ON A PROBLEM BY M. KAZARIAN

V. A. VASSILIEV

1. Results

We prove that the group of symmetries of a function germ with zero 2-jet and isolated singularity has no compact positive-dimensional subgroups.

Denote by $Diff_0(\mathbf{C}^n)$ (respectively, $Diff_0(\mathbf{R}^n)$) the group of germs of diffeomorphisms $(\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ (respectively, $(\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$).

THEOREM $1_{\mathbf{C}}$. *Suppose that $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ is the germ of a holomorphic function with isolated singularity at 0, and its 2-jet $j^2 f$ is equal to zero. Let G be a compact Lie group of its symmetries, i.e. a subgroup of $Diff_0(\mathbf{C}^n)$ such that $f \circ g \equiv f$ for any $g \in G$. Then G is discrete.*

THEOREM $1_{\mathbf{R}}$. *Suppose that $f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ is the germ of a C^∞-smooth function with a finite Milnor number, and its 2-jet $j^2 f$ is equal to zero. Let G be a compact Lie group of its symmetries, i.e. a subgroup of $Diff_0(\mathbf{R}^n)$ such that $f \circ g \equiv f$ for any $g \in G$. Then G is discrete.*

The last statement was conjectured by M. Kazarian in a relation with his works in the equivariant homology theory, see [3] and his Appendix to the Russian version of [5]. Namely, according to him, this theorem terminates the proof of the following statement:
The cohomology groups of the Universal Vassiliev Complex of classes of stable right equivalence of function germs (see [5]*) are isomorphic (modulo torsion or with coefficients in a field of characteristic* 0*) to the cohomology group of the stable Lagrangian Grassmannian. Therefore, all integral Lagrange characteristic classes can be expressed as classes, Poincaré dual to certain cycles of Lagrangian singularities.*

2. Proofs

Suppose that G is a compact positive-dimensional group of symmetries of a (real or coomplex) function f with finite Milnor number.

PROPOSITION 1 (H. Cartan, S. Bochner, see e.g. [4]). *The holomorphic (respectively, C^∞-smooth) action of a compact Lie group on a complex (respectively, real) smooth manifold in a neighborhood of its fixed point is holomorphically (respectively, C^∞-) equivalent to a linear one.*

So, we can assume that our group G is a subgroup of $GL(\mathbf{C}^n)$ (respectively, of $GL(\mathbf{R}^n)$). Then by the *sufficient jet theorem* (see e.g. [2]) it is also a group of symmetries of of some polynomial function with finite Milnor number, namely of any sufficiently high Taylor expansion of f at 0. So, we can assume that f is a

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polynomial. If f is a real function, then the obvious extension of the action of G into the complex domain preserves the obvious complexification of f, thus Theorem $1_{\mathbf{R}}$ is reduced to Theorem $1_{\mathbf{C}}$.

Finally, we can assume that $G = S^1$: indeed, any non-discrete compact Lie group contains such a subgroup.

Let us identify it with the unit circle in $\mathbf{C}^1$, so that the coordinate t in S^1 will be a complex number with $|t| = 1$.

The Schur's lemma and the Peter-Weyl theorem imply immediately that any representation of an Abelian compact Lie group in $\mathbf{C}^n$ splits into a sum of one-dimensional representations.

For the group S^1 this means that there are integer numbers $a_1, \dots, a_n$ such that in some linear coordinates $z_1, \dots, z_n$ in $\mathbf{C}^n$ this action is as follows: $G_t : (z_1, z_2, \dots, z_n) \mapsto (t^{a_1} z_1, t^{a_2} z_2, \dots, t^{a_n} z_n)$.

Consider the Taylor expansion of f in this coordinate system, $f = \sum_{\alpha \in \mathbf{Z}^n_+} f_\alpha z^\alpha$, where $\alpha = (\alpha_1, \dots, \alpha_n)$ are multiindices, f_α are coefficients, and $z^\alpha \equiv z_1^{\alpha_1} \cdot \dots \cdot z_n^{\alpha_n}$.

As we assume that the action of S^1 is non-trivial, at least one of numbers a_i is different from zero. Therefore all multiindices α corresponding to non-zero coefficients f_α lie in the hyperplane $L(a)$ given by the equation $a_1\alpha_1 + \dots + a_n\alpha_n = 0$.

Consider the intersection of this hyperplane with the simplex in $\mathbf{R}^n_+$ distinguished by the equation $\sum \alpha_i = 3$.

This simplex contains n smaller simplices $\Delta_1, \dots, \Delta_n$, homotetic to it and placed at all its wedges; namely, Δ_i is the convex hull of the point $(0, \dots, 0, 3, 0, \dots, 0)$ ("3" on the i-th place) and $n-1$ points obtained from it by replacing "3" by "2" and one of zeros by "1".

Now Theorem $1_{\mathbf{C}}$ follows from the next two propositions.

Consider the convex hull of the union of all points $\alpha \in \mathbf{Z}^n$ such that $f_\alpha \neq 0$ and the origin in $\mathbf{Z}^n$. Obviously, this convex hull lies in the plane $L(a)$.

PROPOSITION 2. *If f has an isolated singularity at 0 and $j^2 f = 0$, then this convex hull should have noinempty intersections with all n simplices Δ_i.*

PROPOSITION 3. *A hyperplane in $\mathbf{R}^n$ passing through the origin cannot intersect all n simplices Δ_i simultaneously.*

Proof of Proposition 2. The polynomial f defines an isolated singularity, therefore for any i it contains a monomial $f_\alpha z^\alpha$, $f_\alpha \neq 0$, whose exponent α lies at disctance at most 1 from the i-th coordinate axis, see e.g. [1], [2]. But if deg $\alpha \geq 3$, then the segment joining such a point α and 0 intersects the simplex Δ_i. □

Proposition 3 is equivalent to the following assertion: the sign of the determinant of any vectors $v_1, \dots, v_n$, $v_i \in \Delta_i$, does not depend on the choice of such n vectors.

To prove this assertion, let us introduce the norm $\sum |a_i|$ in $\mathbf{R}^n$. Then the matrix, formed by the coordinate expansion of our n vectors v_i, is equal to $2I + A$, where I is the unity matrix and $\|A\| \leq 1$. The determinant of any such matrix always is positive, and our proposition is proved.

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