Crossing probabilities in the critical 2D Ising model

Dmitry Chelkak (PDMI Steklov, St.Petersburg)

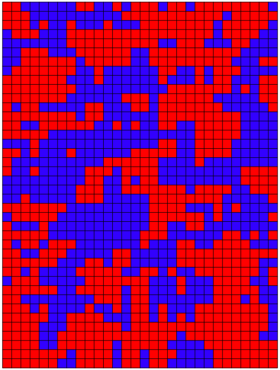
joint work with Stanislav Smirnov (Geneva)

arXiv:0910.2045: *"Universality in the 2D Ising model and conformal invariance of fermionic observables"*, 50pp.

CONFORMAL STRUCTURES AND DYNAMICS (CODY)

SEILLAC, FRANCE, MAY 2–8, 2010

2D Ising model:
(square grid)



Spins $\sigma_i = +1$ or -1.

Hamiltonian:

$$H = -\sum_{\langle ij \rangle} \sigma_i \sigma_j \, .$$

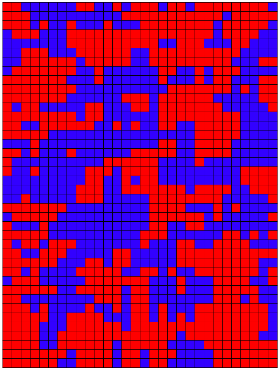
Partition function:

$$\mathbb{P}(conf.) \sim e^{-\beta H} \sim x^{\# \langle +- \rangle},$$

where

$$x = e^{-2\beta} \in [0, 1] \, .$$

2D Ising model:
(square grid)



Spins $\sigma_i = +1$ or -1.

Hamiltonian:

$$H = -\sum_{\langle ij \rangle} \sigma_i \sigma_j \,.$$

Partition function:

$$\mathbb{P}(conf.) \sim e^{-\beta H} \sim x^{\#\langle +- \rangle},$$

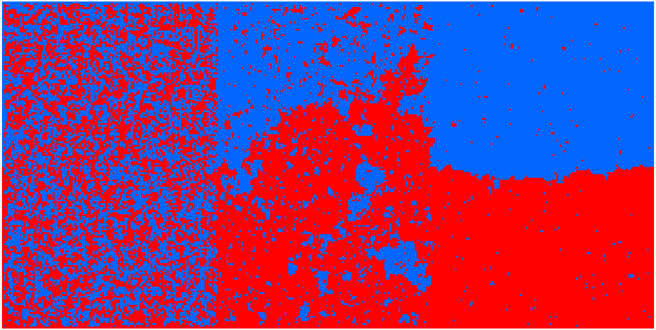
where

$$x = e^{-2\beta} \in [0,1]\,.$$

Other "lattices" (planar graphs): $H = -\sum_{\langle ij \rangle} J_{ij} \sigma_i \sigma_j \,.$

$$\mathbb{P}(conf.) \sim \prod_{\langle ij \rangle : \sigma_i \neq \sigma_j} x_{ij}, \quad x_{ij} \in [0,1].$$

Phase transition, criticality:



$x > x_{\mathrm{crit}}$ $\quad x = x_{\mathrm{crit}}$ $\quad x < x_{\mathrm{crit}}$

(Dobrushin boundary values: two marked points a, b on the boundary; $+1$ on the arc (ab), -1 on the opposite arc (ba))

[Peierls '36; Kramers-Wannier '41]: $x_{\mathrm{crit}} = \frac{1}{\sqrt{2}+1}$

Conformal invariance:

Quantities (spin correlations, crossing probabilities, etc.)

[Cardy's formula for percolation, etc.]

$\updownarrow$

Geometry (interfaces, loop ensembles, etc.)

[Schramm's SLEs, CLEs, etc.]

Conformal invariance:

Quantities (spin correlations, crossing probabilities, etc.)

[Cardy's formula for percolation, etc.]

$\updownarrow$

Geometry (interfaces, loop ensembles, etc.)

[Schramm's SLEs, CLEs, etc.]

"$\Uparrow$": SLE computations

Conformal invariance:

Quantities (spin correlations, crossing probabilities, etc.)
[Cardy's formula for percolation, etc.]

$\updownarrow$

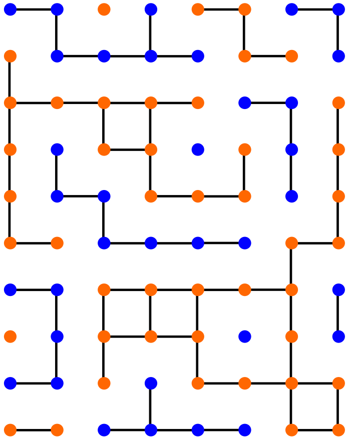
Geometry (interfaces, loop ensembles, etc.)
[Schramm's SLEs, CLEs, etc.]

"$\Uparrow$": SLE computations

"$\Downarrow$": Conformal martingale principle

Ref: S. Smirnov. *Towards conformal invariance of 2D lattice models.* [Proceedings of the international congress of mathematicians (ICM), Madrid, Spain, August 22–30, 2006]

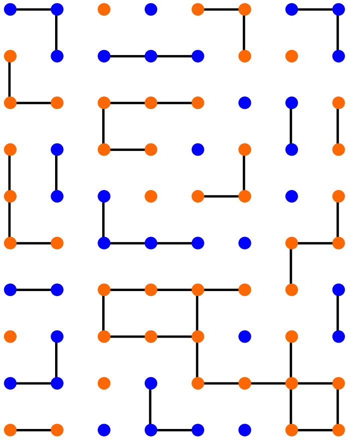
Spin- and FK-Ising models (random cluster representation):



$$\mathbb{P}(\text{spins conf.}) \sim x^{\#\langle +- \rangle}$$

$$= \prod_{<ij>} \left[x + (1-x) \cdot \chi_{s(i)=s(j)} \right]$$

Spin- and FK-Ising models (random cluster representation):

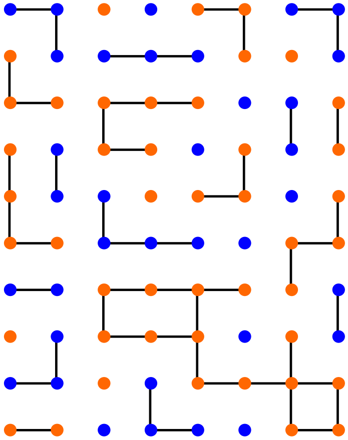


$$\mathbb{P}(\text{spins conf.}) \sim x^{\#\langle +- \rangle}$$

$$= \prod_{<ij>} \left[x + (1-x) \cdot \chi_{s(i)=s(j)} \right]$$

$$= \sum_{edges\ conf.} (1-x)^{\#open} x^{\#closed}$$

Spin- and FK-Ising models (random cluster representation):



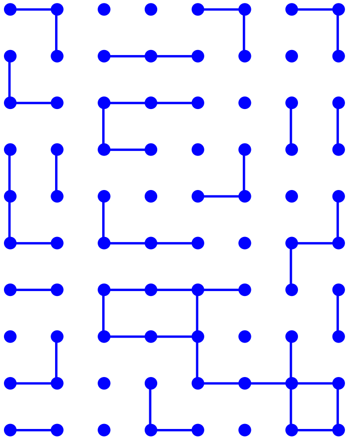
$\mathbb{P}(\text{spins\&edges conf.})$

$\sim (1-x)^{\#open} x^{\#closed}$

Open edges connect equal spins (but not all)

Erase spins:

Spin- and FK-Ising models (random cluster representation):



$\mathbb{P}$(spins&edges conf.)

$\sim (1-x)^{\#open} x^{\#closed}$

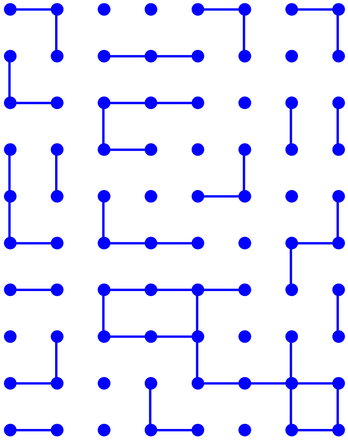
Open edges connect equal spins (but not all)

Erase spins:

$\mathbb{P}$(edges conf.)

$\sim 2^{\#clusters}(1-x)^{\#open} x^{\#closed}$

Spin- and FK-Ising models (random cluster representation):

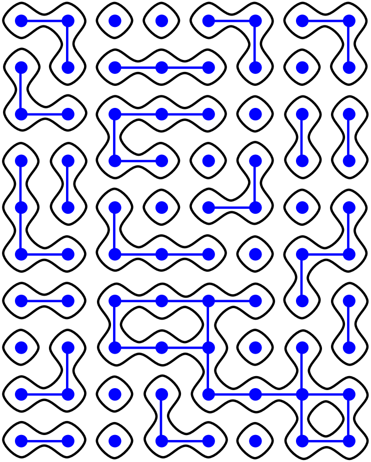


$\mathbb{P}(\text{edges conf.})$

$\sim 2^{\#clusters}(1-x)^{\#open}x^{\#closed}$

$\sim 2^{\#clusters}[(1-x)/x]^{\#open}$

Spin- and FK-Ising models (random cluster representation):



$\mathbb{P}(\text{edges conf.})$

$\sim 2^{\#clusters}(1{-}x)^{\#open}x^{\#closed}$

$\sim 2^{\#clusters}[(1-x)/x]^{\#open} \sim$

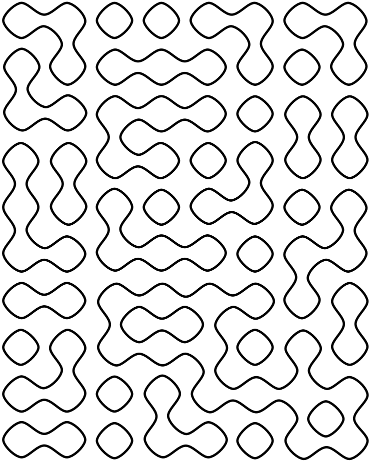
$\sqrt{2}^{\#loops}[(1-x)/(x\sqrt{2})]^{\#open}$

since

$\#loops - \#open\ edges$

$= 2\#clusters + const$

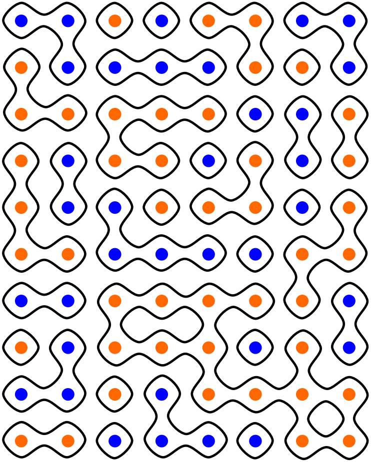
Spin- and FK-Ising models (random cluster representation):



Self-dual case ($x = x_{\mathrm{crit}}$):

$$\mathbb{P}(\text{loops conf.}) \sim \sqrt{2}^{\#loops}$$

Spin- and FK-Ising models (random cluster representation):



Self-dual case ($x = x_{\mathrm{crit}}$):

$$\mathbb{P}(\text{loops conf.}) \sim \sqrt{2}^{\#loops}$$

Then

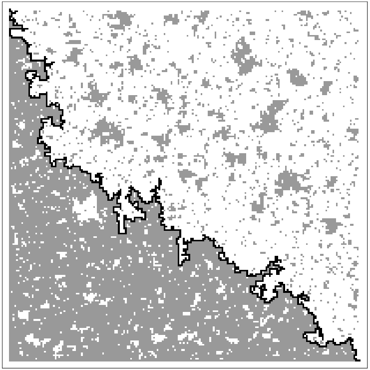
$$\mathbb{P}_{\mathrm{spin}}(s(i) = s(j))$$

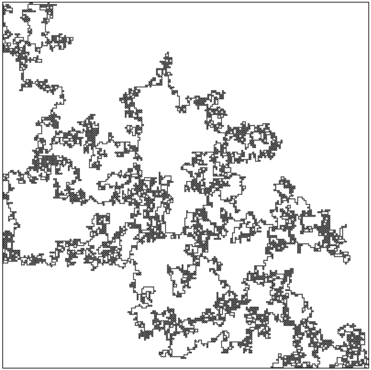
$$= \frac{1}{2}(1 + \mathbb{P}_{\mathrm{FK}}(i \leftrightarrow j))$$

Convergence to SLE. Square lattice (Smirnov):

Spin-Ising <u>Theorem</u>:
Interface → SLE(3)

FK-Ising <u>Theorem</u>:
Interface → SLE(16/3)

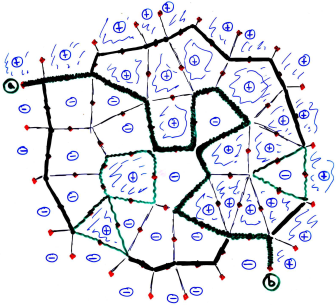




Universality. Isoradial graphs/rhombic lattices:

Spin-Ising Theorem:

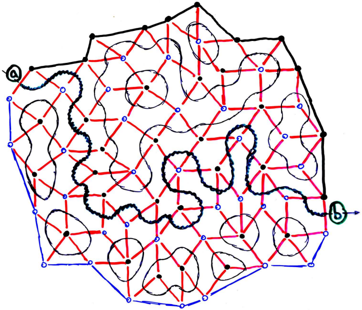
Interface → SLE(3)



$$Z = \sum_{\text{config.}} \prod_{z:\oplus\leftrightarrow\ominus} \tan\frac{\theta(z)}{2}$$

FK-Ising Theorem:

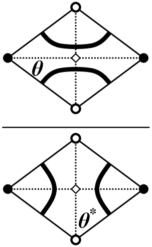
Interface → SLE(16/3)



$$Z = \sum_{\text{config.}} \sqrt{2}^{\#\text{loops}} \prod_{z} \sin\frac{\theta(z)}{2}$$

Universality. Isoradial graphs/rhombic lattices:

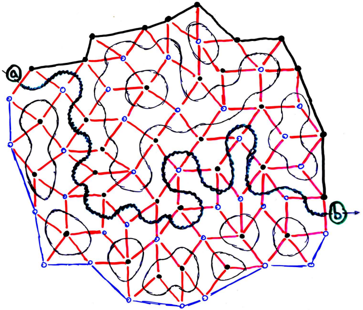
FK-Ising local weights:



$$= \frac{\sin\frac{\theta}{2}}{\sin(\frac{\pi}{4}-\frac{\theta}{2})}$$

FK-Ising Theorem:

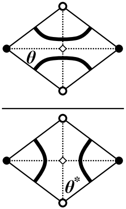
Interface $\longrightarrow$ SLE(16/3)



$$Z = \sum_{\text{config.}} \sqrt{2}^{\,\#\text{loops}} \prod_{z} \sin\frac{\theta(z)}{2}$$

Universality. Isoradial graphs/rhombic lattices:

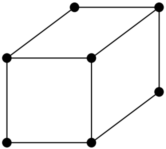
FK-Ising local weights:



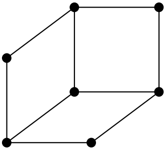
$$= \frac{\sin\frac{\theta}{2}}{\sin(\frac{\pi}{4}-\frac{\theta}{2})} =: r(\theta)$$

satisfies $r(0)=0$ and $Y-\Delta$ invariance: if $\alpha+\beta+\gamma=\frac{\pi}{2}$, then

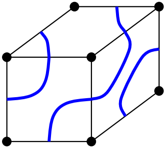
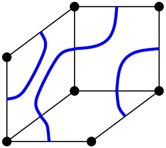
$$1 = r(\alpha)r(\beta)+r(\alpha)r(\gamma)+r(\beta)r(\gamma)+\sqrt{2}\cdot r(\alpha)r(\beta)r(\gamma).$$



$\leftrightarrow$

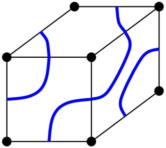


Universality. Isoradial graphs/rhombic lattices:

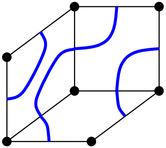
Y-Δ INVARIANCE:  $\leftrightarrow$ 

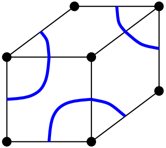
Universality. Isoradial graphs/rhombic lattices:

Y-Δ INVARIANCE:

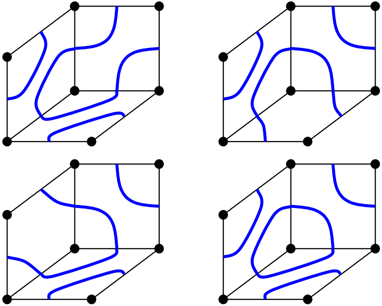


$\longleftrightarrow$





$\longleftrightarrow$

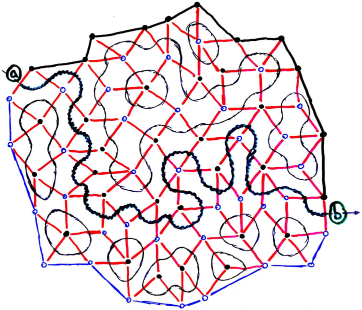


$$1 = r(\alpha)r(\beta) + r(\alpha)r(\gamma) + r(\beta)r(\gamma) + \sqrt{2}\cdot r(\alpha)r(\beta)r(\gamma)$$

Conformal martingale (discrete fermionic observable):

FK-Ising <u>Theorem</u>:

Interface → SLE(16/3)



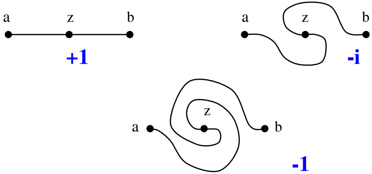
$$Z = \sum_{\text{config.}} \sqrt{2}^{\#\text{loops}} \prod_{z} \sin \frac{\theta(z)}{2}$$

Conformal martingale (discrete fermionic observable):

Discrete holomorphic observable having the martingale property:

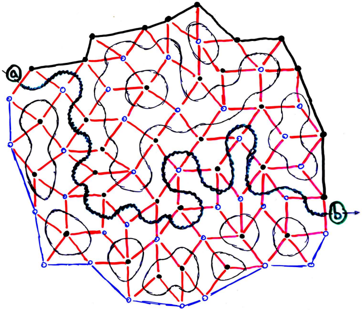
$$F^{\delta} = \mathbb{E}\,\chi[z \in \gamma]\cdot e^{-\frac{i}{2}\cdot \mathrm{wind}(\gamma, b \to z)},$$

where $z \in \diamondsuit$.



FK-ISING THEOREM:

Interface $\to$ *SLE(16/3)*



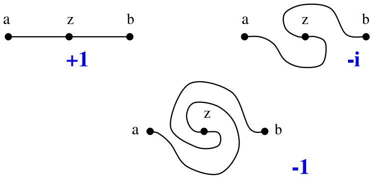
$$Z = \sum_{\text{config.}} \sqrt{2}^{\#\text{loops}} \prod_{z} \sin\frac{\theta(z)}{2}$$

Conformal martingale (discrete fermionic observable):

Discrete holomorphic observable having the martingale property:

$$F^\delta = \mathbb{E}\,\chi[z \in \gamma]\cdot e^{-\frac{i}{2}\cdot \mathrm{wind}(\gamma, b\to z)},$$

where $z \in \diamondsuit$.



Boundary Value Problem:

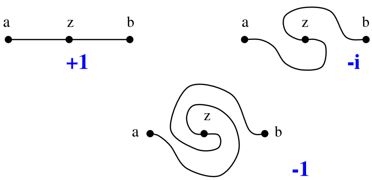
- $F(z)$ is holomorphic in Ω;
- $\mathrm{Im}[F(\zeta)(\tau(\zeta))^{\frac{1}{2}}] = 0$ for $\zeta \in \partial\Omega \setminus \{a, b\}$, where $\tau(\zeta)$ goes from a to b;
- (mult.) normalization.

Conformal martingale (discrete fermionic observable):

Discrete holomorphic observable having the martingale property:

$$F^{\delta} = \mathbb{E}\,\chi[z \in \gamma] \cdot e^{-\frac{i}{2} \cdot \mathrm{wind}(\gamma, b \to z)},$$

where $z \in \diamondsuit$.



<u>Boundary Value Problem</u>:

- $F(z)$ is holomorphic in Ω;
- $\mathrm{Im}[F(\zeta)(\tau(\zeta))^{\frac{1}{2}}] = 0$ for $\zeta \in \partial\Omega \setminus \{a, b\}$, where $\tau(\zeta)$ goes from a to b;
- (mult.) normalization.

<u>Solution</u>: $F(z) = \sqrt{\Phi'(z)}$,

$$\Phi : (\Omega; a, b) \to (S, -\infty, +\infty),$$

$$S = \mathbb{R} \times (0, 1).$$

Universality. Convergence to SLE (FK-Ising):

F^δ is a *discrete holomorphic* martingale. Then:

- Take a "discrete integral" $H^\delta := \mathrm{Im} \int (F^\delta)^2(z) d^\delta z$ (miraculously, it is well defined);
- H^δ is NOT discrete harmonic, so prove that it is "approximately" harmonic;

Universality. Convergence to SLE (FK-Ising):

F^δ is a *discrete holomorphic* martingale. Then:

- Take a "discrete integral" $H^\delta := \mathrm{Im} \int (F^\delta)^2(z) d^\delta z$ (miraculously, it is well defined);
- H^δ is NOT discrete harmonic, so prove that it is "approximately" harmonic;
- Prove that H^δ is *uniformly (w.r.t. Ω)* close to its eventual limit $\mathrm{Im}\,\Phi = \omega(\cdot, ba, \Omega)$ inside Ω;
- Prove that F^δ is uniformly close to $\sqrt{\Phi'}$ inside Ω;

This needs some work (see `arXiv:0910.2045,0810.2188`*).*

Universality. Convergence to SLE (FK-Ising):

F^δ is a *discrete holomorphic* martingale. Then:

- Take a "discrete integral" $H^\delta := \operatorname{Im} \int (F^\delta)^2(z) d^\delta z$ (miraculously, it is well defined);
- H^δ is NOT discrete harmonic, so prove that it is "approximately" harmonic;
- Prove that H^δ is *uniformly (w.r.t. Ω)* close to its eventual limit $\operatorname{Im} \Phi = \omega(\cdot, ba, \Omega)$ inside Ω;
- Prove that F^δ is uniformly close to $\sqrt{\Phi'}$ inside Ω;

This needs some work (see `arXiv:0910.2045,0810.2188`*).*

- Then deduce convergence of an interface to $SLE(16/3)$ from the convergence of the martingale observable.

Universality. Convergence to SLE (FK-Ising):

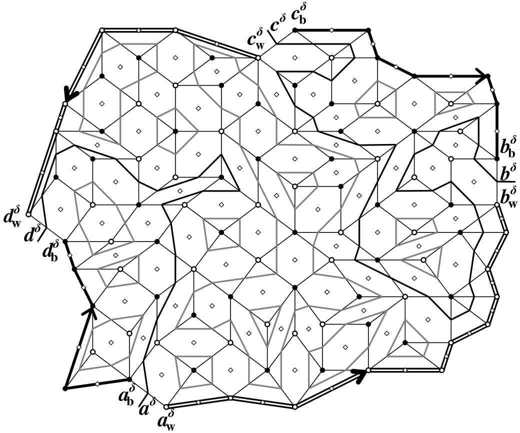
- Then deduce convergence of an interface to $SLE(16/3)$ from the convergence of the martingale observable.

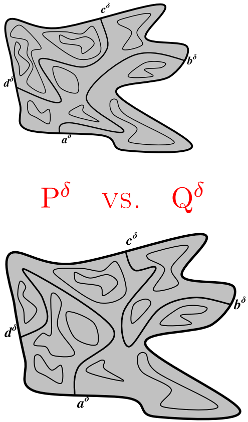
INTERFACES $\to$ SLE(16/3). In which topology?

- Convergence of driving forces in the Loewner equation. Directly follows from the convergence of observable.
- Convergence of curves themselves. Needs some a priori information (estimates of some crossing probabilities).

  (Aizenman, Burchard, '99; Kemppainen, Smirnov '09)

FK-Ising crossing probability:

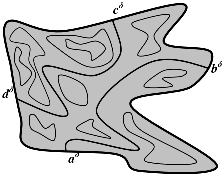




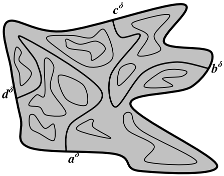
P^δ vs. Q^δ

FK-Ising crossing probability:

P^δ



Q^δ



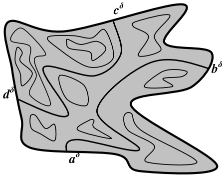
Theorem: *For all* $r, R, t > 0$ *there exists* $\varepsilon(\delta) \to 0$ *as* $\delta \to 0$ *such that if* $B(0,r) \subset \Omega^\delta \subset B(0,R)$ *and either both* $\omega(0;\Omega^\delta;a^\delta b^\delta), \omega(0;\Omega^\delta;c^\delta d^\delta)$ *or both* $\omega(0;\Omega^\delta;b^\delta c^\delta), \omega(0;\Omega^\delta;d^\delta a^\delta)$ *are* $\geqslant t$ *(i.e., quadrilateral* Ω^δ *has no neighboring small arcs), then*

$$|\mathrm{P}^\delta - \mathrm{P}(\Omega^\delta; a^\delta, b^\delta, c^\delta, d^\delta)| \leqslant \varepsilon(\delta)$$

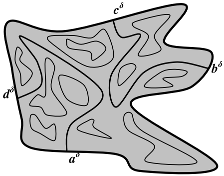
(uniformly w.r.t. Ω^δ *and* $\diamondsuit^\delta$*), where* P *depends only on the conformal modulus of* $(\Omega^\delta; a^\delta, b^\delta, c^\delta, d^\delta)$.

FK-Ising crossing probability:

P^{δ}



Q^{δ}



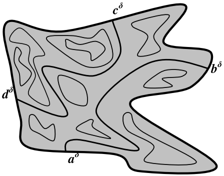
In the half-plane $\mathbb{H}$: for $u \in [0,1]$,

$$\mathrm{P}(\mathbb{H};[1-u,1] \leftrightarrow [\infty,0]) = \frac{\sqrt{1-\sqrt{1-u}}}{\sqrt{1-\sqrt{u}}+\sqrt{1-\sqrt{1-u}}}.$$

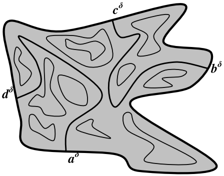
This is a special case of a hypergeometric formula for crossings in a general FK model. In the Ising case it becomes algebraic and furthermore can be rewritten in several ways.

FK-Ising crossing probability:

P^{δ}



Q^{δ}



In the unit disc $\mathbb{D}$: for $\theta \in [0, \frac{\pi}{2}]$,

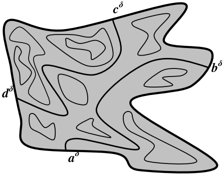
$$\frac{\mathrm{P}(\mathbb{D}; [-e^{-i\theta}, -e^{i\theta}] \leftrightarrow [e^{-i\theta}, e^{i\theta}])}{\mathrm{P}(\mathbb{D}; [e^{i\theta}, -e^{-i\theta}] \leftrightarrow [-e^{i\theta}, e^{-i\theta}])}$$

$$= \frac{\sin \frac{\theta}{2}}{\sin(\frac{\pi}{4} - \frac{\theta}{2})} =: r(\theta).$$

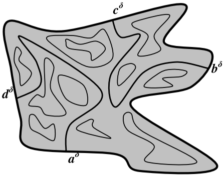
Remark: *This macroscopic formula formally coincides with the relative weights corresponding to two different possibilities of crossings inside microscopic rhombi in the FK-Ising model on isoradial graphs.*

FK-Ising crossing probability:

P^δ



Q^δ



In the unit disc $\mathbb{D}$: for $\theta \in [0, \frac{\pi}{2}]$,

$$\frac{\mathrm{P}(\mathbb{D}; [-e^{-i\theta}, -e^{i\theta}] \leftrightarrow [e^{-i\theta}, e^{i\theta}])}{\mathrm{P}(\mathbb{D}; [e^{i\theta}, -e^{-i\theta}] \leftrightarrow [-e^{i\theta}, e^{-i\theta}])}$$

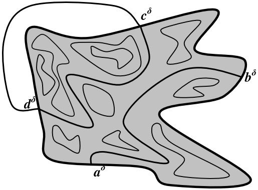
$$= \frac{\sin\frac{\theta}{2}}{\sin(\frac{\pi}{4} - \frac{\theta}{2})} =: r(\theta).$$

Remark: In particular, the $Y - \Delta$ relation holds, i.e.,

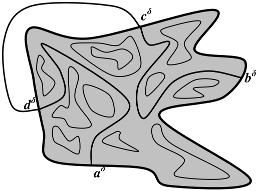
$$r(\alpha+\beta) = \frac{r(\alpha)+r(\beta)+\sqrt{2}\cdot r(\alpha)r(\beta)}{1 - r(\alpha)r(\beta)}$$

FK-Ising crossing probability. External coupling.

$$\frac{\sqrt{2}\mathrm{P}^{\delta}}{\sqrt{2}\mathrm{P}^{\delta}+\mathrm{Q}^{\delta}}$$

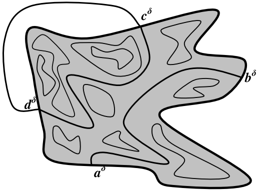


$$\frac{\mathrm{Q}^{\delta}}{\sqrt{2}\mathrm{P}^{\delta}+\mathrm{Q}^{\delta}}$$

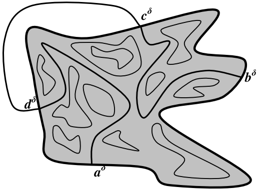


FK-Ising crossing probability. External coupling.

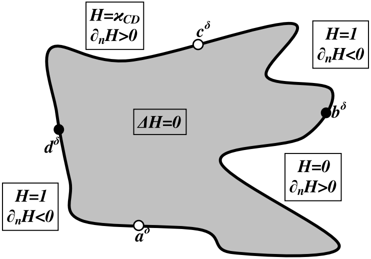
$\frac{\sqrt{2}\mathrm{P}^{\delta}}{\sqrt{2}\mathrm{P}^{\delta}+\mathrm{Q}^{\delta}}$



$\frac{\mathrm{Q}^{\delta}}{\sqrt{2}\mathrm{P}^{\delta}+\mathrm{Q}^{\delta}}$

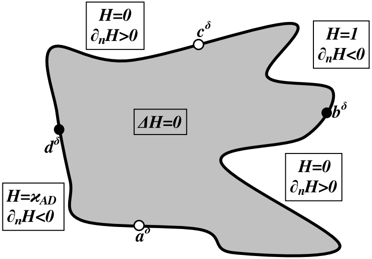


Construct a discrete holomorphic observable F^{δ}_{CD}. Then for an (almost) discrete harmonic function $H_{CD} = \operatorname{Im} \int (F^{\delta}_{CD}(z))^2 d^{\delta} z$:

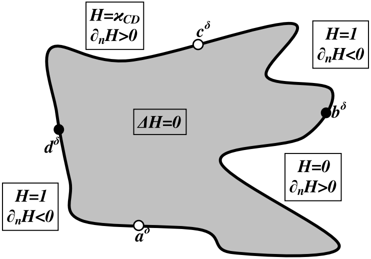


FK-Ising crossing probability. External coupling.

Construct a discrete holomorphic observable F^{δ}_{AD}. Then for an (almost) discrete harmonic function $H_{AD} = \mathrm{Im} \int (F^{\delta}_{AD}(z))^2 d^{\delta} z$:

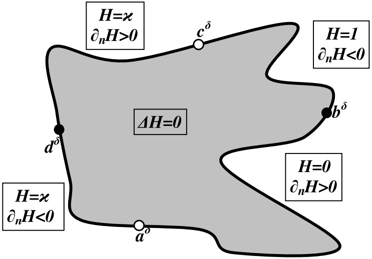


Construct a discrete holomorphic observable F^{δ}_{CD}. Then for an (almost) discrete harmonic function $H_{CD} = \mathrm{Im} \int (F^{\delta}_{CD}(z))^2 d^{\delta} z$:



FK-Ising crossing probability. Conformal mapping.

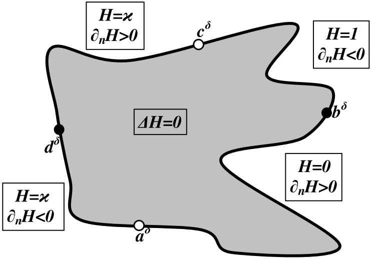
For some linear combination of observables $F^\delta := \alpha F^\delta_{AD} + \beta F^\delta_{CD}$ and $H = \operatorname{Im} \int (F^\delta(z))^2 d^\delta z$ one has:



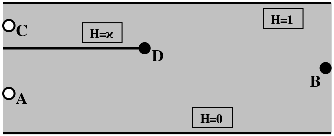
where the value $\varkappa^\delta$ is determined by the ratio of crossing probabilities $\mathrm{P}^\delta / Q^\delta$.

FK-Ising crossing probability. Conformal mapping.

For some linear combination of observables $F^\delta := \alpha F^\delta_{AD} + \beta F^\delta_{CD}$ and $H = \mathrm{Im} \int (F^\delta(z))^2 d^\delta z$ one has:



Uniformization:

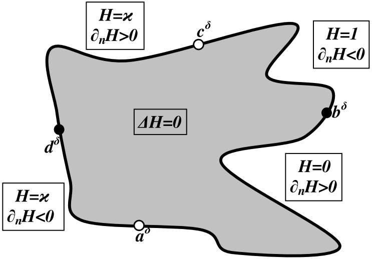


where the value $\varkappa^\delta$ is determined by the ratio of crossing probabilities $\mathrm{P}^\delta / Q^\delta$.

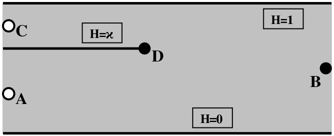
$\varkappa$ is uniquely determined by the conformal modulus of $(\Omega^\delta, a^\delta, b^\delta, c^\delta, d^\delta)$

FK-Ising crossing probability. Conformal mapping.

For some linear combination of observables $F^\delta := \alpha F^\delta_{AD} + \beta F^\delta_{CD}$ and $H = \mathrm{Im} \int (F^\delta(z))^2 d^\delta z$ one has:



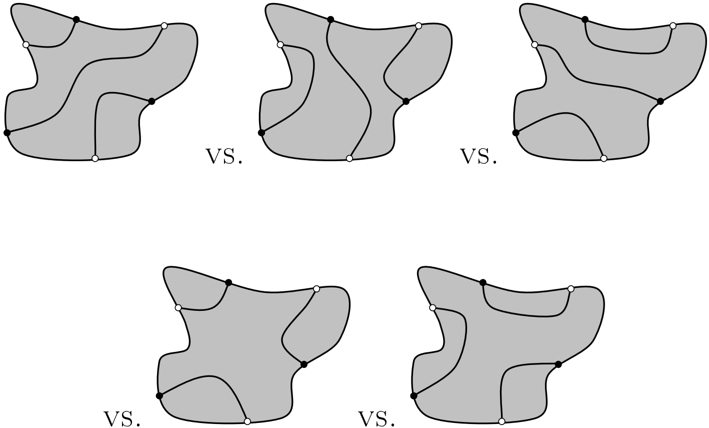
Uniformization:



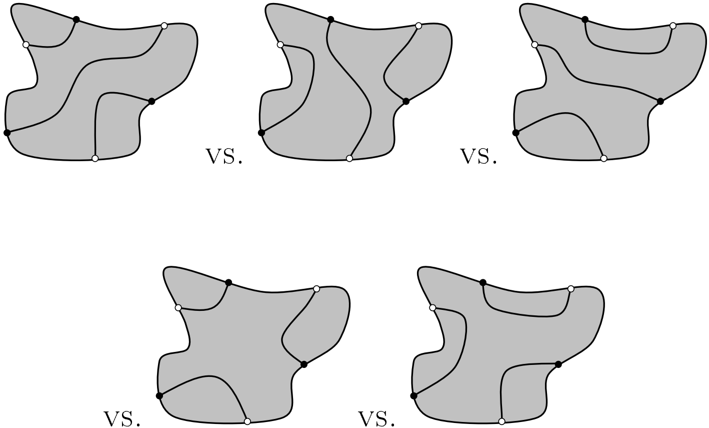
where the value $\varkappa^\delta$ is determined by the ratio of crossing probabilities $\mathrm{P}^\delta / Q^\delta$.

Convergence $H^\delta \to H$ for rough domains needs some work (see `arXiv:0910.2045`*).*

FK-Ising crossing probabilities: more points?



FK-Ising crossing probabilities: more points?



THANK YOU!