of Theorems 3 and 4 — Lemmas 1 and 2 imply the inequality $\dim \mathcal{S}(V) \leq \mathrm{ind}(V)$. In view of these two lemmas it is enough to show that $\dim \mathcal{S}(V)$ is actually equal to $\mathrm{ind}(V)$.

We prove this statement by induction on $|V|$. If the linear span of vectors in V is one-dimensional, then both $\dim \mathcal{S}(V)$ and $\mathrm{inv}(V)$ are equal to the number of non-zero vectors in V plus one. This establishes the base of induction.

Assume that $v = v_N$ is a nonzero vector in V. Let $V' = V \setminus \{v\} = \{v_1, \ldots, v_{N-1}\}$, and let V'' be the collection of images of $v_1, \ldots, v_{N-1}$ in the quotient space $E/\langle v \rangle$. It is immediate from the definition of independent subset that $\mathrm{ind}(V) = \mathrm{ind}(V') + \mathrm{ind}(V'')$. Assume by induction that $\dim \mathcal{S}(V') = \mathrm{ind}(V')$ and $\dim \mathcal{S}(V'') = \mathrm{ind}(V'')$.

Clearly, $\mathcal{S}(V)$ is spanned by $\mathcal{S}(V') + \mathcal{S}(V')v$. Both $\mathcal{S}(V')$ and $\mathcal{S}(V')v$ have same dimensions. Hence, $\dim \mathcal{S}(V) = 2 \dim \mathcal{S}(V') - \dim(\mathcal{S}(V') \cap \mathcal{S}(V')v)$. Let $p : \mathcal{S}(V') \to \mathcal{S}(V'')$ be the natural projection. Then $\mathcal{S}(V') \cap \mathcal{S}(V')v \subset \mathrm{Ker}(p)$. Thus $\dim \mathcal{S}(V') - \dim(\mathcal{S}(V') \cap \mathcal{S}(V')v) \geq \dim \mathcal{S}(V'')$ and $\dim \mathcal{S}(V) \geq \dim \mathcal{S}(V') + \dim \mathcal{S}(V'') = \mathrm{ind}(V') + \mathrm{ind}(V'') = \mathrm{ind}(V)$. Coupled with the inequality $\dim \mathcal{S}(V) \leq \mathrm{ind}(V)$, this produces the required statement.

This finishes the proof of Theorems 3 and 4 and thus of Theorems 1 and 2. □

4 Remarks and Open Problems

The algebra $\mathrm{C}(X)$ for type A flag manifolds $X = SL(n)/B$ was studied in more details in [5] and [4]. In this case, we first proved Theorem 1 in [4] using a different approach based on a presentation of $\mathrm{C}(X)$ as a quotient of a polynomial ring (cf. conjecture below). The theorem claims that the dimension of $\mathrm{C}(X)$ is equal to the number forests on n labelled vertices whereas the dimension of $\mathrm{C}^k(X)$ is equal to the number of forests with $\binom{n}{2} - k$ inversions. The statement about forests was initially conjectured in [5] and the statement concerning inversions was then guessed by R. Stanley. In [4] we also discuss various generalizations of the ring $\mathrm{C}(X)$.

There is a natural projection $\mathrm{Sym}(\mathfrak{h}^*) \to \mathrm{C}(X)$ that send $\lambda \in \hat{T}$ to the curvature form $\Theta(L_\lambda)$. Let J be the kernel of this homomorphism. Then $\mathrm{C}(X)$ is naturally isomorphic to $\mathrm{Sym}(\mathfrak{h}^*)/J$. It would be interesting to describe the ideal J explicitly. Let $\omega_1, \omega_2, \ldots, \omega_l$ be the fundamental weights. They generate the weight lattice $\hat{T}$. Also let d_i be the number of positive roots $\alpha \in \Delta_+$ such that $\alpha(\omega_i) \neq 0$.

Conjecture 1 *The ideal $J \in \mathrm{Sym}(\mathfrak{h}^*)$ is generated by the elements $(w \cdot \omega_i)^{d_i+1}$, where $i = 1, \ldots, l$ and w is an element of the Weyl group W.*

It is clear from a dimension argument that $(w \cdot \omega_i)^{d_i+1} \in J$. The nontrivial part is to show that these elements generate the ideal J. This statement was proved for type A flag manifolds in [5].

Another natural open problem is to extend the results to homogeneous manifolds G/P, where P is a parabolic subgroup. Formulas for curvature forms on G/P can be found in [2].

It is also intriguing to find the links between the algebra $\mathrm{C}(X)$ and the arithmetic Schubert calculus developed in [6, 7].

B. Kostant pointed out that the algebra $\mathrm{C}(X)$ is related to the $S(\mathfrak{g})$-module $\wedge\mathfrak{g}$ studied in [3]. It would be interesting to investigate this relationship.

The authors are grateful to Richard Stanley, Alek Vainshtein, and Andrei Zelevinsky for stimulating discussions.

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