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HOLONOMIC LINKS AND SMALE PRINCIPLES FOR MULTISINGULARITIES

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ABSTRACT
A loop $S^1 \to \mathbb{R}^n$ is holonomic if it is the $(n-1)$-jet extension of a function $S^1 \to \mathbb{R}^1$. We prove that for $n = 3$ any tame link in $\mathbb{R}^n$ is isotopy equivalent to a holonomic one; for $n > 3$ the space of holonomic links is homotopy equivalent to the space of all differentiable links.

1. Introduction

Denote by C_k the disjoint union of k circles with fixed parametrization, and by $\mathbb{R}^n$ the real arithmetic space with coordinates $x_1, \ldots, x_n$. Any smooth function $f : C_k \to \mathbb{R}^1$ defines a map $j^{(n-1)}f : C_k \to \mathbb{R}^n$: the $(n-1)$-jet extension of f given by the formula

$$x_1(t) = f(t), x_2(t) = f'(t), \ldots, x_n(t) = f^{(n-1)}(t). \tag{1.1}$$

A link in $\mathbb{R}^n$ is *holonomic* if it appears in this way from some f.

Theorem 1. *For any $n \geq 4$ and $k \geq 1$, the space of holonomic links of k strings in $\mathbb{R}^n$ is homotopy equivalent to the space of all differentiable k-string links; this homotopy equivalence is induced by the identical embedding.*

Theorem 2. *Any tame link in $\mathbb{R}^3$ is isotopy equivalent to a holonomic one.*

Problem. Is it true that any two holonomic links in $\mathbb{R}^3$ are isotopy equivalent if and only if they are isotopic in the space of holonomic links? More generally, does the obvious embedding define an isomorphism of all homology groups of the spaces of holonomic and usual links?

This conjecture is motivated by the fact that the theory of finite-order invariants of holonomic links is isomorphic to that for usual links, in particular the spectral sequences from [8] calculating homology classes of both spaces are isomorphic beginning with the term E_1.

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These theorems are an illustration of two fundamental principles of S. Smale. The first one (= the Smale–Hirsch principle, or parametric h-principle) in its general form states that in many situations the spaces of smooth maps $M \to N$ without certain singularities are topologically similar to the corresponding spaces of admissible sections of the jet bundle $J^{\cdot}(M, N) \to M$, see ([7]), ([5]), ([6]) ([9]), ([3]) etc. Our Theorems 1 and 2 are an extension of this fact to the case of multisingularities and multijet bundles.

The second principle of Smale asserts that in the higher dimensions many things (especially the ones related to the links) are easier than their low-dimensional analogues. In fact, our Theorem 1 is much more standard than Theorem 2.

Theorem 2 has a braid-theoretical interpretation. All crossing points of a link or braid diagram are naturally divided into two types. Indeed, given a crossing point, the *ordered* pair of corresponding tangent vectors (where the lower string is taken first) defines an orientation of the plane. Our crossing point is called positive if and only if this orientation coincides with a standard one.

Definition 1. A braid diagram is *normal*, if on any of its strings all the positive crossing points lie higher than all the negative ones.

Then we have the following specialization of the Alexander theorem.

Theorem 3. *Any tame link in* $\mathbb{R}^3$ *can be represented by a normal braid.*

2. Graphical Calculus for Holonomic Links in $\mathbb{R}^3$

Let us project our holonomic curve $j^2 f(C_k) \subset \mathbb{R}^3$ to the plane (x_1, x_2); the composition $p \circ j^2 f : C_k \to \mathbb{R}^2$ of this projection and the holonomic embedding $j^2 f$ is no other but the 1-jet extension of f.

Proposition 1. (*i*) *For any generic smooth function* $f : C_k \to \mathbb{R}^1$, *the image of* C_k *in* $\mathbb{R}^2$ *under the corresponding 1-jet extension map* $j^1 f$ *satisfies the following conditions:*

(*a*) *it is a smooth closed oriented immersed curve;*

(*b*) *in the points of intersection with the axis* $x_2 = 0$ *it is orthogonal to this axis and has nonzero curvature; in the half-plane* $\{x_2 > 0\}$ (*respectively,* $\{x_2 < 0\}$) *the coordinate* x_1 *strictly increases* (*respectively, decreases*) *along it;*

(*c*) *all its self-intersection points are the double transversal points* (*in particular, they do not lie in the axis* $x_2 = 0$);

(*d*) *if we supply these intersection points with the over- and under-crossing information defined by the behavior of the third coordinate* x_3 *at the corresponding points of the holonomic curve* $j^2 f(C_k) \subset \mathbb{R}^3$, *then all crossing points in the half-plane* $\{x_2 > 0\}$ (*respectively,* $\{x_2 < 0\}$) *are positive* (*respectively, negative*) *with respect to the orientation* $dx_1 \wedge dx_2$.

(*ii*) *Any link diagram in* $\mathbb{R}^2$ *satisfying these conditions* (*a–d*) *is the diagram of some holonomic link.*

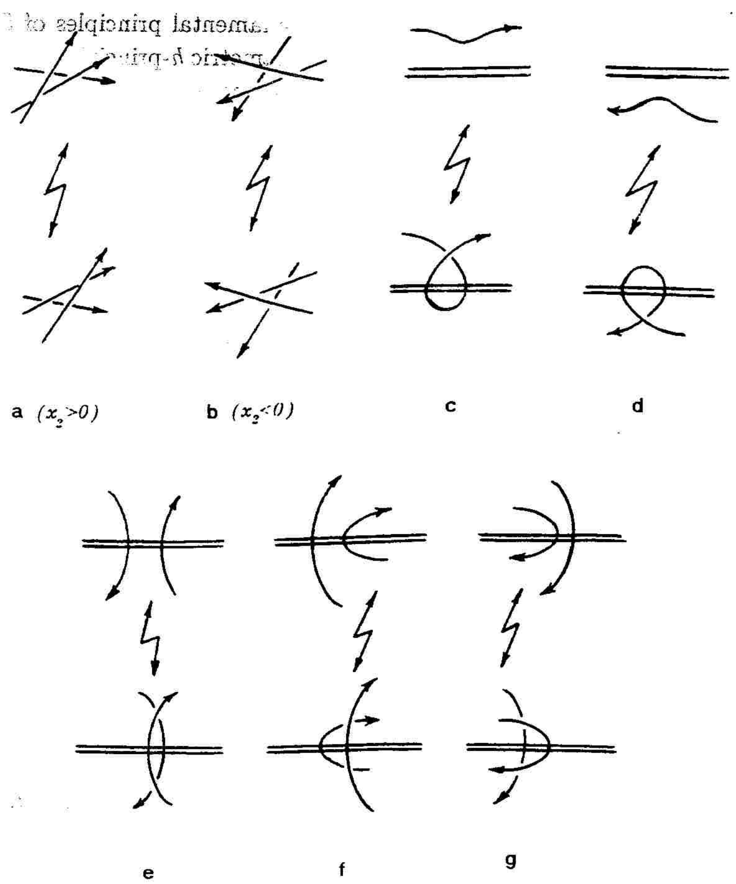

Fig. 1. Reidemeister moves for holonomic links.

(iii) Two holonomic links in $\mathbb{R}^3$*, whose diagrams satisfy these conditions, are isotopic in the space of holonomic links if and only if these diagrams can be connected by a finite chain of local moves shown in Figs. 1a–1g (where the double horizontal line denotes the axis* $x_2 = 0$*).*

(In particular, the one- and two-string Reidemeister moves can happen only at the points of that axis, and the third move only outside of it.)

Proof. In the terms of the function f, condition (a) and the part of condition (b) concerning the non-vanishing curvature mean that f' and f'' nowhere vanish simultaneously, condition (d) and the rest of condition (b) are obvious, and condition (c) claims that there are no two points x, y such that $f(x) = f(y)$, $f'(x) = f'(y)$, $f''(x) = f''(y)$ or $f(x) = f(y)$, $f'(x) = f'(y) = 0$, and there are no three points x, y, z such that $f(x) = f(y) = f(z)$, $f'(x) = f'(y) = f(z)$. All these conditions are satisfied for generic f and statement (i) is proved.

In a generic one-parametric family of functions f_λ only the following additional degenerations can appear for particular values of the parameter λ.

1. The Morse surgery, connecting two function germs locally equivalent to $x^3 + \epsilon x + c$ and $x^3 - \epsilon x + c$ or $-x^3 + \epsilon x + c$ and $-x^3 - \epsilon x + c$; the corresponding metamorphoses of the curve are exactly the ones shown in Figs. 1c and 1d.

2. A pair of points x, y, at which $f(x) = f(y)$, $f'(x) = f'(y) = 0$, but $f''(x) \neq f''(y)$: see Figs. 1e, 1f and 1g.

3. A triple of points x, y, z, at which $f(x) = f(y) = f(z)$, $f'(x) = f'(y) = f(z) \neq 0$ and all values $f'(x), f'(y)$ and $f(z)$ are distinct, see Figs. 1a and 1b.

4. Two points x, y, at which $f(x) = f(y)$, $f'(x) = f'(y)$, $f''(x) = f''(y)$: this situation defines a self-intersection of the holonomic curve and thus does not define any admissible move.

This proves statement (iii). Finally, given a closed plane curve satisfying conditions (a–c), we lift it to a holonomic knot in the following way. Close to any point (x_1, x_2) of this curve with $x_2 \neq 0$, we can parametrize it by the coordinate x_1 and express x_2 as $x_2 = \phi(x_1)$. Then the "interior" parameter t of the curve is locally defined by the equation $t = \int(1/\phi(x_1))dx_1 + const$, and the coordinate x_3 at the corresponding point is equal to $x_2 d\phi/dx_1$. It is easy to see that these functions t and x_3 can be continued smoothly to the points where $x_2 = 0$. □

Note that the *interior length* of our curve (i.e., the period of the parametrization $\mathbf{R}^1 \to S^1$) is defined uniquely by its image in $\mathbf{R}^2$. However, dilating our space $\mathbf{R}^3$ we can replace any holonomic knot by equivalent one of arbitrary other length. Moreover, using the admissible deformations we can easily replace any holonomic link by equivalent one, all whose components have prescribed interior lengths.

3. Proof of Theorem 2

Lemma 1. *Any tame oriented link in $\mathbb{R}^3$ is isotopy equivalent to a link whose diagram satisfies conditions (a–c) of Proposition 1.*

This is an immediate corollary of the Alexander theorem, see e.g. [2].

Definition 2. A crossing point of a link diagram satisfying conditions (a–c) of Proposition 1 is *irregular* if the condition (d) is not satisfied at this point.

Proposition 2. *Any irregular crossing point of a diagram satisfying conditions (a–c) can be killed by some usual Reidemeister moves in such a way that the new diagram satisfies again these conditions and no new irregular points appear in it.*

Proof. We consider only irregular points in the half-plane $x_2 > 0$: the points from the other half-plane can be killed in exactly the same way.

Such irregular points always look as in Fig. 2a (i.e., they can be locally reduced to this picture by a diffeomorphism of the plane (x_1, x_2) preserving the axis $x_2 = 0$, the orientation and the foliation of this plane into the lines $\{x_1 = const\}$). The elimination of this point A consists of three pictures. First, we suppose that the

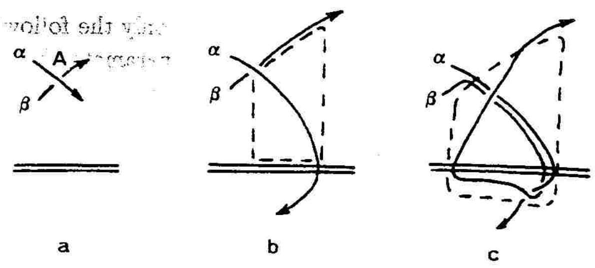

Fig. 2. The surgery in the simplest case.

upper local string at A (i.e. the one denoted by α in our picture) extends without crossings to the intersection with the axis $x_2 = 0$ at some point B and, moreover, there are no other strings in the curvilinear quadrilateral bounded by this axis (from bottom), vertical lines $x_1 = x_1(A)$ and $x_1 = x_1(B)$ and the string β, see Fig. 2b. Then the desired surgery lies in the similar quadrilateral bounded from below by the line $x_2 = -\epsilon$, ϵ arbitrarily small, and its result looks as in Fig. 2c.

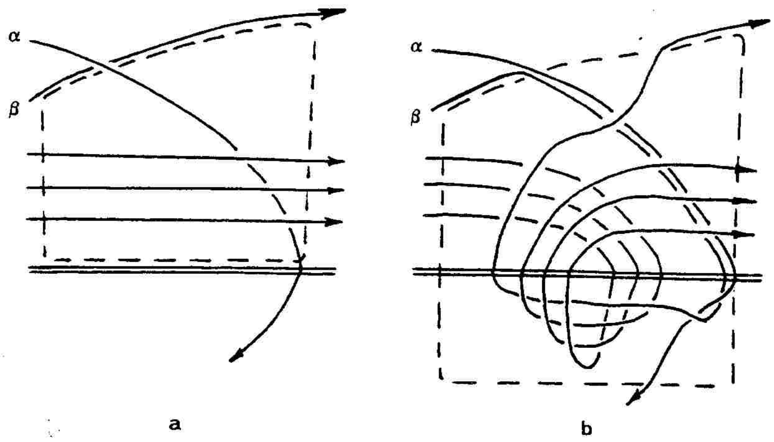

Fig. 3. Missing the trivial lower strings.

More generally, suppose that there are several other strings intersecting the first quadrilateral, namely, all of them enter it in the points of the line $x_1 = x_1(A)$, leave it in the points of the line $x_1 = x_1(B)$, have no intersections in it and intersect the string α *regularly* at a unique point each, i.e. are situated as shown in Fig. 3a. The corresponding move again lies in the similar quadrilateral as in Fig. 2c and its result is as shown in Fig. 3b. We may assume that the number ϵ defining the bottom line $x_2 = -\epsilon$ of this quadrilateral is so small that there appear no new crossing points other than the ones indicated in our picture.

Now we show that for any irregular point the geometry of the corresponding string α and of the domain bounded by it from above can be reduced to the special form indicated in Fig. 3a. By an arbitrarily small deformation of the link we can ensure that for some $\delta > 0$ its diagram has no other crossing points and intersections with the axis $x_2 = 0$ in the strip $x_1 \in [x_1(A) - \delta, x_1(A) + \delta]$, in particular it behaves as in Fig. 4a in the pentagon bounded by the axis $x_2 = -\epsilon$, the boundary lines of this strip and the strings α and β. Then we replace it by the curve shown in Fig. 4b. The resulting link is the desired one. Proposition 2 is proved, and Theorem 2 is a direct corollary of it. □

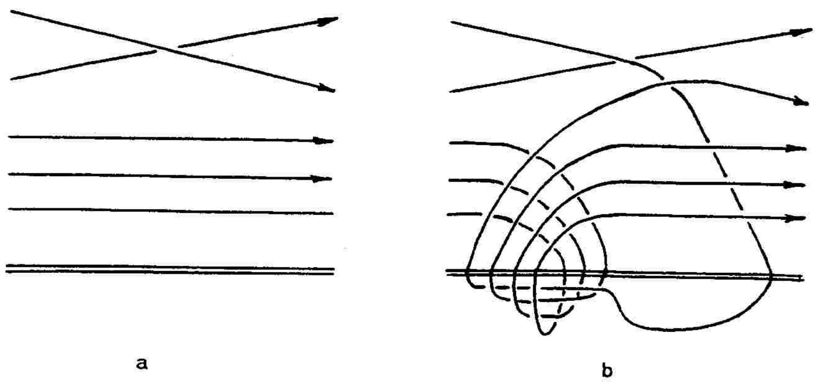

Fig. 4. Cutting the upper string.

To prove Theorem 3, we take the diagram of a holonomic link equivalent to the given one and apply to it several surgeries indicated in Figs. 1e, 1f and 1g in such a way that all the intersection points of the axis $x_2 = 0$ with the resulting link diagram, at which the corresponding string goes from the negative half-plane into the positive one, are separated by some point C of this axis from all the intersection points where the strings go down. Then we cut our plane (x_1, x_2) along the ray $L = \{x_2 = 0, x_1 \in (-\infty, C)\}$; the resulting braid in the domain $\mathbb{R}^2 \setminus L \sim \mathbb{R}^1 \times (0,1)$ is normal. □

4. On the Proof of Theorem 1

This proof essentially repeats the proof of the Smale–Hirsch principle for the spaces of functions without (mono)singularities of large codimension, see [9], therefore we only outline it emphasizing the features of our present situation.

The number of strings k and the dimension n of the target space $\mathbb{R}^n$ are fixed throughout this section.

Denote by $\mathcal{F}$ the space of all smooth functions $C_k \to \mathbb{R}^1$ supplied with the C^n-topology, by $\mathcal{K}$ the space of all smooth maps $C_k \to \mathbb{R}^n$ with the C^1-topology, and by Σ the subset in $\mathcal{K}$ consisting of all maps which are not the knots, i.e. have either self-intersection points or the points of vanishing derivative.

The set Σ has a natural partition into the strata of finite codimension (any stratum consists of the maps having equivalent (up to the orientation-preserving diffeomorphisms of C_k) finite configurations of points in C_k which are glued together by these maps or at which these maps have singular points) plus some strata of infinite codimension, consisting of maps having infinitely many singular points; see [8] and [9], Chapter V.

Definition 3. A finite-dimensional affine subspace in $\mathcal{K}$ is *generic* if it does not intersect strata of infinite codimension and is transversal to all finite-dimensional strata of the canonical stratification of Σ.

Let $L = L(d)$ be an affine subspace in $\mathcal{F}$ of finite dimension d, and $L^{(i)}$, $i = 0, 1, \ldots, n-1$, the spaces of all functions of the form $f^{(i)}$ where $f \in L$. If L is generic, then all these spaces have the same dimension d. Denote by L^{*n} the space of all maps $\Phi : C_k \to \mathbb{R}^n$ of the form $\Phi = (\phi_0, \ldots, \phi_{n-1})$, $\phi_i \in L^{(i)}$ for any i.

There is a natural embedding j^{n-1} of the space $L(d)$ into $L(d)^{*n}$, namely, to any function $\phi \in L(d)$ there corresponds the map

$$(\phi, \phi', \ldots, \phi^{(n-1)}) \in L(d)^{*n}.$$

The image of $L(d)$ under this embedding will be denoted by $\tilde{L} \equiv \tilde{L}(d)$.

The spaces $L(d)^{*n} \setminus \Sigma$ approximate the space of all links in $\mathbb{R}^n$ in the following sense.

Lemma 2. *There is a sequence of subspaces*
$\cdots \subset L(d) \subset L(d+1) \subset \cdots$ *in $\mathcal{F}$ such that*

a) all the spaces $L(d)^{*n}$ and $\tilde{L}(d) \subset L(d)^{*n}$ defined by it are generic in the sense of Definition 3,

b) for any class $\gamma \in H_*(\mathcal{K} \setminus \Sigma)$ (respectively, for any continuous map χ of a finite-dimensional CW-complex to $\mathcal{K} \setminus \Sigma$) there is a number d such that γ can be represented by a cycle lying in $L(d)^{*n} \setminus \Sigma$ (respectively, χ is homotopic in $\mathcal{K} \setminus \Sigma$ to a map whose image lies in $L(d)^{*n}$), and

c) for any d and any two cycles $\gamma_1, \gamma_2 \in H_*(L(d)^{*n} \setminus \Sigma)$ which are homologous in $\mathcal{K} \setminus \Sigma$ (respectively, for two maps χ_1, χ_2 of the same finite-dimensional complex into $L(d)^{*n} \setminus \Sigma$ which are homotopic as the maps into $\mathcal{K} \setminus \Sigma$) there is $d' \geq d$ such that γ_1 and γ_2 are homologous (respectively, χ_1 and χ_2 are homotopic) to each other in $L(d')^{*n} \setminus \Sigma$.

Moreover, any system of subspaces $\cdots \subset L'(d) \subset L'(d+1) \subset \cdots$ which is sufficiently close to this one in the Grassmannian metric also satisfies all these conditions.

This follows directly from the Weierstrass approximation theorem and Thom multijet transversality theorem, cf. [8], [9]. □

Proposition 3. *If all subspaces $L(d)^{*n}$ and $\tilde{L}(d)$, $d \to \infty$, are in general position with respect to the canonical stratification of Σ, then for any $d' \geq d$ the* *identical embeddings $\tilde{L}(d) \to L(d)^{*n}$ and $L(d) \to L(d')$ induce four isomorphisms*

$$\begin{array}{ccc} H^i(L(d')^{*n} \setminus \Sigma) & \to & H^i(L(d)^{*n} \setminus \Sigma), \\ \downarrow & & \downarrow \\ H^i(\tilde{L}(d') \setminus \Sigma) & \to & H^i(\tilde{L}(d) \setminus \Sigma) \end{array} \tag{4.1}$$

for all $i < (n-3)(d/(n+2)+1)$.

Proposition 4. *For any $n \geq 4$, any $d \geq 4n$ and any generic $L(d)$ the spaces $L(d)^{*n} \setminus \Sigma$ and $\tilde{L}(d) \setminus \Sigma$ are simple-connected.*

Theorem 1 follows immediately from these two propositions and from the Whitehead's theorem.

Proof of Proposition 3. As in [1], [8], [9], we use the Alexander isomorphisms

$$H^i(L(d)^{*n} \setminus \Sigma) \simeq \bar{H}_{nd-1-i}(\Sigma \cap L(d)^{*n}),$$

$$H^i(\tilde{L}(d) \setminus \Sigma) \simeq \bar{H}_{d-1-i}(\Sigma \cap \tilde{L}(d))$$

(where $\bar{H}.$ denotes the homology of the one-point compactification reduced modulo the compactifying point). We construct a simplicial resolution $\sigma_{d,n}$ of the variety $\Sigma \cap L(d)^{*n}$ and similar resolution σ_d of $\Sigma \cap \tilde{L}(d)$ exactly as in [8], [9]; by the construction σ_d is a subspace in $\sigma_{d,n}$.

Denote by $\bar{X}$ the one-point compactification of the space X, then there are canonical homotopy equivalences

$$\begin{array}{l} \overline{\sigma_d} \simeq \overline{\Sigma \cap \tilde{L}(d)} \\ \overline{\sigma_{d,n}} \simeq \overline{\Sigma \cap L(d)^{*n}}. \end{array} \tag{4.2}$$

Both spaces $\overline{\sigma_d}$, $\overline{\sigma_{d,n}}$ admit a natural increasing filtration by closed subsets defined by the complexities of the corresponding singular strata, see [9].

Lemma 3. (see [8], [9]) *If the spaces $L(d)$, $L(d)'$ are generic, then for $m = d/(n+2)$ the m-th term $F_m(\overline{\sigma_{d',n}})$ of the natural filtration of the one-point compactification of the space $\sigma_{d',n}$ is homotopy equivalent to the $(nd'-d')$-fold (respectively, $n(d'-d)$-fold, respectively, $(nd'-d)$-fold) suspension of the similar term of $\overline{\sigma_{d'}}$, (respectively, of $\overline{\sigma_{d,n}}$, respectively, of $\overline{\sigma_d}$) while the corresponding quotient spaces $\overline{\sigma_{d',n}}/F_m(\overline{\sigma_{d',n}})$, $\overline{\sigma_{d'}}/F_m(\overline{\sigma_{d'}})$, $\overline{\sigma_{d,n}}/F_m(\overline{\sigma_{d,n}})$, $\overline{\sigma_d}/F_m(\overline{\sigma_d})$ of all these four spaces are the CW-complexes whose homology groups H_t are trivial for all t which are at least by $(n-3)(d/(n+2)+1)$ smaller than the dimensions of the corresponding spaces of maps (i.e. than $d'n$, d', dn and d respectively).*

The proof repeats the proof of Proposition 4.2.4 of the book [9], see pages 102–106 there. This lemma together with the Alexander duality implies that for any $i < (n-3)(d/(n+2)+1)$ all cohomology groups H^i participating in the diagram (4.1) are actually isomorphic to each other; the fact that these isomorphisms are induced by the identical embeddings follows from the constructions of these isomorphisms as in [9], p. 106, and Proposition 3 is proved.

Proof of Proposition 4. For any generic affine subspace $\Lambda \subset \mathcal{K}$, the discriminant $\Sigma \cap \Lambda$ is a subvariety of codimension $\geq n-2$ in Λ. Therefore our proposition is true if $n > 4$, and for $n = 4$ the fundamental groups of the complements of Σ in these subspaces are generated by finitely many simple loops embracing Σ at several nonsingular points of Σ.

It follows from the construction of resolutions $\sigma_{d,n}$, σ_d that the first terms $F_1(\sigma_{d,n})$, $F_1(\sigma_d)$ of their natural filtrations are the spaces of normalizations of discriminants $\Sigma \cap L(d)^{*n}$ and $\Sigma \cap \tilde{L}(d)$ respectively; these spaces are nonsingular manifolds with boundaries, and the pre-images in them of the set of singular points of Σ have codimension $\geq n-2$. In particular, the smooth part of Σ is path-connected. Hence in the case $n = 4$ our simple loops can be unhooked from Σ by some deformations which cover the paths in the non-singular parts of $\Sigma \cap L(d)^{*n}$, $\Sigma \cap \tilde{L}(d)$ connecting the points of embracing with the boundary of Σ. □

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