VERLINDE ALGEBRAS AND THE INTERSECTION FORM ON VANISHING CYCLES

S.M. Gusein-Zade *   A. Varchenko †

September 27, 1998

Dedicated to E.Brieskorn on his sixtieth birthday

Abstract

We prove Zuber's conjecture [Z] establishing connections of the fusion rules of the $su(N)_k$ WZW model of conformal field theory and the intersection form on vanishing cycles of the associated fusion potential.

1 Introduction

We prove Zuber's conjecture [Z] establishing connections between the fusion rules of the $su(N)_k$ WZW model of conformal field theory and the intersection form on vanishing cycles of the associated fusion potential.

The fusion rules of the $su(N)_k$ model is the multiplication law of the Verlinde algebra associated to the model. The fusion rules can be encoded into a weighted graph, the Dynkin diagram, see Section 2.6. The Dynkin diagram defines in a standard way a lattice with a symmetric bilinear form, a group generated by reflections, a Coxeter element of the group.

According to the Gepner theorem [Gep] the Verlinde algebra can be presented as a quotient algebra $\mathbb{C}[x_1, ..., x_{N-1}]/(\partial_i V_{N,k})$, where $(\partial_i V_{N,k})$ is the ideal generated by the first partial derivatives of the fusion potential

$$V_{N,k}(x) = \frac{(-1)^{N+k}}{(N+k)!}(\frac{d}{dt})^{N+k} \log(1 - tx_1 + t^2 x_2 - \ldots + (-t)^{N-1} x_{N-1} + (-t)^N)|_{t=0}.$$

*Supported in part by grants INTAS-94-4373 and RFBR-95-01-01122a. Independent University of Moscow, Russia. E-mail: sabir@ium.ips.ras.ru.

†Supported in part by NSF grant 9501290. Address: Department of Mathematics, University of North Carolina, Chapel Hill, N.C. 27599, USA. E-mail: av@math.unc.edu.

The fusion potential is a lower order deformation of a quasi-homogeneous polynomial

$$V^0_{N,k}(x) = \frac{(-1)^{N+k}}{(N+k)!}\left(\frac{d}{dt}\right)^{N+k} \log(1 - tx_1 + t^2x_2 - \ldots + (-t)^{N-1}x_{N-1})|_{t=0}$$

called the short fusion potential. The short fusion potential has an isolated critical point at the origin.

We show that the lattice of the Verlinde algebra of $su(N)_k$ with the bilinear form, the reflection group, and the Coxeter element is isomorphic to the lattice of vanishing cycles of the critical point of the short fusion potential together with the symmetric intersection form, the monodromy group, and the operator of classical monodromy, Theorem 3.

As a corollary we prove another Zuber's conjecture in [Z], the level-rank duality. We show that the lattice of $su(N+1)_k$ with the bilinear form, the reflection group, and the Coxeter element is isomorphic to the lattice of $su(k+1)_N$ with the bilinear form, the reflection group, and the Coxeter element. This conjecture was motivated in [Z] by analogies with the $N=2$ superconformal theories.

The second author thanks V.Petkova for attracting his attention to articles [Gep] and [Z] and J.-B. Zuber for explaining his conjectures and related results. This work would not be written without his help. The authors thank the IHES and the organizers of the Singularity conference at Oberwolfach, July 1996, for hospitality and stimulating atmosphere. The authors thank R.-O.Buchweitz and W.Ebeling for useful discussions.

2 Verlinde algebras

The Verlinde algebra or the fusion ring is a finite dimensional commutative associative algebra assigned to a model of conformal field theory. It is an analog of the ring of finite dimensional representations of a simple Lie algebra. In this section we define the Verlinde algebra of the $su(N)_k$ WZW model of conformal field theory. We define its fusion potential, bilinear form, Dynkin diagram and reflection group. We follow here [Gep] and [Z].

2.1 The representation ring of $su(2)$

Finite dimensional representations of the Lie algebra $su(2)$ are labelled by non-negative integers, highest weights. The representation with highest weight $m \in \mathbb{Z}_{\geq 0}$ has dimension $m+1$. Denote it by $[m]$. We have

$$[n] \otimes [m] = [m+n] \oplus [m+n-2] \oplus [m+n-4] \oplus \ldots \oplus [|m-n|]. \qquad (1)$$

In particular, $[1] \otimes [m] = [m+1] \oplus [m-1]$ if $m > 0$.

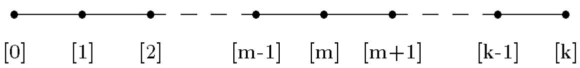

Figure 1: Fusion graph of $su(2)_k$, Dynkin diagram A_{k+1}.

2.2 The Verlinde algebra of the $su(2)_k$ model

The Verlinde algebra $R = R(su(2)_k)$ of $su(2)$ at level k is generated by the elements $[0], [1], \cdots, [k]$ as a vector space. The multiplication (the fusion rules) is defined by the formula

$$[n] \times [m] = \sum_{\substack{\ell=|m-n| \\ m+n-\ell \text{ even}}}^{\min(2k-m-n, m+n)} [\ell].$$

This product is equal to the product defined by (1) if $m + n \leq k$ and does not contain representations with top highest weights otherwise. In particular, we have

$$\begin{aligned} &[1] \times [0] = [1], \\ &[1] \times [m] = [m-1] + [m+1], \quad 0 < m < k, \\ &[1] \times [k] = [k-1]. \end{aligned} \tag{2}$$

These formulae show that the Verlinde algebra is generated by the elements [0] and [1].

It is convenient to represent the formulae (2) by a fusion graph, Figure 1. The edges of the graph connecting the vertex $[m]$ with vertices $[m-1]$ and $[m+1]$ represent the fact that $[m-1]$ and $[m+1]$ enter $[1] \times [m]$ with coefficient 1. The element [0] is the unit element of the Verlinde algebra.

Let us express an element $[m]$ as a polynomial of the element [1]. Denote the element [1] by x and the element [0] by 1. Using (2) we get $[2] = x^2 - 1$, $[3] = x^3 - 2x$. More generally, if $[m] = P_m(x)$, then

$$x P_m(x) = P_{m-1}(x) + P_{m+1}(x).$$

The solution to this recursive relation with the initial condition $P_0(x) = 1$ and $P_1(x) = x$ is given by the Chebyshev polynomials of the second kind,

$$P_m(2\cos\theta) = \frac{\sin(m+1)\theta}{\sin\theta}.$$

According to Gepner [Gep], the Verlinde algebra of $su(2)_k$ is the quotient algebra

$$R(su(2)_k) = \frac{\mathbb{C}[x]}{(dV_{2,k}/dx)}$$

where $(dV_{2,k}/dx)$ is the ideal generated by the derivative of the fusion potential $V_{2,k}(x)$, and the fusion potential $V_{2,k}(x)$ is the Chebyshev polynomial of the first kind,

$$(2+k)\,V_{2,k}(2\cos\theta) \;=\; 2\cos(2+k)\theta.$$

Remark. The Chebyshev polynomial of the first kind is a polynomial with non-degenerate critical points and only two critical values.

Remark. The fusion graph of the multiplication by element [1] is the Dynkin diagram of type A_{k+1}. At the same time the fusion potential $V_{2,k}$ is a lower order deformation of the monomial x^{k+2} which has a critical point of type A_{k+1} in the terminology of singularity theory.

2.3 The representation ring of $su(N)$

Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_{N-1}$ be the fundamental weights of the Lie algebra $su(N)$. They are linear independent vectors in the dual to the Cartan subalgebra of $su(N)$,

$$\mathfrak{h}^* = \mathbb{R}\{e_1, e_2, \ldots, e_N\}/\mathbb{R}\{e_1 + e_2 + \cdots + e_N\}.$$

The vector Λ_i is the image of the vector $(1_1, ..., 1_i, 0, \ldots, 0)$. An irreducible finite dimensional representation of $su(N)$ is determined by its highest weight $\lambda = \lambda_1\Lambda_1 + \cdots + \lambda_{N-1}\Lambda_{N-1}$, where λ_i are non-negative integers. Let $[\lambda]$ denote the irreducible representation with highest weight λ. The tensor product $[\lambda]\otimes[\mu]$ of two representations is a sum of irreducible representations. The representation ring of $su(N)$ is generated by representations $[0]$, $[\Lambda_1]$, ..., $[\Lambda_{N-1}]$.

The Verlinde algebra of $su(N)_k$ is obtained from the representation ring by suitable cutting off representations with top highest weights.

2.4 Verlinde algebra of the $su(N)_k$ model

The Verlinde algebra $R = R(su(N)_k)$ of the Lie algebra $su(N)$ at level k is a finite dimensional commutative associative algebra generated as a vector space by elements $[\lambda]$ of the set

$$P_{N,k} = \{\sum_{i=1}^{N-1} \lambda_i\Lambda_i : \; \lambda_i \in \mathbb{Z}_{\geq 0}, \; \sum_{i=1}^{N-1} \lambda_i \leq k\}.$$

The multiplication is defined by the fusion rules. We use the formula in [K], Sec. 13.35, as the definition of the multiplication, see the same formula in [G] and [Wal]. The element [0] is the unit element of the algebra. As an algebra, the Verlinde algebra is generated by the elements $[0]$, $[\Lambda_1]$, ...,$[\Lambda_{N-1}]$. To define the multiplication it is enough to define the multiplication by the generators $[\Lambda_1]$, ...,$[\Lambda_{N-1}]$.

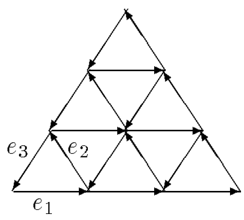

Figure 2: Fusion graph γ_1 of $su(3)_k$, $k = 3$.

Let $e_1 = \Lambda_1$, $e_2 = \Lambda_2 - \Lambda_1$, $\ldots$, $e_{N-1} = \Lambda_{N-1} - \Lambda_{N-2}$, $e_N = -\Lambda_{N-1}$. The vectors e_1, e_2, $\ldots$,e_N are linear dependent: $e_1 + e_2 + \ldots + e_N = 0$. For any $p = 1, ..., N-1$, we have

$$[\Lambda_p] \times [\lambda] = \sum [\mu]$$

where the sum is over all $\mu \in P_{N,k}$ such that $\mu = \lambda + e_{i_1} + \cdots + e_{i_p}$, $1 \leq i_1 < ... < i_p \leq N$.

The multiplication by an element $[\Lambda_p]$ is useful to represent by a fusion graph γ_p and by an adjacency matrix G_p. The fusion graph γ_p is an oriented graph with the vertex set $P_{N,k}$ and such that an edge goes from a vertex λ to a vertex μ if and only if $\mu = \lambda + e_{i_1} + \cdots + e_{i_p}$, $1 \leq i_1 < ... < i_p \leq N$. The fusion graph γ_1 of $su(3)_k$ is shown in Figure 2.

For any $p = 1, ..., N-1$ the fusion graph γ_{N-p} is obtained from the fusion graph γ_p by reversing all the orientations of the edges.

The set of vertices $P_{N,k}$ has a natural $\mathbb{Z}_N$-valued grading:

$$\lambda \qquad \mapsto \qquad \tau(\lambda) = \sum_{n=1}^{N-1} n\,\lambda_n \mod N.$$

If an edge of the fusion graph γ_p goes from a vertex λ to a vertex μ, then $\tau(\mu) = \tau(\lambda) + p$.

The adjacency matrix G_p is a matrix with entries $(G_p)_{\lambda,\mu}$ where $\lambda, \mu \in P_{N,k}$. The entries of the adjacency matrix are given by the formula: $(G_p)_{\lambda,\mu} = 1$ if $\mu = \lambda + e_{i_1} + \cdots + e_{i_p}$, $1 \leq i_1 < ... < i_p \leq N$, and $(G_p)_{\lambda,\mu} = 0$ otherwise.

If $(G_p)_{\lambda,\mu} = 1$, then $\tau(\mu) = \tau(\lambda) + p$.

For any $p = 1, ..., N-1$ the matrix G_{N-p} is the transpose of the matrix G_p, $G_{N-p} = (G_p)^t$.

2.5 The fusion potential of $su(N)_k$

Denote the elements $[\Lambda_1]$, ..., $[\Lambda_{N-1}]$ of the Verlinde algebra by x_1, ..., x_{N-1}, respectively, and the element $[0]$ by 1. Then any element of the Verlinde algebra can be written as a polynomial in x_1, ..., x_{N-1}, see in [Gep] a general formula. These polynomials are multidimensional analogs of the Chebyshev polynomials of the second kind. For instance, for $su(3)_k$ we have $[2\Lambda_1] = (x_1)^2 - x_2$, $[\Lambda_1 + \Lambda_2] = x_1 x_2 - 1$, $[2\Lambda_2] = (x_2)^2 - x_1$, ...

There is a beautiful Gepner's theorem.

Theorem 1 ([Gep]) *The Verlinde algebra of $su(N)_k$ is the quotient algebra*

$$R(su(N)_k) = \frac{\mathbb{C}[x]}{(\partial_i V_{N,k})}$$

where $(\partial_i V_{N,k})$ is the ideal generated by the first partial derivatives of the fusion potential $V_{N,k}(x)$, and the fusion potential is a polynomial in x_1, ..., x_{N-1} given below.

There are three formulae for the fusion potential $V_{N,k}(x)$. Let $h = N + k$. According to the first formula,

$$V_{N,k}(x) = \frac{(-1)^h}{h!} \left(\frac{d}{dt}\right)^h \log(1 - tx_1 + t^2 x_2 - \ldots + (-t)^{N-1} x_{N-1} + (-t)^N)|_{t=0}.$$

According to the second, the fusion potential $V_{N,k}(x)$ is the symmetric polynomial $\sum_{i=1}^N (y_i)^h / h$ written as a polynomial in the elementary symmetric functions x_1, ..., x_N,

$$x_n = \sum_{1 \le i_1 < \ldots < i_n \le N} y_{i_1} \cdots y_{i_n}, \tag{3}$$

where we assume that $x_N = 1$. Finally, the multiple $(N + k)\, V_{N,k}(x)$ of the fusion potential can be written as the determinant of an $h \times h$ matrix D with non-zero elements given by

$$\begin{array}{ll} D_{n,n+1} = 1 & n = 1, ..., h-1, \\ D_{n,1} = n\, x_n & n = 1, ..., h, \\ D_{i+n-1,i} = x_n & n = 1, ..., h-1,\ i = 2, ..., h-n+1, \end{array}$$

where we assume that $x_N = 1$ and $x_n = 0$ for $n > N$. (One can find this formula for the polynomial $\sum_{i=1}^N (y_i)^h$ in [M].)

An important role is played by the top quasi-homogeneous part of the fusion potential. Namely, consider the polynomial

$$V^0_{N,k}(x) = \frac{(-1)^h}{h!} \left(\frac{d}{dt}\right)^h \log(1 - tx_1 + t^2 x_2 - \ldots + (-t)^{N-1} x_{N-1})|_{t=0}. \tag{4}$$

The polynomial $V^0_{N,k}(x)$ can be defined as the symmetric polynomial $\sum_{i=1}^{N-1} (y_i)^h / h$ written as a function in x_1, ..., x_{N-1}. Finally, $(N + k)\, V^0_{N,k}(x)$ is equal to the determinant of the $h\times h$ matrix D with non-zero elements given by

$$\begin{array}{ll} D_{n,n+1}=1 & n=1,...,h-1, \\ D_{n,1}=n\,x_n & n=1,...,h, \\ D_{i+n-1,i}=x_n & n=1,...,h-1,\ i=2,...,h-n+1, \end{array}$$

where we assume that $x_n=0$ for $n>N-1$.

The polynomial $V^0_{N,k}$ will be called the short fusion potential. The short fusion potential is a quasi-homogeneous polynomial of degree h with the weight of the variable x_n equal to n, $n=1,...,N-1$. The fusion potential $V_{N,k}$ is a lower order deformation of the short fusion potential by terms of degree less than h.

The short fusion potential has an isolated critical point at the origin, since the polynomial $\sum_{i=1}^{N-1}(y_i)^h$ has an isolated critical point at the origin. The Milnor number of the critical point of the short fusion potential is equal to the dimension of the Verlinde algebra.

Example. For $su(3)_1$, $4V_{3,1}(x_1,x_2)=(x_1)^4-4(x_1)^2x_2+4x_1+2(x_2)^2$, $4V^0_{3,1}(x_1,x_2)=(x_1)^4-4(x_1)^2x_2+2(x_2)^2$. The critical point of the short fusion potential has type A_3, see in [AGV] the classification of critical points.

Example. For $su(3)_2$, $5V_{3,2}(x_1,x_2)=(x_1)^5-5(x_1)^3x_2+5x_1(x_2)^2+5(x_1)^2-5x_2$, $5V^0_{3,2}(x_1,x_2)=(x_1)^5-5(x_1)^3x_2+5x_1(x_2)^2$. The critical point of the short fusion potential has type D_6.

Remark. The fusion potential $V_{N,k}(x_1,...,x_{N-1})$ is a multidimensional analog of the Chebyshev polynomials of the first kind. The fusion potential has the following remarkable property (which we will not use). Namely, the fusion potential has only N critical values, and the critical values are $N-$th roots of unity multiplied by N/h.

In fact, a critical point of the fusion potential corresponds to a critical point of the function $\sum_{i=1}^N(y_i)^h/h$ restricted to the hypersurface $y_1\cdots y_N=1$. Such a critical point corresponds to a critical point of the function $\lambda(1-y_1\cdots y_N)/h+\sum_{i=1}^N(y_i)^h/h$, where λ is the Lagrange multiplier. The critical points of the last function are solutions to the system of equations $y_1\cdots y_N=1$ and $y_n^h=\lambda/h$, $n=1,...,N$. The system implies that $\lambda^N=h^N$ and the critical value is equal to $N\lambda/h^2$.

Notice also that there is another lower order deformation of the short fusion potential with only N critical values. That polynomial is the direct sum of the Chebyshev polynomials of one variable, $\sum_{i=1}^{N-1}V_{2,h-2}(y_i)$, written as a function of the elementary symmetric functions x_n. Its critical values are $-2(N-1)/h$, $-2(N-3)/h$, $\dots$, $2(N-3)/h$, $2(N-1)/h$.

Remark. There is a general problem of finding the maximal number of critical points with the same critical value of a lower order deformation of a quasihomogeneous polynomial of given degree and weights, see [AGV], Sec. 14.3.2, [Ch], [Gor] and references therein.

The direct sum $\widetilde{V}=\sum_{i=1}^{N-1}V_{2,h-2}(y_i)$ of the Chebyshev polynomials of the first kind, written as a function of the elementary symmetric functions x_n, provides an example of a lower order deformation of a quasihomogeneous polynomial of degree h and weights $1, 2, ..., N-1$ with many critical points on one level. Namely, denote by $\#$ the number of critical points of this function with critical value 0 for N odd and with critical value $-2/h$ for N even. Then $\#$ is equal to

$$\begin{array}{ll} \binom{h/2}{N/2}\binom{h/2-1}{N/2-1} & \text{for } N \text{ and } h \text{ even;} \\ \binom{(h-1)/2}{N/2}\binom{(h-1)/2}{N/2-1} & \text{for } N \text{ even, } h \text{ odd;} \\ \binom{h/2}{(N-1)/2}\binom{h/2-1}{(N-1)/2} & \text{for } N \text{ odd, } h \text{ even;} \\ \binom{(h-1)/2}{(N-1)/2}\binom{(h-1)/2}{(N-1)/2} & \text{for } N \text{ and } h \text{ odd,} \end{array}$$

while the total number of critical points is equal to $\binom{h-1}{N-1}$.

Considering sums of Chebyshev polynomials of the first kind is a standard way to construct polynomials with many critical points on one level . In particular, if h is divisible by 2, 3, ..., $(N-1)$, then a sum of the form

$$\sum_{i=1}^{N-1} \pm V_{2,h/i-2}(x_i)/i \tag{5}$$

gives another example of a lower order deformation of a quasihomogeneous polynomial of degree h with weights of variables equal to 1, 2, 3, ..., $(N-1)$ and many critical points on one level.

It turns out that the maximal number of critical points on one level of the function $\widetilde{V}$ is greater than the maximal number of critical points on one level of the sums of Chebyshev polynomials indicated in (5). For example, for $N = 3$, i.e. for functions of two variables, and $h = 2k$ we have $\# = k^2 - k$, while the maximal number of critical points of functions (5) on one level is $k^2 - (3/2)k + 1$ for k even and $k^2 - (3/2)k + (1/2)$ for k odd. For $N = 4$, i.e. for functions of three variables, and $h = 6k$ we have $\# = \frac{1}{2}(27k^3 - 18k^2 + 3k)$, while the maximal number of critical points of functions (5) on one level is equal to $\frac{1}{2}(27k^3 - 18k^2)$ for k even and to $\frac{1}{2}(27k^3 - 21k^2 + 4k)$ for k odd, and so on.

2.6 The Dynkin diagram and the reflection group of $su(N)_k$

Consider a lattice $L = L(su(N)_k)$ over $\mathbb{Z}$ with a basis α_λ, $\lambda \in P_{N,k}$. Introduce a bilinear symmetric form $B_{N,k}$ on L,

$$B_{N,k}\left(\alpha_\lambda, \alpha_\mu\right) = 2\delta_{\lambda\mu} + (G_1 + G_2 + ... + G_{N-1})_{\lambda\mu}. \tag{6}$$

The bilinear form can be described by its Dynkin diagram. The vertices of the Dynkin diagram are elements of $P_{N,k}$. Any two vertices λ and μ are connected by an edge if and only if $B_{N,k}\left(\alpha_\lambda, \alpha_\mu\right) \neq 0$. If $B_{N,k}\left(\alpha_\lambda, \alpha_\mu\right) \neq 0$, then the vertices are connected by an edge of multiplicity $-B_{N,k}\left(\alpha_\lambda, \alpha_\mu\right)$. This graph will be called the Dynkin diagram of the $su(N)_k$ model.

According to the definition of the adjacency matrices, all edges of the Dynkin diagram have multiplicity -1. Two edges λ and μ are connected by an edge if and only if $\mu = \lambda + e_{i_1} + \cdots + e_{i_p}$, $1 \leq i_1 < ... < i_p \leq N$, for some p, $0 < p < N$.

Fix a lexicographical ordering of the set of vertices of the Dynkin diagram. We assume that $\lambda < \mu$ if there is i such that $\lambda_n = \mu_n$ for $n = i+1, ..., N-1$ and $\lambda_i < \mu_i$.

The Dynkin diagram can be obtained from the fusion graphs $\gamma_1, \ldots, \gamma_{N-1}$. Namely, the Dynkin diagram and the fusion graphs have the same set of vertices. The union of edges of the graphs γ_1, ..., γ_{N-1} is naturally decomposed into pairs of edges with the opposite orientations. Forgetting the orientation we construct from each pair an edge of multiplicity -1 of the Dynkin diagram.

Notice also that knowing the Dynkin diagram we can reconstruct the fusion graphs, since an edge of the fusion graph γ_p goes from λ to μ only if $\tau(\mu) = \tau(\lambda)+p$.

Thus, the Dynkin diagram encodes the operators of multiplication by $[\Lambda_1]$, ..., $[\Lambda_{N-1}]$ in the Verlinde algebra.

Introduce the group $\Gamma_{N,k}$ as the group of linear automorphisms of the lattice L generated by reflections s_λ at the hyperplanes orthogonal to the basis elements $\alpha_\lambda, \lambda \in P_{N,k}$,

$$s_\lambda \,:\, x \,\mapsto\, x - (x, \alpha_\lambda)\alpha_\lambda.$$

The group $\Gamma_{N,k}$ will be called the reflection group of $su(N)_k$.

The product of all basis reflections,

$$M_{N,k} = s_0 \cdots s_{k\Lambda_{N-1}} = \prod_{\lambda \in P_{N,k}} s_\lambda,$$

written in the lexicographical order will be called the Coxeter element of the reflection group of the Verlinde algebra.

Notice that this definition of the Coxeter element differs from Zuber's definition in [Z].

Theorem 2 [Z]. *The form $B_{N,k}$ is positive definite if and only if $N = 2$ or $N = 3$, $k = 0, 1, 2$, or $k = N, N+1$.*

The finite reflection groups $\Gamma_{2,k}$, $\Gamma_{3,0}$, $\Gamma_{3,1}$, $\Gamma_{3,2}$, $\Gamma_{N,N}$, $\Gamma_{N,N+1}$ have types A_{k+1}, A_1, A_3, D_6, A_1, A_{N+1}, respectively. For other pairs N, k the groups $\Gamma_{N,k}$ are infinite.

It turns out that the bilinear form $B_{N,k}$, the reflection group $\Gamma_{N,k}$ and the Coxeter element $M_{N,k}$ of the $su(N)_k$ model can be defined in terms of the topology of the level hypersurfaces of the short fusion potential of $su(N)_k$. We recall the necessary definitions in the next section.

3 The intersection form and the monodromy group of an isolated critical point, [AGV]

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function with an isolated critical point at the origin. Let $B_\delta \subset \mathbb{C}^n$ denote the ball of radius δ with the center at the origin. Fix small $\delta \gg |\varepsilon| > 0$, $\varepsilon \in \mathbb{C}$. The manifold $V_\varepsilon = f^{-1}(\varepsilon) \cap B_\delta$ is called the Milnor fiber of the germ f. The Milnor fiber is homotopy equivalent to the bouquet of $(n-1)$-dimensional spheres. The number $\mu = \mu(f)$ of the spheres is called the Milnor number of f. The middle homology group $L_f = H_{n-1}(V_\varepsilon; \mathbb{Z})$ of the Milnor fiber is called the Milnor lattice. The intersection form is a bilinear form on the Milnor lattice. The intersection form is symmetric for n odd and is skew-symmetric for n even.

Consider a deformation $f_t : \mathbb{C}^n \to \mathbb{C}$, $t \in [0,1]$, of the germ f such that for small non-zero t the functions f_t have only non-degenerate critical points with pair-wise distinct critical values. The number of critical points equals the Milnor number. Fix such t_0, and set $\widetilde{f} = f_{t_0}(x)$. Let $p_1, p_2, \ldots, p_\mu$ denote the critical points of $\widetilde{f}$, and let $z_i = \widetilde{f}(p_i)$ denote the critical values.

Let $\widetilde{V}_z = \widetilde{f}^{-1}(z) \cap B_\delta$ denote the local level set of the function $\widetilde{f}$. Let z_0 be a non–critical value of the function $\widetilde{f}$ such that $|z_0| > |z_i|$ for $i = 1, 2, \cdots, \mu$. The manifold $\widetilde{V}_{z_0}$ is diffeomorphic to the Milnor fiber V_ε of the germ f.

Let u_i, $i = 1, 2, \ldots, \mu$, be smooth non-self-intersecting paths connecting the critical values z_i with the non-critical value z_0 and such that $u_i(0) = z_i$, $u_i(1) = z_0$. We assume that the paths lie inside the circle $\{z \in \mathbb{C} : |z| \leq |z_0|\}$ and any two of them intersect only at the point z_0. We enumerate the paths (and thus the critical values and critical points) in the order they enter the point z_0 counting clockwise and starting from the boundary of the circle.

By the Morse lemma, for each critical point p_i of the function $\widetilde{f}$ there exists a system of local coordinates $y_1, y_2, \ldots, y_n$ centered at p_i such that the function $\widetilde{f}$ can be written in the form $\widetilde{f}(y_1, y_2, \ldots, y_n) = z_i + \sum_{j=1}^n (y_j)^2$. For a small τ, the level manifold $\widetilde{V}_{u_i(\tau)}$ contains the $n-1$-dimensional sphere $S_i(\tau)$ given by the equations $\sum_{j=1}^n (y_j)^2 + z_i = u_i(\tau)$, $\mathrm{Im}\,\left(y_j/(u_i(\tau) - z_i)^{1/2}\right) = 0$, $j = 1, ..., n$. For $\tau = 0$ the sphere degenerates to the critical point p_i. Lifting the homotopy of τ from 0 to 1, we construct a family of $(n-1)$-dimensional spheres $S_i(\tau) \subset \widetilde{V}_{u_i(\tau)}$ for all τ between 0 and 1. The homology class $\delta_i \in H_{n-1}(\widetilde{V}_{z_0}; \mathbb{Z})$ defined (up to orientation) by the sphere $S_i(1)$ is called the vanishing cycle corresponding to the path u_i. The vanishing cycles $\delta_1, \delta_2, \ldots, \delta_\mu$ form a basis of the homology group $H_{n-1}(\widetilde{V}_{z_0}; \mathbb{Z}) \cong H_{n-1}(V_\varepsilon; \mathbb{Z}) = L_f$. A basis constructed in this way is called distinguished. The distinguished basis depends on the choice of the paths u_i.

The system of non-singular level manifolds $\widetilde{V}_z$ forms a locally trivial bundle over the complement to the set of critical values, $\mathbb{C} \setminus \{z_1, z_2, \ldots, z_\mu\}$. Any loop in the complement with the end points at z_0 can be lifted to an isotopy of the fibers over the loop. The isotopy induces a linear automorphism of the homology group

$H_{n-1}(\widetilde{V}_{z_0}) \cong L_f$ called the monodromy transformation. The set of the monodromy transformations corresponding to all loops forms a group called the monodromy group of the germ f.

The set of paths u_i , $i = 1, 2, \ldots, \mu$, gives a set of generators, $s_i : L_f \to L_f$, of the monodromy group,

$$s_i : a \mapsto a + (-1)^{n(n+1)/2}(a, \delta_i)\delta_i,$$

here (a, δ_i) is the intersection number of the cycles a and δ_i. The transformation s_i is called the Picard-Lefschetz transformation. Thus, the monodromy group is determined by the intersection form and a distinguished basis.

We have $(\delta_i, \delta_i) = (-1)^{(n-1)(n-2)/2}(1 + (-1)^{n-1})$. This means that this self-intersection number is equal to 0 for even n and to $(-1)^{(n-1)/2}$ for n odd. Hence, for an odd number of variables, n, a Picard-Lefschetz transformation, s_i, is the reflection at the hyperplane orthogonal to the vanishing cycle δ_i.

The monodromy transformation corresponding to the path going counterclockwise around all the critical values is called the operator of classical monodromy. It equals the product $s_1 \circ s_2 \circ \cdots \circ s_\mu$ of the Picard-Lefschetz transformations.

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ and $g : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ be two germs of holomorphic functions of n and m variables, respectively. The germ of the function

$$f \oplus g : (\mathbb{C}^{n+m}, 0) \to (\mathbb{C}, 0), \qquad (x, y) \mapsto f(x) + g(y),$$

is called the direct sum of the germs f and g. Let $\{\delta_i\}$, $i = 1, \ldots, \mu(f)$, and $\{\delta'_j\}$, $j = 1, \ldots, \mu(g)$, be distinguished bases of vanishing cycles of the germs f and g. Gabrielov's theorem ([Gab], [AGV]) describes a distinguished basis of vanishing cycles of the germ $f \oplus g$ and the corresponding intersection form. Consider the lattice $L_f \otimes L_g$ with the basis $\Delta_{ij} = \delta_i \otimes \delta'_j$ ordered lexicographically: $(i, j) < (i', j')$ if either $j < j'$ or $j = j'$ and $i < i'$. Introduce a bilinear form on the lattice $L_f \otimes L_g$ by the formulae:

$$\begin{array}{ll}
(\Delta_{i j_1}, \Delta_{i j_2}) = sgn(j_2 - j_1)^n(-1)^{nm+n(n-1)/2}(\delta'_{j_1}, \delta'_{j_2}) & j_1 \neq j_2, \\
(\Delta_{i_1 j}, \Delta_{i_2 j}) = sgn(i_2 - i_1)^m(-1)^{nm+m(m-1)/2}(\delta_{i_1}, \delta_{i_2}) & i_1 \neq i_2, \\
(\Delta_{i_1 j_1}, \Delta_{i_2 j_2}) = 0 & (i_2 - i_1)(j_2 - j_1) < 0, \\
(\Delta_{i_1 j_1}, \Delta_{i_2 j_2}) = sgn(i_2 - i_1)(-1)^{nm}(\delta_{i_1}, \delta_{i_2})(\delta'_{j_1}, \delta'_{j_2}) & (i_2 - i_1)(j_2 - j_1) > 0, \\
(\Delta_{ij}, \Delta_{ij}) = (-1)^{(n+m-1)(n+m-2)/2}(1 + (-1)^{n+m-1}). &
\end{array}$$

Then there is a natural isomorphism of the lattice $L_f \otimes L_g$ and the Milnor lattice $L_{f \oplus g}$ of the direct sum sending the bilinear form on the lattice $L_f \otimes L_g$ to the intersection form and the basis Δ_{ij} to a distinguished basis.

Gabrielov's theorem also describes a system of paths which defines the distinguished basis of $L_{f \oplus g}$ corresponding to $\{\Delta_{ij}\}$, see [Gab].

If $f = g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, then $f \oplus f$ is a germ of a function on $\mathbb{C}^n \times \mathbb{C}^n$. The permutation of the factors acts on the Milnor lattice $L_{f \oplus f}$ by the formula

$\sigma_*(\delta_i \otimes \delta_j) = (-1)^n(\delta_j \otimes \delta_i)$. This follows from the description in [AGV] of the Milnor fiber of a direct sum as the joint of the Milnor fibers of summands.

Let $Q(y)$ be a non-degenerate quadratic form of variables $y = (y_1, ..., y_m)$. The germ of the function

$$f \oplus Q \,:\, (\mathbb{C}^{n+m}, 0) \to (\mathbb{C}, 0), \qquad (x, y) \mapsto f(x) + Q(y)\,,$$

is called a stabilization of the germ f. The Milnor number of the stabilization is equal to the Milnor number of the initial germ. The corresponding Milnor lattices are naturally isomorphic. The intersection form and the set of distinguished bases of the germ f defines the intersection form and the set of distinguished bases of the stabilization.

Namely, there exists a natural isomorphism between the Milnor lattices L_f and $L_{f\oplus Q}$ which establishes a one-to-one correspondence between the distinguished bases. If $\{\delta_i\}$ is a distinguished basis of L_f and $\{\widetilde{\delta}_i\}$ is the corresponding distinguished basis of $L_{f\oplus Q}$, then

$$(\widetilde{\delta}_i, \widetilde{\delta}_j) = [\mathrm{sgn}\,(j-i)]^m(-1)^{nm+m(m-1)/2}(\delta_i, \delta_j) \quad \text{for } i \neq j. \tag{7}$$

Formula (7) is simplified for even m, then $(\widetilde{\delta}_i, \widetilde{\delta}_j) = (-1)^{m/2}(\delta_i, \delta_j)$. In what follows we identify the Milnor lattices of the critical points f and $f \oplus Q$ and thus elements of a distinguished basis of L_f and of the corresponding distinguished basis of $L_{f\oplus Q}$ (using the same notations for them).

Formula (7) shows that there are only four different intersection forms of stabiliations. The two of the four forms, corresponding to stabilizations with an odd number of variables, are symmetric and the two, corresponding to stabilizations with an even number of variables, are skew-symmetric. The two forms of the same type (symmetric or skew-symmetric) differ only by the common sign.

Let the number of variables, $n+m$, satisfy $n + m \equiv 1 \bmod 4$. Then the corresponding intersection form will be called the quadratic form of the germ f (or the symmetric bilinear form). We denote it by $(\cdot, \cdot)_q$. In this case the square of each vanishing cycle equals 2.

Given a distinguished basis $\{\delta_i\}$, the quadratic form can be described by a weighted graph, the Dynkin diagram of f. The vertices of the graph are elements of the distinguished basis. Any two vertices δ_i and δ_j are connected by an edge if and only if $(\delta_i, \delta_j)_q \neq 0$. If $(\delta_i, \delta_j)_q \neq 0$, then the vertices are connected by an edge of multiplicity $-(\delta_i, \delta_j)_q$.

The monodromy groups of stabilizations with an odd number of variables are naturally isomorphic. The isomorphism identifies the corresponding classical monodromy operators. The same is true for the monodromy groups of stabilizations with an even number of variables.

The monodromy group of a stabiliation with an odd number of variables is a group generated by reflections. The group will be called the reflection group of the germ f. In this case the operator of classical monodromy will be called the Coxeter element of f.

Gabrielov's theorem implies the following description of the quadratic form of the direct sum $f \oplus g$ of the germs f and g. It is naturally isomorphic to the quadratic form on the tensor product $L_f \otimes L_g$ of the Milnor lattices L_f and L_g defined by the formulae ($\Delta_{ij} = \delta_i \otimes \delta'_j$):

$$\begin{array}{ll}
(\Delta_{i\,j_1}, \Delta_{i\,j_2})_q = (\delta'_{j_1}, \delta'_{j_2})_q & \text{for } j_1 \neq j_2, \\
(\Delta_{i_1 j}\,, \Delta_{i_2 j}\,)_q = (\delta_{i_1}, \delta_{i_2})_q & \text{for } i_1 \neq i_2, \\
(\Delta_{i_1 j_1}, \Delta_{i_2 j_2})_q = 0 & \text{for } (i_2 - i_1)(j_2 - j_1) < 0, \\
(\Delta_{i_1 j_1}, \Delta_{i_2 j_2})_q = (\delta_{i_1}, \delta_{i_2})_q (\delta'_{j_1}, \delta'_{j_2})_q & \text{for } (i_2 - i_1)(j_2 - j_1) > 0, \\
(\Delta_{ij}, \Delta_{ij})_q = 2. &
\end{array}$$

These formulae imply a description of the corresponding Dynkin diagram of the direct sum of two germs.

Let $f_t(x)$ be a continuous family of germs of functions having an isolated critical point. Assume that the Milnor number of the critical point does not depend on t. Then the Milnor lattices, the quadratic forms, the reflection groups, and the Coxeter elements of all the germs f_t are naturally isomorphic. Moreover, they have the same Dynkin diagrams.

For instance, consider the family of germs at the origin of quasi-homogeneous polynomials of a given degree, given weights of variables, and having an isolated critical point. Then the Milnor lattices, the quadratic forms, the reflection groups, and the Coxeter elements of all the germs are naturally isomorphic.

4 Multiplication in the Verlinde algebra, topology of the short fusion potential, and the level-rank duality

Theorem 3 *For any $N > 1$ and $k \geq 0$, consider the Verlinde algebra of $su(N)_k$ and the critical point of the short fusion potential $V^0_{N,k}(x_1, ..., x_{N-1})$. Then there is an isomorphism of the lattice of the Verlinde algebra and the Milnor lattice of the critical point,*

$$\psi\ :\ L(su(N)_k)\ \to\ L_{V^0_{N,k}},$$

sending the bilinear form $B_{n,k}$ to the quadratic form on the Milnor lattice and sending the basis α_λ, $\lambda \in P_{N,k}$, with the lexicographical ordering to a distinguished basis of the Milnor lattice.

Corollary 1 *The isomorphism ψ sends the reflection group $\Gamma_{N,k}$ and the Coxeter element $M_{N,k}$ to the reflection group of the critical point and its Coxeter element, respectively.*

Theorem 4 *Let $N \geq k$. Then there is a continuous family of germs of holomorphic functions at an isolated critical point with a constant Milnor number,*

$f_s : (\mathbb{C}^N, 0) \to (\mathbb{C}, 0)$, $s \in [0,1]$, *such that the germ* f_0 *is the germ of the short fusion potential* $V^0_{N+1,k}(x_1, ..., x_N)$ *at the origin and the germ* f_1 *is stable equivalent to the germ at the origin of another short fusion potential* $V^0_{k+1,N}(x_1, ..., x_k)$.

Corollary 2 , Level-Rank Duality. *There exists an isomorphism* $\psi : L(su(N+1)_k) \to L(su(k+1)_N)$ *of the lattices sending the bilinear form* $B_{N+1,k}$, *the reflection group* $\Gamma_{N+1,k}$, *and the Coxeter element* $M_{N+1,k}$ *to the bilinear form* $B_{k+1,N}$, *the reflection group* $\Gamma_{k+1,N}$, *and the Coxeter element* $M_{k+1,N}$, *respectively.*

Theorem 3 allows us to apply results on Milnor lattices of critical points to the lattices of Verlinde algebras, in particular, see in [E] a description of the reflection groups of critical points and in [S] a formula for the signature of the intersection form of a quasi-homogeneous critical point.

Theorem 3 and Corollaries 1, 2 were conjectured by J.-B. Zuber in [Z]. The level-rank duality conjecture was motivated in [Z] by analogies with $N = 2$ superconformal theories. Theorem 3 and Corollary 2 were proved for $N = 3$ by N. Warner [War]. Warner's proof helped us to invent a proof for general N. Theorem 4 for $N = k + 1$ could be recognized in [Z].

5 Proofs

5.1 Lemmas

The following two lemmas will be used in the proof of Theorem 3.

It is well-known [AGV] that the critical point of the germ y^h of type A_{h-1} has a distinguished basis δ_i, $i = 1, 2, \ldots, h-1$, with the intersection numbers $(\delta_i, \delta_i) = 2$, and $(\delta_i, \delta_j) = 0$ for $|j - i| > 1$, $(\delta_i, \delta_{i+1}) = 1$. Any distinguished basis with this property will be called special.

Lemma 1 *There exists a continuous family* $g_s(y)$, $s \geq 0$, *of polynomials of degree* h *with the coefficients of* y^h *and* y^{h-1} *equal to 1 and 0, respectively, and such that:*
1) $g_s(y)$ *is a deformation of* y^h, *i.e.* $g_0(y) = y^h$;
2) all the critical points $y^{(j)}$, $j = 1, 2, \ldots, h-1$, *of* g_s *are non-degenerate and the critical values* $g_s(y^{(j)})$ *are pair-wise distinct;*
3) the critical value $g_s(y^{(j)})$ *is equal to* $(\frac{1}{3N})^j s$, *where* $j = 1, ..., h-1$;
4) fix a non-critical value $z^{(0)}$ *in the upper half-plane, consider the system of intervals* $u_j(t) = tz^{(0)} + (1-t)g_s(y^{(j)})$ *connecting the non-critical value and the critical values, then the distinguished basis corresponding to this system of paths is special.*

Proof of Lemma 1. Let $\mathbb{C}^{h-1}_a$ be the space of polynomials of the form $y^h + a_{h-2}y^{h-2} + \ldots + a_1 y + a_0$, $a = (a_0, a_1, \ldots a_{h-2})$. Let $\Sigma \subset \mathbb{C}^{h-1}_a$ be the subset of polynomials with multiple critical values, i.e. the polynomials having a degenerate critical point or a pair of critical points with equal critical values.

Let $\mathbb{C}_z^{h-1}$ be the space of unordered sets of $h-1$ complex numbers $(z_1, \ldots, z_{h-1})$. Let $\Delta \subset \mathbb{C}_z^{h-1}$ be the subspace of unordered sets $(z_1, \ldots, z_{h-1})$ with $z_i = z_j$ for some i and j.

Let $p : \mathbb{C}_a^{h-1} \to \mathbb{C}_z^{h-1}$ be the map sending a polynomial to the set of its critical values. p maps the complement of Σ to the complement of Δ. By the O.Lyashko and E.Looijenga theorem ([A], [L]) the map p is proper (i.e. the preimage of a compact subspace is compact), the preimage of 0 consists of one point corresponding to the polynomial y^h, and the restriction of p to $\mathbb{C}_a^{h-1} \setminus \Sigma$ is a covering map over $\mathbb{C}_z^{h-1} \setminus \Delta$.

Let $f(y) \in \mathbb{C}_a^{h-1}$ be a deformation of y^h with nondegenerate critical points and pair-wise distinct critical values $z_1, \ldots z_{h-1}$. Fix a non-critical value z_0 of f lying in the upper half-plane. Fix a system of paths $u_i(t)$, $i = 1, \ldots, h-1$, $u_i(0) = z_i$, $u_i(1) = z_0$, connecting the non-critical value with the critical values and defining a special distinguished basis of vanishing cycles $\delta_1, \ldots, \delta_{h-1}$. We have $p(f) = (z_1, \ldots, z_{h-1})$.

For $\tau \in [0, 1)$ let $z_i(\tau) = u_i(\tau)$, and $u_{\tau,i}(t) = u_i((1-\tau)t + \tau)$. The family of unordered sets $(z_1(\tau), \ldots, z_{h-1}(\tau))$ of points of the space $\mathbb{C}_z^{h-1}$ is a homotopy of the point $(z_1, \ldots, z_{h-1})$.

For τ_0 close to 1, the paths $u_{\tau_0,i}$ are close to straight lines. The system of points $z_1(\tau_0), \ldots, z_{h-1}(\tau_0)$ in $\mathbb{C}$ and paths $u_{\tau_0,i}$ can be easily deformed into the system described in Lemma 1. By the theorem of O.Lyashko and E.Looijenga every homotopy of $h-1$ distinct points in $\mathbb{C}$ gives rise to a unique homotopy of the corresponding polynomial. So deforming the initial system of paths $\{u_i\}$ into the system of paths described in section 4 of Lemma 1 we construct a deformation of the initial polynomial $f(y)$ to a polynomial which we denote by $g_s(y)$. This polynomial considered with the system of paths described in section 4 of Lemma 1 satisfies the conditions of Lemma 1. The fact that $g_s(y)$ tends to y^h for $s \to 0$ follows from the fact that the map p is proper. Lemma 1 is proved. □

Let $\mathbb{C}^n$ be the complex linear space with the standard S_n-action, $C \subset \mathbb{C}^n$ the union of non-regular orbits of the action, i.e. the union of its mirrors. Let f be an S_n-invariant holomorphic function defined in a neighbourhood of a point $y \in C$. Let $B_\delta(y)$ be the ball of radius δ in $\mathbb{C}^n$ with the center at y, D_ε the disk of radius ε in $\mathbb{C}$ with the center at $f(y)$. Suppose that the point y is not a critical point of the function f. Then there exist positive δ and ε such that the restriction of f to $B_\delta(y) \cap f^{-1}(D_\varepsilon)$ is a locally trivial bundle over D_ε.

Lemma 2 *There exist positive δ and ε such that f defines a locally trivial bundle of pairs, $f : (B_\delta(y) \cap f^{-1}(D_\varepsilon), C \cap B_\delta(y) \cap f^{-1}(D_\varepsilon)) \to D_\varepsilon$.*

Proof of Lemma 2. C is the union of hyperplanes in $\mathbb{C}^n$, the mirrors $\{y_i = y_j\}$ of the S_n-action. In order to prove that f defines a locally trivial bundle of pairs it suffices to show that the restriction of f to any intersection M of some mirrors does not have a critical point at y. Suppose that y is a critical point of $f|_M$, i.e. $d f|_M = 0$ at y. Let G be the subgroup of S_n generated by the reflections at the mirrors containing M. Let $M^\perp$ be the orthogonal complement to M at y. The action of G on $M^\perp$ is a group generated by reflections. The intersection of the corresponding mirrors in $M^\perp$ is trivial, coincides with y.

Let a linear function ℓ on $M^\perp$ be invariant with respect to the action of G. Then it is invariant with respect to each reflection in G and thus, its gradient lies in the corresponding mirror. This implies that $\ell = 0$.

The restriction $f|_{M^\perp}$ is an G-invariant function on $M^\perp$ and thus, its differential $d\,f|_{M^\perp}$ at y equals zero. By assumptions we have $d\,f|_M = 0$ and $d\,f|_{M^\perp} = 0$ at y. Hence, $d\,f|_y = 0$ and y is a critical point of f. Lemma 2 is proved. □

5.2 Proof of Theorem 3

The symmetric group S_{N-1} acts on the space $\mathbb{C}_y^{N-1}$ permuting coordinates. Let

$$\pi : \mathbb{C}_y^{N-1} \to \mathbb{C}_x^{N-1}, \qquad (y_1, ..., y_{N-1}) \mapsto (x_1, ..., x_{N-1}),$$

be the quotient map, where x_n are the elementary symmetric functions defined in (3).

Let $h = N + k$. Consider the lifting of the short fusion potential to the preimage of π, $f(y) = V^0_{N,k}(\pi(y)) = (y_1^h + y_2^h + \ldots + y_{N-1}^h)/h$. The Milnor fiber of the germ f is invariant with respect to the S_{N-1}-action and thus, the group S_{N-1} acts on the Milnor lattice of f. The map π maps the Milnor fiber of f onto the Milnor fiber of $V^0_{N,k}$.

The Milnor lattice of the germ $f(y)$ is the tensor product of $N-1$ copies of the Milnor lattice of y^h (section 3). A permutation σ from the group S_{N-1} acts on the tensor product permuting the factors and multiplying the result by $(-1)^{\varepsilon(\sigma)}$, where $\varepsilon(\sigma)$ is the parity of σ. According to Gabrielov's theorem [Gab] the germ f has a distinguished basis consisting of the tensor products of the basis vanishing cycles of the germ y^h, $\Delta_{\bar\imath} = \delta_{i_1} \otimes \ldots \otimes \delta_{i_{N-1}}$, where $\bar\imath = (i_1, \ldots, i_{N-1}) \in \{1, 2, \ldots, h-1\}^{N-1}$, ordered lexicographically. Namely, $\bar\imath = (i_1, \ldots, i_{N-1}) < \bar\jmath = (j_1, \ldots, j_{N-1})$, if there exists n such that $i_m = j_m$ for $m > n$ and $i_n < j_n$. To describe the quadratic form of f on the Milnor lattice we describe the corresponding bilinear symmetric form by the rule: for $\bar\imath < \bar\jmath$, we have $(\Delta_{\bar\imath}, \Delta_{\bar\jmath})_q = 1$ if j_n equals either i_n or $i_n + 1$ for all $n = 1, 2, \ldots, N-1$, and $(\Delta_{\bar\imath}, \Delta_{\bar\jmath})_q = 0$ otherwise.

Denote by $Q_{N,k}$ the set $\{\bar\imath = (i_1, \ldots, i_{N-1}) : 1 \le i_{N-1} < \ldots < i_1 \le h-1\}$ and by $\bar Q_{N,k}$ the set $\{\bar\imath : 1 \le i_{N-1} \le \ldots \le i_1 \le h-1\}$.

Lemma 3 *The cycles $\widetilde\Delta_{\bar\imath} = \pi_\star(\Delta_{\bar\imath})$ with $\bar\imath \in Q_{N,k}$ form a distinguished basis of the Milnor lattice of the short fusion potential $V^0_{N,k}(y)$. Moreover the cycles $\Delta_{\bar\imath}$ with $\bar\imath \in Q_{N,k}$ can be realized geometrically so that $\Delta_{\bar\imath} \cap C = \emptyset$.*

We finish the proof of Theorem 3 and then prove Lemma 3.

The intersection number of the vanishing cycles $\widetilde\Delta_{\bar\imath}$, $\widetilde\Delta_{\bar\jmath}$ of the short fusion potential for $\bar\imath, \bar\jmath \in Q_{N,k}$ can be obtained in the following way. By Lemma 3 the cycles $\Delta_{\bar{\imath}}$ and $\Delta_{\bar{\jmath}}$ can be realized geometrically so that $\Delta_{\bar{\imath}} \cap C = \emptyset$ and $\Delta_{\bar{\jmath}} \cap C = \emptyset$. The intersection points of the geometric cycles $\widetilde{\Delta}_{\bar{\imath}}$ and $\widetilde{\Delta}_{\bar{\jmath}}$ do not lie in $\pi(C)$ and are the images under π of the intersection points of the cycle $\Delta_{\bar{\imath}}$ with all the cycles of the form $\sigma\Delta_{\bar{\jmath}}$ for $\sigma \in S_{N-1}$. Thus, $(\widetilde{\Delta}_{\bar{\imath}}, \widetilde{\Delta}_{\bar{\jmath}}) = \sum_{\sigma \in S_{N-1}} (\Delta_{\bar{\imath}}, \sigma\Delta_{\bar{\jmath}})$. It is not difficult to see that for $\bar{\imath}$ and $\bar{\jmath}$ in $Q_{N,k}$ and $\sigma \neq 1$ we have $(\Delta_{\bar{\imath}}, \sigma\Delta_{\bar{\jmath}}) = 0$. Hence $(\widetilde{\Delta}_{\bar{\imath}}, \widetilde{\Delta}_{\bar{\jmath}}) = (\Delta_{\bar{\imath}}, \Delta_{\bar{\jmath}})$ and thus, the Dynkin diagram of the short fusion potential $V_{N,k}(x)$ in the distinguished basis $\{\widetilde{\Delta}_{\bar{\imath}}\}$ is a part of the described Dynkin diagram of the function f, the part corresponding to the vanishing cycles $\Delta_{\bar{\imath}}$, $\bar{\imath} \in Q_{N,k}$. In particular this implies that $(\widetilde{\Delta}_{\bar{\imath}}, \widetilde{\Delta}_{\bar{\jmath}})_q = (\Delta_{\bar{\imath}}, \Delta_{\bar{\jmath}})_q$ $(\bar{\imath}, \bar{\jmath} \in Q_{N,k})$.

For $\lambda \in P_{N,k}$, set

$$\bar{\imath}(\lambda) = (1 + \lambda_1 + \lambda_2 + \ldots + \lambda_{N-1}, 1 + \lambda_2 + \ldots + \lambda_{N-1}, \ldots, 1 + \lambda_{N-1}) \in Q_{N,k}.$$

It is not difficult to see that the correspondence $\alpha_\lambda \mapsto \widetilde{\Delta}_{\bar{\imath}(\lambda)}$ between the basis elements α_λ, $\lambda \in P_{N,k}$, of the lattice of the Verlinde algebra of $su(N)_k$ and the basis vanishing cycles $\{\widetilde{\Delta}_{\bar{\imath}}\}$, $\bar{\imath} \in Q_{N,k}$, defines the isomorphism of Theorem 3. □

Proof of Lemma 3. Fix a small $s \in (0, 1]$ and let $g(y) = g_s(y)$ be the deformation of the function y^h described in Lemma 1. Set $\widetilde{f}(y) = \frac{1}{h}\sum_{i=1}^{N-1} g(y_i)$. The function $\widetilde{f}$ is an S_{N-1}-invariant deformation of $f(y)$ with only non-degenerate critical points. Moreover, $\widetilde{f}$ has different critical values at different orbits of critical points. The critical points of $\widetilde{f}$ are the points $p_{\bar{\imath}} = (y^{(i_1)}, y^{(i_2)}, \ldots, y^{(i_{N-1})})$ for $\bar{\imath} = (i_1, \ldots, i_{N-1}) \in \{1, 2, \ldots, h-1\}^{N-1}$. All the critical values $z_{\bar{\imath}} = \frac{1}{h}\sum_{j=1}^{N-1} g(y^{(i_j)})$ are real and if $\bar{\imath}$ and $\bar{\jmath}$ are in $\bar{Q}_{N,k}$ and $\bar{\imath} < \bar{\jmath}$, then $z_{\bar{\imath}} > z_{\bar{\jmath}}$.

Let $\widetilde{V}(x)$ be the deformation of the short fusion potential $V^0_{N,k}$ such that $\widetilde{f}(y) = \widetilde{V}(\pi(y))$. The critical points of $\widetilde{V}$ correspond to the critical points $p_{\bar{\imath}}$ of $\widetilde{f}$ with $\bar{\imath} \in Q_{N,k}$. Take the number $\frac{N-1}{h}z^{(0)}$ as a non-critical value of $\widetilde{f}$ (and thus, of $\widetilde{V}$), see Lemma 1. It follows from the proof of the Gabrielov theorem [Gab] that the vanishing cycle $\Delta_{\bar{\imath}}$ with $\bar{\imath} \in \bar{Q}_{N,k}$ vanishes along the path $v_{\bar{\imath}}(t)$, $0 \le t \le N-1$, defined by the formula: $v_{\bar{\imath}}(t) = \frac{n}{h}z^{(0)} + \frac{1}{h}\left(\sum_{j=1}^{N-n-2} g(y^{(i_j)}) + u_{i_{N-n-1}}(t-n)\right)$ for $n \le t \le n+1$. The path $v_{\bar{\imath}}(t)$ is composed from shifts of the corresponding intervals u_{i_n}.

The paths $v_{\bar{\imath}}$, $\bar{\imath} \in \bar{Q}_{N,k}$, can be deformed inside the upper half-plane so that they will not intersect each other except at their end point $\frac{N-1}{h}z^{(0)}$. The paths $v_{\bar{\imath}}$ with $\bar{\imath} \in Q_{N,k}$ as well as their small deformation do not go through the critical values of $\widetilde{f}$ at the critical points lying on the union of mirrors, i.e. through $z_{\bar{\jmath}}$ with $\bar{\jmath} \in \bar{Q}_{N,k} \setminus Q_{N,k}$.

For any $\imath \in Q_{N,k}$ the union of mirrors C forms a subbundle in the bundle of local level manifolds of $\widetilde{f}$ over the path $v_{\bar{\imath}}$, see Lemma 2. Therefore, the spheres defining the vanishing cycle $\Delta_{\bar{\imath}}$ (and thus, the cycle $\widetilde{\Delta}_{\bar{\imath}}$ itself) can be chosen in such a way that they do not intersect C. It implies that the cycle $\widetilde{\Delta}_{\bar{\imath}} = \pi(\Delta_{\bar{\imath}})$ vanishes along the path $v_{\bar{\imath}}$. The system of paths $\{v_{\bar{\imath}}(t) : \bar{\imath} \in Q_{N,k}\}$ satisfies the conditions for a system of paths to define a distinguished basis of vanishing cycles. This implies the statement of Lemma 3. □

5.3 Proof of Theorem 4

Consider a function

$$F(x_1, \ldots, x_N) = V^0_{k+1,N}(x_1, \ldots, x_k) + \sum_{i=k+1}^{N} x_i x_{N+k-i+1},$$

which is a stabilization of the short fusion potential $V^0_{k+1,N}(x_1, ..., x_k)$. Both functions: the function $F(x_1, ..., x_N)$ and the short fusion potential $V^0_{N+1,k}(x_1, ..., x_N)$ are quasi-homogeneous polynomials of degree $N + k + 1$ with the weight of the variable x_n equal to n, $n = 1, ..., N$. Both functions have an isolated critical point at the origin. This proves Theorem 4. □

References

[A] Arnold, V.I.: Critical points of functions and classification of caustics. Uspekhi Matematicheskikh Nauk, Vol. 29, N 3, 243–244 (1974, in Russian).

[AGV] Arnold, V. I., Gusein-Zade, S. M., Varchenko, A. N.: Singularities of Differentiable Maps, Volume II. Birkhäuser, Boston, Basel, Berlin, 1988.

[Ch] Chmutov S.V.: Examples of projective surfaces with many singularities. Journal of Algebraic Geometry, Vol. 1, 191–196 (1992).

[E] Ebeling, W.: Quadratische Formen und Monodromee–Gruppen von Singularitäten. Mathematische Annalen, Vol. 255, 463–498 (1981).

[Gab] Gabrielov, A.M.: Intersection matrices for certain singularities. Functional Analysis and its Applications, Vol. 7, 182–193 (1973).

[Gep] Gepner D.: Fusion Rings and Geometry. Comm. Math. Phys., Vol. 141, 381–411 (1991).

[G] Goodman F.M., Wenzl H.: Littlewood-Richardson coefficients for Hecke algebras at roots of unity. Adv. in Math., Vol. 82, 244–265 (1990).

[Gor] Goryunov V.V.: Symmetric quartics with many nodes. In: Advances of Soviet Math., vol.21, 147–161, AMS, Providence R.I. (1994).

[K] Kac V.: Infinite dimensional Lie algebras. Cambr. Univ. Press, 3d Edition, 1990.

[L] Looijenga E.: The complement of the bifurcation variety of a simple singularity. Invent. Math. Vol. 23:2, 105–116 (1974).

[M] Macdonald I.: Symmetric functions and Hall polynomials. Oxford Univ. Press, 1995.

[S] Steenbrink J.H.M.: Intersection forms for quasihomogeneous singularities. Compositio Math., Vol. 34:2, 211–223 (1977).

[Wal] Walton M.A.: Algorithm for WZW fusion rules: A proof. Physics Letters B, Vol. 241, No. 3, 365–368 (1990).

[War] Warner N., Private communication.

[Z] Zuber J.-B.: Graphs and reflection groups. Comm. Math. Phys., Vol.179, 265–294 (1996).