2D Ising model: correlations via boundary value problems

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[Sample of a critical 2D Ising configuration (with two disorders), © Clément Hongler (EPFL)]

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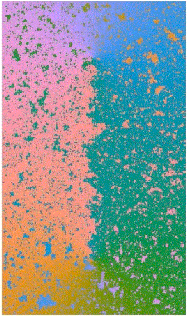
2D Ising model: correlations via boundary value problems

- Nearest-neighbor Ising model in 2D
  - dimers and fermionic observables
  - discrete holomorphicity at criticality
  - spinor observables and spin correlations
- A classical computation revisited: explicit formulae for "diagonal" two-point correlations in $\mathbb{Z}^2$ via full-plane spinors
- Conformal covariance at criticality
  - Riemann boundary value problems for holomorphic spinors in continuum
  - Explicit formulae (CFT prediction)
  - Convergence (Ch.–Hongler–Izyurov)
  - Other fields (convergence, fusion rules)

Extended version:

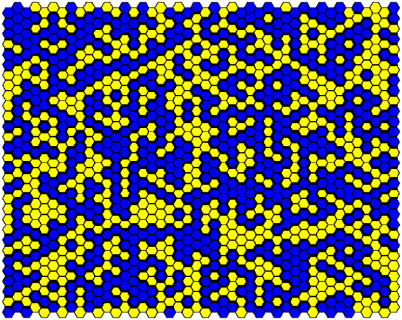
arXiv:1605.09035



Nearest-neighbor Ising (or Lenz-Ising) model in 2D

Definition: *Lenz-Ising model* on a planar graph G^* (dual to G) is a random assignment of $+/-$ spins to vertices of G^* (faces of G)

Q: I heard this is called a (site) percolation?



[sample of a honeycomb percolation]

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A: .. according to the following probabilities:

$$\mathbb{P}\left[\text{conf. } \sigma \in \{\pm 1\}^{V(G^*)}\right] \propto \exp\left[\beta \sum_{e=\langle uv\rangle} J_{uv}\sigma_u\sigma_v\right] \propto \prod_{e=\langle uv\rangle:\sigma_u\neq\sigma_v} x_{uv},$$

where $J_{uv} > 0$ are interaction constants assigned to edges $\langle uv\rangle$, $\beta = 1/kT$ is the inverse temperature, and $x_{uv} = \exp[-2\beta J_{uv}]$.

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Disclaimer:

no external magnetic field.

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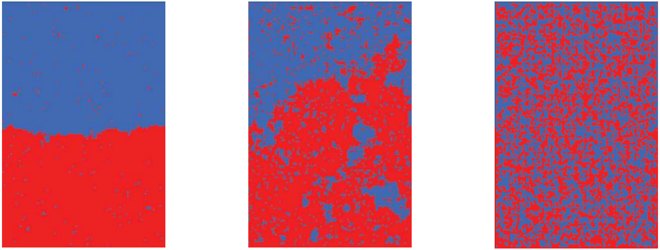
- It is also convenient to use the parametrization $x_{uv} = \tan(\frac{1}{2}\theta_{uv})$.
- Working with subgraphs of *regular lattices*, one can consider the *homogeneous model* in which all x_{uv} are equal to each other.

Phase transition (e.g., on $\mathbb{Z}^2$)

E.g., Dobrushin boundary conditions: $+1$ on (ab) and -1 on (ba):

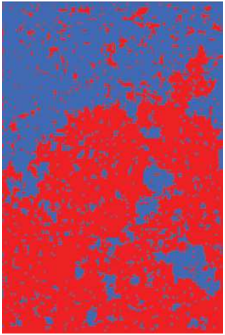


$x < x_{\mathrm{crit}}$ $\quad$ $x = x_{\mathrm{crit}}$ $\quad$ $x > x_{\mathrm{crit}}$

- Ising (1925): no phase transition in 1D $\rightsquigarrow$ doubts about 2+D;
- Peierls (1936): existence of the phase transition in 2D;
- Kramers-Wannier (1941): $x_{\text{self-dual}} = \sqrt{2} - 1 = \tan(\frac{1}{2} \cdot \frac{\pi}{4})$;
- Onsager (1944): sharp phase transition at $x_{\text{crit}} = \sqrt{2} - 1$.

At criticality (e.g., on $\mathbb{Z}^2$):

- Kaufman-Onsager(1948-49), Yang(1952): **scaling exponent $\frac{1}{8}$** for the magnetization. [via spin-spin correlations in $\mathbb{Z}^2$ at $x \uparrow x_{\mathrm{crit}}$]
- At criticality, for $\Omega_\delta \to \Omega$ and $u_\delta \to u \in \Omega$, it should be $\mathbb{E}_{\Omega_\delta}[\sigma_{u_\delta}] \asymp \delta^{\frac{1}{8}}$ as $\delta \to 0$.



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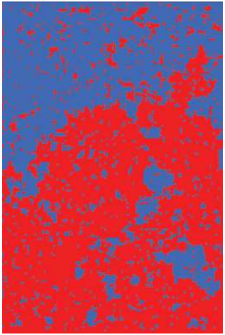

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- **Question:** Convergence of (rescaled) spin correlations and conformal covariance of their **scaling limits in arbitrary planar domains:**

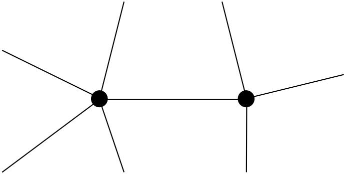
$$\begin{aligned} \delta^{-\frac{n}{8}} \cdot \mathbb{E}_{\Omega_\delta}[\sigma_{u_{1,\delta}} \ldots \sigma_{u_{n,\delta}}] \quad &\to \quad \langle \sigma_{u_1} \ldots \sigma_{u_n} \rangle_\Omega \\ &= \quad \langle \sigma_{\varphi(u_1)} \ldots \sigma_{\varphi(u_n)} \rangle_{\varphi(\Omega)} \cdot \prod_{s=1}^n |\varphi'(u_s)|^{\frac{1}{8}} \end{aligned}$$

- In the infinite-volume setup other techniques are available, notably "exact bosonization" approach due to J. Dubédat.

2D Ising model as a dimer model (on a non-bipartite graph) [Fisher, Kasteleyn ('60s+),..., Kenyon, Dubédat ('00s+),...]

- **Partition function** $\mathcal{Z} = \sum_{\sigma\in\{\pm1\}^{V(G^*)}} \prod_{e=\langle uv\rangle:\sigma_u\neq\sigma_v} x_{uv}$

- There exist various representations of the 2D Ising model via dimers on an auxiliary graph



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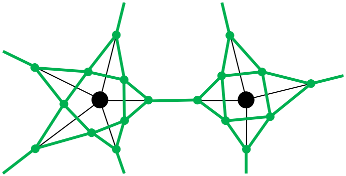

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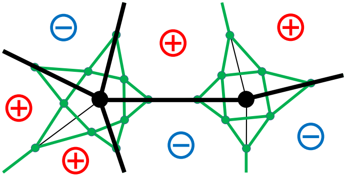

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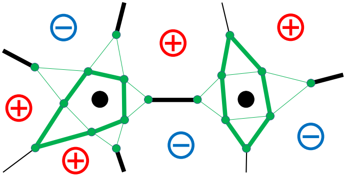

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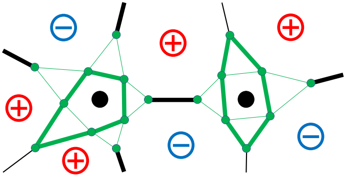

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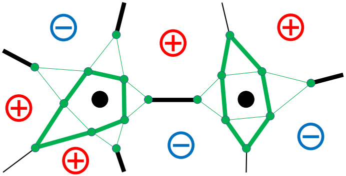

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- **Kac–Ward formula (1952–..., 1999–...):** $\mathcal{Z}^2 = \det[\mathrm{Id}-\mathrm{T}]$,

$$\mathrm{T}_{e,e'} = \begin{cases} \exp[\tfrac{i}{2}\mathbf{wind}(\boldsymbol{e},\boldsymbol{e'})]\cdot(x_e x_{e'})^{1/2} & \text{if } e'\neq\overline{e} \text{ prolongs } e; \\ 0 & \text{otherwise.}\end{cases}$$

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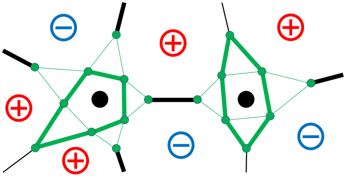

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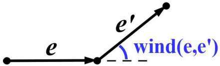

[**is equivalent** to the **Kasteleyn theorem for dimers on G_F**]
[more details in `arXiv:1507.08242` (w/ *Cimasoni & Kassel*)]

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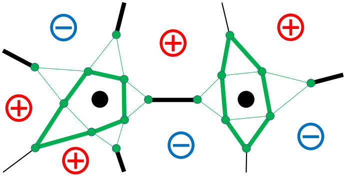

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- Note that $V(G_F) \cong$ ***{oriented edges and corners of G}***

- **Local relations** for the entries $\mathbf{K}^{-1}_{a,e}$ and $\mathbf{K}^{-1}_{a,c}$ of the inverse Kasteleyn (or the inverse Kac–Ward) matrix:

(an equivalent form of) the identity $\mathbf{K} \cdot \mathbf{K}^{-1} = \mathbf{Id}$

Fermionic observables: combinatorial definition [Smirnov'00s]

For an oriented edge a and a midedge z_e (similarly, for a corner c),

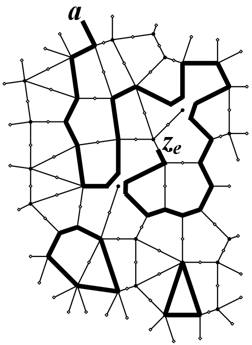
$$F_G(a, z_e) \;:=\; \overline{\eta}_a \sum_{\omega\in\mathrm{Conf}_G(a,z_e)} \Big[e^{-\frac{i}{2}\mathrm{wind}(a\rightsquigarrow z_e)} \prod_{\langle uv\rangle\in\omega} x_{uv} \Big]$$

where η_a denotes the (once and forever fixed) square root of the direction of a.

- The factor $e^{-\frac{i}{2}\mathrm{wind}(a\rightsquigarrow z_e)}$ does not depend on the way how ω is split into non-intersecting loops and a path $a \rightsquigarrow z_e$.

- **Via dimers on G_F:** $F_G(a, c) = \overline{\eta}_c \mathrm{K}^{-1}_{c,a}$

$$F_G(a, z_e) = \overline{\eta}_e \mathrm{K}^{-1}_{e,a} + \overline{\eta}_{\overline{e}} \mathrm{K}^{-1}_{\overline{e},a}$$



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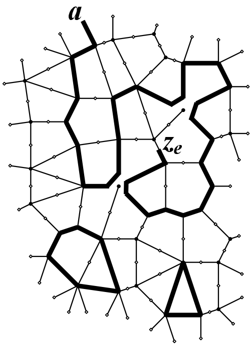

• The factor $e^{-\frac{i}{2}\mathrm{wind}(a \rightsquigarrow z_e)}$ does not depend on the way how ω is split into non-intersecting loops and a path $a \rightsquigarrow z_e$.

• When both a and e are "boundary" edges, the factor $\overline{\eta}_a e^{-\frac{i}{2}\mathrm{wind}(a \rightsquigarrow z_e)} = \pm \overline{\eta}_{\bar{e}}$ is fixed and $F_G(a, z_e)$ becomes the partition function of the Ising model (on G^*) with Dobrushin boundary conditions.

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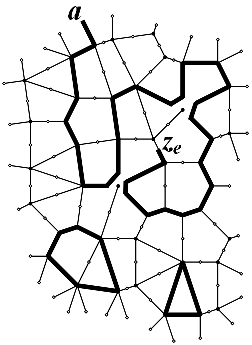

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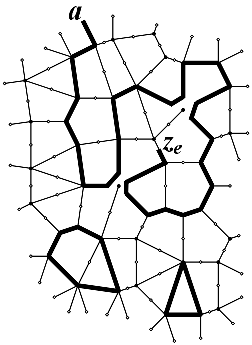

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• **Boundary conditions $F(a, z_e) \in \overline{\eta}_{\overline{e}}\mathbb{R}$** ($\overline{e}$ is oriented outwards) uniquely determine F as a solution to an appropriate **discrete Riemann-type boundary value problem.**

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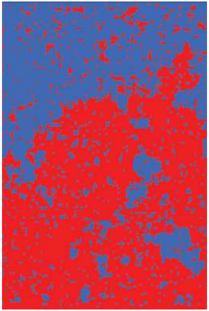
Fermionic observables *per se* can be used

- to construct (discrete) martingales for growing **interfaces** and then to study their convergence to SLE curves *[Smirnov(2006), ..., Ch.–Duminil-Copin –Hongler–Kemppainen–Smirnov(2013)]*
- to analyze the **energy density field** *[Hongler–Smirnov, Hongler (2010)]*



$$\varepsilon_e \;:=\; \delta^{-1} \cdot [\, \boldsymbol{\sigma}_{e^-} \boldsymbol{\sigma}_{e^+} - \varepsilon_e^{\infty} \,]$$

where $e^{\pm}$ are the two neighboring faces separated by an edge e

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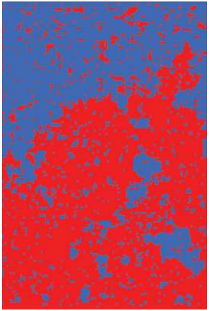

- **but more involved ones are needed** to study spin correlations and their limits *[Ch.–Izyurov(2011), Ch.–Hongler–Izyurov(2012)]*

Spinor observables and spin correlations

- spin configurations on G^*
  $\leftrightsquigarrow$ domain walls on G
  $\leftrightsquigarrow$ dimers on G_F
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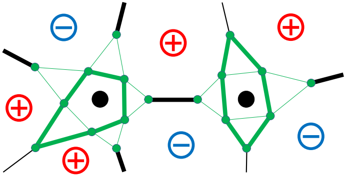

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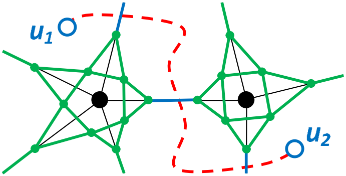

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- **Claim**:

$$\mathbb{E}[\sigma_{u_1} \ldots \sigma_{u_n}] \; = \; \frac{\mathrm{Pf}[\,\mathrm{K}_{[u_1,\ldots,u_n]}\,]}{\mathrm{Pf}[\,K\,]},$$

where $\mathrm{K}_{[u_1,\ldots,u_n]}$ is obtained from K by changing the sign of its entries on **slits linking** $u_1, \ldots, u_n$ (and, possibly, u_{out}) pairwise.

- **More invariant way** to think about entries of $\mathrm{K}^{-1}_{[u_1,\ldots,u_n]}$:

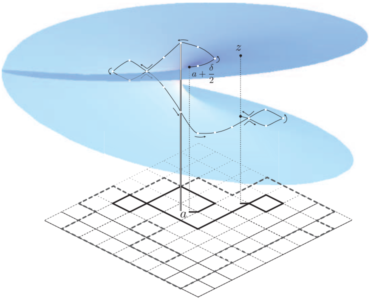
double-covers of G branching over $u_1, \ldots, u_n$

Spinor observables and spin correlations

Main tool: spinors on the double cover $[\Omega_\delta; u_1, \ldots, u_n]$.

$$F_{\Omega_\delta}(z) := \left[\mathcal{Z}^{+}_{\Omega_\delta}[\sigma_{u_1}\ldots\sigma_{u_n}]\right]^{-1} \cdot \sum_{\omega\in\mathrm{Conf}_{\Omega_\delta}(u_1^{\to},z)} \phi_{u_1,\ldots,u_n}(\omega,z)\cdot x_{\mathrm{crit}}^{\#\mathrm{edges}(\omega)},$$

$$\phi_{u_1,\ldots,u_n}(\omega,z) := e^{-\frac{i}{2}\mathrm{wind}(\mathrm{p}(\omega))}\cdot(-1)^{\#\mathrm{loops}(\omega\setminus\mathrm{p}(\omega))}\cdot \mathrm{sheet}(\mathrm{p}(\omega),z).$$



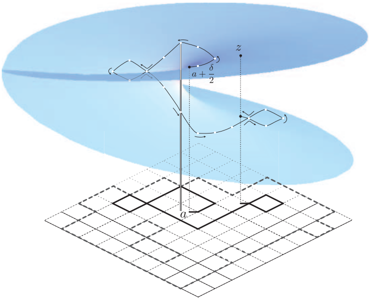
- $\mathrm{wind}(\mathrm{p}(\gamma))$ is the winding of the path $\mathrm{p}(\gamma): u_1^{\to} = u_1 + \frac{\delta}{2} \rightsquigarrow z$;
- $\#\mathrm{loops}$ – those containing an odd number of $u_1, \ldots, u_n$ inside;
- $\mathrm{sheet}(\mathrm{p}(\gamma), z) = +1$, if $\mathrm{p}(\gamma)$ defines z, and -1 otherwise.
- **Note that $F(z^\sharp) = -F(z^\flat)$** if $z^\sharp, z^\flat$ lie over the same edge of Ω_δ.

Spinor observables and spin correlations

Main tool: spinors on the double cover $[\Omega_\delta; u_1, \ldots, u_n]$.

$$F_{\Omega_\delta}(z) := \left[\mathcal{Z}^+_{\Omega_\delta}[\sigma_{u_1} \ldots \sigma_{u_n}]\right]^{-1} \cdot \sum_{\omega \in \mathrm{Conf}_{\Omega_\delta}(u_1^{\rightarrow}, z)} \phi_{u_1,\ldots,u_n}(\omega, z) \cdot x_{\mathrm{crit}}^{\#\mathrm{edges}(\omega)},$$

$$\phi_{u_1,\ldots,u_n}(\omega, z) := e^{-\frac{i}{2}\mathrm{wind}(\mathrm{p}(\omega))} \cdot (-1)^{\#\mathrm{loops}(\omega \setminus \mathrm{p}(\omega))} \cdot \mathrm{sheet}(\mathrm{p}(\omega), z).$$



Claim:

$$F_{\Omega_\delta}(u_1 + \tfrac{3\delta}{2}) = \frac{\mathbb{E}^+_{\Omega_\delta}[\sigma_{u_1+2\delta} \ldots \sigma_{u_n}]}{\mathbb{E}^+_{\Omega_\delta}[\sigma_{u_1} \ldots \sigma_{u_n}]}$$

In other words, *spatial derivatives* of spin correlations are particular values of spinor observables.

- **Remark:** Both fermionic and spinor observables can be introduced using **spin-disorder formalism of Kadanoff and Ceva.**

"Diagonal" correlations in $\mathbb{Z}^2$: classical computation revisited

Let $x = \tan \frac{1}{2}\theta \leqslant x_{\mathrm{crit}} = \tan \frac{\pi}{8}$ and $D_n(x) := \mathbb{E}_{\mathbb{C}^\diamond}[\sigma_{(0,0)}\sigma_{(2n,0)}]$ where $\mathbb{C}^\diamond = \{(k,s) : k,s \in \mathbb{Z}, k+s \in 2\mathbb{Z}\}$ is the $\frac{\pi}{4}$-rotated $\mathbb{Z}^2$.

Theorem: [B. Kaufman – L. Onsager '48-49, C.N. Yang '52]

$\lim_{n\to\infty} D_n(x) = (1-\tan^4\theta)^{\frac{1}{4}} \sim \mathrm{const} \cdot (x_{\mathrm{crit}} - x)^{\frac{1}{4}}$ for $x < x_{\mathrm{crit}}$

[T.T.Wu'66] $D_n(x_{\mathrm{crit}}) = \left(\frac{2}{\pi}\right)^n \prod_{s=1}^{n-1}\left(1-\frac{1}{4s^2}\right)^{s-n} \sim \mathrm{const} \cdot (2n)^{-\frac{1}{4}}$

Classical reference for many explicit computations (1973):

B.M. McCoy and T.T. Wu *"The two-dimensional Ising model"*

“Diagonal” correlations in $\mathbb{Z}^2$: classical computation revisited

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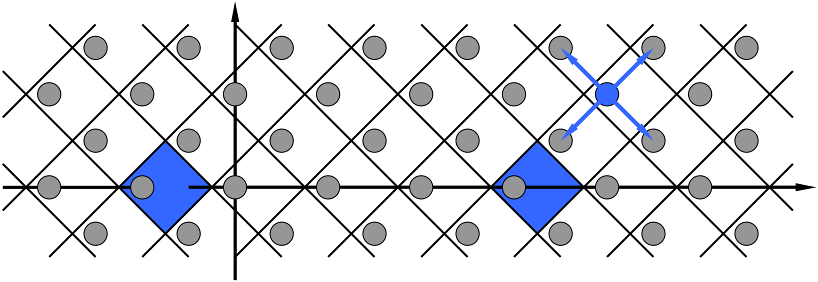
Historical comments: [see R.J. Baxter, arXiv:1103.3347 & 1211.2665 for more details]

Onsager: ... I have found a general formula for the evaluation of Toeplitz matrices. The only thing I did not know was how to fill out the holes in the mathematics and show the epsilons and the deltas and all of that...

... we talked to Kakutani and Kakutani talked to Szego, and the mathematicians got there first.

"Diagonal" correlations in $\mathbb{Z}^2$: classical computation revisited

Let $x = \tan \frac{1}{2}\theta \leqslant x_{\mathrm{crit}} = \tan \frac{\pi}{8}$ and $D_{n+1}(x) := \mathbb{E}_{\mathbb{C}^\diamond}[\sigma_{(-\frac{3}{2},0)}\sigma_{(2n+\frac{1}{2},0)}]$

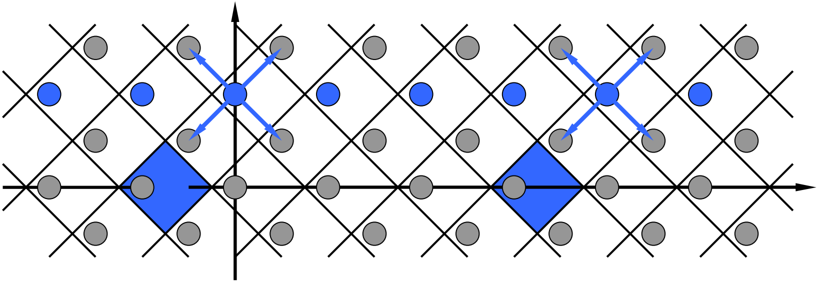


Local relations: $F_{\mathbb{C}^\diamond}(d) = \frac{m}{4}\sum_{d'\sim d} F_{\mathbb{C}^\diamond}(d')$, $\quad m := \sin(2\theta) \leqslant 1.$

[Above, we focus on purely real values of the spinor observable on one particular type of corners.] **Note that $m = 1$ iff $x = x_{\mathrm{crit}}$.**

“Diagonal” correlations in $\mathbb{Z}^2$: classical computation revisited

Let $x = \tan\frac{1}{2}\theta \leqslant x_{\mathrm{crit}} = \tan\frac{\pi}{8}$ and $D_{n+1}(x) := \mathbb{E}_{\mathbb{C}^\diamond}[\sigma_{(-\frac{3}{2},0)}\sigma_{(2n+\frac{1}{2},0)}]$



Local relations: $F_{\mathbb{C}^\diamond}(d) = \frac{m}{4}\sum_{d'\sim d} F_{\mathbb{C}^\diamond}(d'),\quad m := \sin(2\theta) \leqslant 1.$

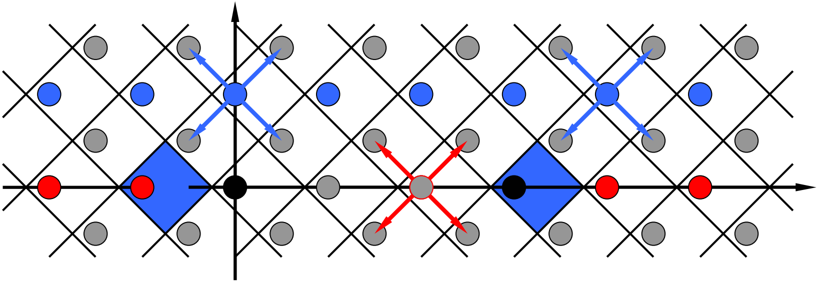
For $s \geqslant 0$, denote $Q_{n,s}(e^{it}) := \sum_{k\in\mathbb{Z}:k+s\in 2\mathbb{Z}} e^{\frac{1}{2}ikt} F_{\mathbb{C}^\diamond}(k,s).$

Local relations $\Rightarrow Q_{n,s}(e^{it}) = (\frac{m}{2}\cos\frac{t}{2})\cdot(Q_{n,s-1}(e^{it}) + Q_{n,s+1}(e^{it})).$

Boundedness as $s\to\infty \Rightarrow Q_{n,1}(e^{it}) = \left[\frac{1-(1-(m\cos\frac{t}{2})^2)^{\frac{1}{2}}}{m\cos\frac{t}{2}}\right] Q_{n,0}(e^{it}).$

“Diagonal” correlations in $\mathbb{Z}^2$: classical computation revisited

Let $x = \tan\frac{1}{2}\theta \leqslant x_{\mathrm{crit}} = \tan\frac{\pi}{8}$ and $D_{n+1}(x) := \mathbb{E}_{\mathbb{C}^\diamond}[\sigma_{(-\frac{3}{2},0)}\sigma_{(2n+\frac{1}{2},0)}]$



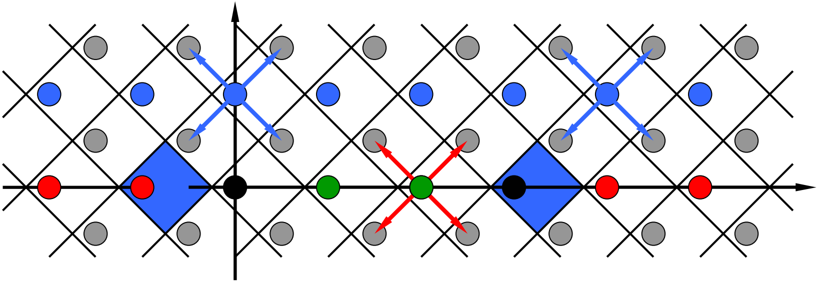
Combinatorics of spinor observables $\Rightarrow$ the following values on $\mathbb{R}$:

$$\begin{aligned} D_{n+1}Q_{n,0}(e^{it}) &= \mathbf{0} + D_n + \ldots + D_n^* e^{int} + \mathbf{0} \\ w(e^{it}) \cdot D_{n+1}Q_{n,0}(e^{it}) &= \ldots + D_{n+1} + \mathbf{0} + q^2 D_{n+1}^* e^{int} + \ldots \end{aligned}$$

where $\boldsymbol{w(e^{it}) = |1 - q^2 e^{it}|}$, $q := \tan\theta \leqslant 1$ and $D_n^* := D_n(\tan(\frac{\pi}{4} - \theta))$.

“Diagonal” correlations in $\mathbb{Z}^2$: classical computation revisited

Let $x = \tan\frac{1}{2}\theta \leqslant x_{\mathrm{crit}} = \tan\frac{\pi}{8}$ and $D_{n+1}(x) := \mathbb{E}_{\mathbb{C}^\diamond}[\sigma_{(-\frac{3}{2},0)}\sigma_{(2n+\frac{1}{2},0)}]$

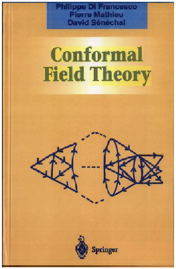


Therefore, **the values of these full-plane spinor observables** on the real line are coefficients of certain **orthogonal polynomials Q_n wrt $w(e^{it})$** [which are simply Legendre polynomials if $x = x_{\mathrm{crit}}$].

$\Longrightarrow$ one can express D_{n+1}, D^*_{n+1} via D_n, D^*_n and norms of Q_n,

where $w(e^{it}) = |1 - q^2 e^{it}|$, $q := \tan\theta \leqslant 1$ and $D^*_n := D_n(\tan(\frac{\pi}{4} - \theta))$.

Conformal covariance of spin correlations at criticality



- Three local primary fields:
  $\mathbf{1}$, $\boldsymbol{\sigma}$ (spin), ε (energy density);
  Scaling exponents: $\mathbf{0}$, $\mathbf{\frac{1}{8}}$, $\mathbf{1}$.

- **CFT prediction:**

  If $\Omega_\delta \to \Omega$ and $u_{k,\delta} \to u_k$ as $\delta \to 0$, then

$$\delta^{-\frac{n}{8}} \cdot \mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_{1,\delta}} \ldots \sigma_{u_{n,\delta}}] \underset{\delta \to 0}{\to} \mathcal{C}_\sigma^n \cdot \langle \boldsymbol{\sigma}_{u_1} \ldots \boldsymbol{\sigma}_{u_n} \rangle^{+}_{\Omega}$$

where $\mathcal{C}_\sigma$ is a lattice-dependent constant,

$$\langle \sigma_{u_1} \ldots \sigma_{u_n} \rangle^{+}_{\Omega} \;=\; \langle \sigma_{\varphi(u_1)} \ldots \sigma_{\varphi(u_n)} \rangle^{+}_{\Omega'} \cdot \prod_{s=1}^{n} |\varphi'(u_s)|^{\frac{1}{8}}$$

for any conformal mapping $\varphi : \Omega \to \Omega'$, and

$$\Big[\langle \boldsymbol{\sigma}_{u_1} \ldots \boldsymbol{\sigma}_{u_n} \rangle^{+}_{\mathbb{H}} \Big]^2 = \prod_{1 \leqslant s \leqslant n} (2\,\mathrm{Im}\, u_s)^{-\frac{1}{4}} \times \sum_{\mu \in \{\pm 1\}^n} \prod_{s<m} \left| \frac{u_s - u_m}{u_s - \overline{u}_m} \right|^{\frac{\mu_s \mu_m}{2}}$$

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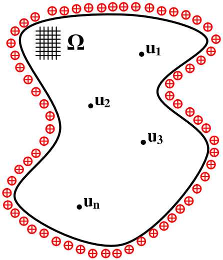

- **Theorem:** *[Ch.–Hongler–Izyurov]*

  If $\Omega_\delta \to \Omega$ and $u_{k,\delta} \to u_k$ as $\delta \to 0$, then

$$\delta^{-\frac{n}{8}} \cdot \mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_{1,\delta}} \dots \sigma_{u_{n,\delta}}] \underset{\delta \to 0}{\to} \mathcal{C}^n_\sigma \cdot \langle \boldsymbol{\sigma_{u_1}} \dots \boldsymbol{\sigma_{u_n}} \rangle^{+}_{\boldsymbol{\Omega}}$$

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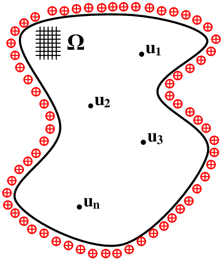

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General strategy: • in discrete: encode **spatial derivatives** as values of discrete holomorphic spinors F^δ that solve some

discrete Riemann-type boundary value problems;

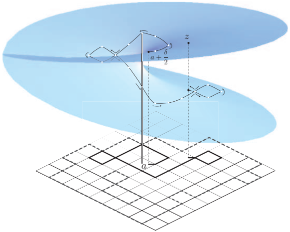
• discrete→continuum: prove convergence of F^δ to the solutions f of the similar continuous b.v.p. [non-trivial technicalities];

• continuum→discrete: find the limit of (spatial derivatives of) using the convergence $F^\delta \to f$ [via coefficients at singularities].

Conformal covariance of spin correlations at criticality

Example: to handle $\mathbb{E}^{+}_{\Omega_\delta}[\sigma_u]$, one should consider the following b.v.p.:

- $f(z^\sharp) \equiv -f(z^\flat)$, branches over u;
- $\mathrm{Im}\big[\, f(\zeta)\sqrt{n(\zeta)}\,\big] = 0$ for $\zeta \in \partial\Omega$;
- $f(z) = \frac{1}{\sqrt{z-u}} + \dots$



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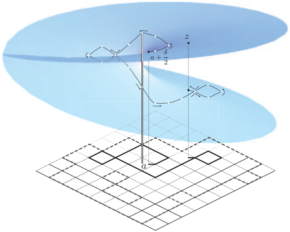

- $f(z^\sharp) \equiv -f(z^\flat)$, branches over u;
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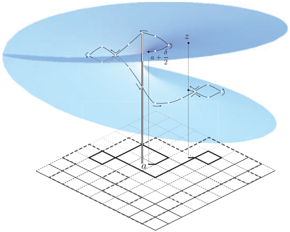
- $f(z) = \frac{1}{\sqrt{z-u}} + 2\mathcal{A}_\Omega(u) \cdot \sqrt{z-u} + \ldots$

Claim: If Ω_δ converges to Ω as $\delta \to 0$, then

- $(2\delta)^{-1} \log \left[\mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_\delta+2\delta}] \,/\, \mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_\delta}] \right] \to \mathrm{Re}[\,\mathcal{A}_\Omega(u)\,]\,;$
- $(2\delta)^{-1} \log \left[\mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_\delta+2i\delta}] \,/\, \mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_\delta}] \right] \to -\,\mathrm{Im}\,[\,\mathcal{A}_\Omega(u)\,]\,.$

Conformal covariance of spin correlations at criticality

Example: to handle $\mathbb{E}^{+}_{\Omega_\delta}[\sigma_u]$, one should consider the following b.v.p.:



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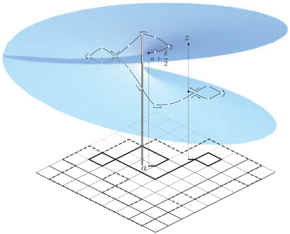
- $(2\delta)^{-1} \log \Big[\mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_\delta+2i\delta}] \,/\, \mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_\delta}]\Big] \to -\,\mathrm{Im}\,[\,\mathcal{A}_\Omega(u)\,]$.

Conformal covariance $\frac{1}{8}$: for any conformal map $\phi : \Omega \to \Omega'$,

- $f_{[\Omega,a]}(w) = f_{[\Omega',\phi(a)]}(\phi(w)) \cdot (\phi'(w))^{1/2}$;
- $\mathcal{A}_\Omega(z) = \mathcal{A}_{\Omega'}(\phi(z)) \cdot \phi'(z) + \frac{1}{8} \cdot \phi''(z)/\phi'(z)$.

Conformal covariance of spin correlations at criticality



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Claim: If Ω_δ converges to Ω as $\delta \to 0$, then

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$$\circ \quad (2\delta)^{-1} \log \left[\mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_\delta+2i\delta}] \,/\, \mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_\delta}] \right] \to -\,\mathrm{Im}\,[\,\mathcal{A}_\Omega(u)\,]\,.$$

Quite a lot of technical work is needed, e.g.:

- to handle tricky boundary conditions [Dirichlet for $\int \mathrm{Re}[f^2dz]$];
- to prove convergence, incl. near singularities [complex analysis];
- to recover the **normalization** of $\mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_1}\ldots\sigma_{u_n}]$ [probability].

Explicit formulae for multi-point spin correlations

We *define* $\langle \sigma_{u_1} \ldots \sigma_{u_n} \rangle_{\Omega}^{+} := \exp[\int \mathcal{L}(u_1, \ldots, u_n)]$, where

$$\mathcal{L}_{\Omega}(u_1, \ldots, u_n) := \textstyle\sum_{s=1}^{n} \mathrm{Re}\,[\,\mathcal{A}_{\Omega}(u_s; u_1, ..., \hat{u}_s, ..., u_n) du_s\,],$$

and the multiplicative normalization is chosen so that

$$\begin{array}{rcll} \langle \sigma_{u_1} ... \sigma_{u_n} \rangle_{\Omega}^{+} & \sim & \langle \sigma_{u_1} .... \sigma_{u_{n-1}} \rangle_{\Omega}^{+} \cdot \langle \sigma_{u_n} \rangle_{\Omega}^{+} & \text{as } \; u_n \to \partial\Omega\,, \\ \langle \sigma_{u_1} \sigma_{u_2} \rangle_{\Omega}^{+} & \sim & |u_2 - u_1|^{-\frac{1}{4}} & \text{as } \; u_2 \to u_1 \in \Omega\,. \end{array}$$

Coefficients $\mathcal{A}_{\Omega}(u_1; u_2, ..., u_n)$ are *defined* via the following b.v.p.:

- $f(z^{\sharp}) \equiv -f(z^{\flat})$ is a holomorphic spinor on $[\Omega; u_1, ..., u_n]$;
- $\mathrm{Im}\big[\, f(\zeta)(n(\zeta))^{\frac{1}{2}}\,\big] = 0$ for $\zeta \in \partial\Omega$;
- $f(z) = ic_s \cdot (z - u_s)^{-\frac{1}{2}} + \ldots$ for some (unknown) $c_s \in \mathbb{R}$, $s \geqslant 2$;
- $f(z) = (z - u_1)^{-\frac{1}{2}} + 2\mathcal{A}_{\Omega}(u_1; u_2, ..., u_n) \cdot (z - u_1)^{\frac{1}{2}} + \ldots$

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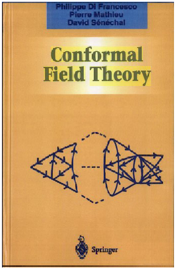
$$\begin{aligned} \langle \sigma_{u_1} \ldots \sigma_{u_n} \rangle_\Omega^+ &\sim \langle \sigma_{u_1} \ldots . \sigma_{u_{n-1}} \rangle_\Omega^+ \cdot \langle \sigma_{u_n} \rangle_\Omega^+ && \text{as} \;\; u_n \to \partial\Omega\,, \\ \langle \sigma_{u_1} \sigma_{u_2} \rangle_\Omega^+ &\sim |u_2 - u_1|^{-\frac{1}{4}} && \text{as} \;\; u_2 \to u_1 \in \Omega\,. \end{aligned}$$

Remarks: • The closedness of the differential form $\mathcal{L}_{\Omega,n}$ and the existence of an appropriate multiplicative normalization are not immediate (can be deduced along the proof of convergence);

• Similar techniques can be applied for more involved boundary conditions and/or in the multiply connected setup (when no explicit formulae are available), as well as to **other fields...**

Correlations at criticality: convergence to CFT predictions



- Three local primary fields:
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- **Theorem:** *[Ch.–Hongler–Izyurov]*
  If $\Omega_\delta \to \Omega$ and $u_{k,\delta} \to u_k$ as $\delta \to 0$, then

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for any conformal mapping $\varphi : \Omega \to \Omega'$, and

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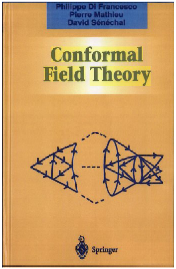

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- **Theorem:** *[Hongler–Smirnov, Hongler (2010)]*
  If $\Omega_\delta \to \Omega$ and $e_{k,\delta} \to z_k$ as $\delta \to 0$, then

$$\delta^{-n} \cdot \mathbb{E}^{+}_{\Omega_\delta}[\varepsilon_{e_{1,\delta}} \dots \varepsilon_{e_{n,\delta}}] \underset{\delta \to 0}{\to} \mathcal{C}^{n}_{\varepsilon} \cdot \langle \varepsilon_{z_1} \dots \varepsilon_{z_n} \rangle^{+}_{\Omega}$$

where $\mathcal{C}_\varepsilon$ is a lattice-dependent constant,

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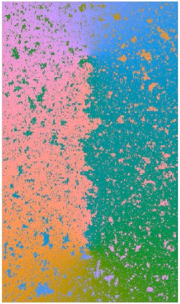
for any conformal mapping $\varphi : \Omega \to \Omega'$, and

$$\langle \varepsilon_{z_1} \dots \varepsilon_{z_n} \rangle^{+}_{\mathbb{H}} = i^n \cdot \mathrm{Pf}\left[(z_s - z_m)^{-1}\right]^{2n}_{s,m=1}, \quad z_s = \overline{z}_{2n+1-s}\,.$$

- **Ingredients:** convergence of basic **fermionic observables** (via Riemann-type b.v.p.) and (built-in) **Pfaffian formalism**

Correlations at criticality: convergence to CFT predictions

[Ch.–Hongler–Izyurov (2016, in progress)]



• Convergence of **mixed correlations: spins (σ), disorders (μ), fermions (ψ), energy densities (ε)** (in multiply connected domains Ω, with mixed free/fixed boundary conditions $\mathfrak{b}$) to conformally covariant limits that can be defined via solutions to appropriate Riemann-type boundary value problems in Ω.

• Standard **CFT fusion rules**

$$\sigma\mu \rightsquigarrow \eta\psi + \overline{\eta}\overline{\psi}, \qquad \psi\sigma \rightsquigarrow \mu, \qquad \psi\mu \rightsquigarrow \sigma,$$
$$i\psi\overline{\psi} \rightsquigarrow \varepsilon, \qquad \sigma\sigma \rightsquigarrow 1+\varepsilon, \qquad \mu\mu \rightsquigarrow 1-\varepsilon$$

can be deduced from properties of solutions to Riemann-type b.v.p.

• **Stress-energy tensor:** *[Ch.–Glazman–Smirnov (2016)]*

Correlations at criticality: convergence to CFT predictions

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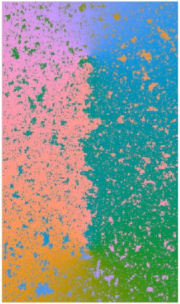

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- Standard **CFT fusion rules**, e.g. $\sigma\sigma \rightsquigarrow 1 + \varepsilon$:

$$\langle \sigma_{u'}\sigma_u \ldots \rangle_\Omega^\mathfrak{b} = |u'-u|^{-\frac{1}{4}} \left[\langle \ldots \rangle_\Omega^\mathfrak{b} + \tfrac{1}{2}|u'-u|\langle \varepsilon_u \ldots \rangle_\Omega^\mathfrak{b} + o(|u'-u|) \right],$$

can be deduced from properties of solutions to Riemann-type b.v.p.

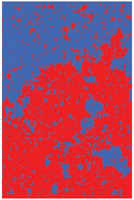
- **More details:** `arXiv:1605.09035`, `arXiv:1[6]??.?????`

Some research routes / open questions

• Better understanding of the CFT description at criticality: other fields, fusion rules, height functions, "geometric" observables (e.g., probabilities of concrete topologies of domain walls)

• Near-critical (massive) regime $x - x_{\mathrm{crit}} = m \cdot \delta$: convergence of correlations, massive SLE_3 curves and loop ensembles etc.

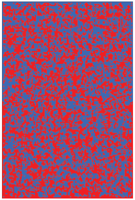
• **Super-critical regime:** e.g., convergence of interfaces to SLE_6 curves for any fixed $x > x_{\mathrm{crit}}$ [known only for $x = 1$ (percolation)]



$x = x_{\mathrm{crit}}$

• **Renormalization**

$$\text{fixed } x > x_{\mathrm{crit}},\ \delta \to 0 \quad \xrightarrow{\qquad} \quad (x - x_{\mathrm{crit}}) \cdot \delta^{-1} \to \infty$$



$x = 1$

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Tool: local relations and spinor observables are always there!

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Extended version of this talk: arXiv:1605.09035

Thank you!