2D Ising model: s-holomorphicity and correlation functions

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[Sample of a critical 2D Ising configuration (with two disorders), © Clément Hongler (EPFL)]

Charles River Lectures

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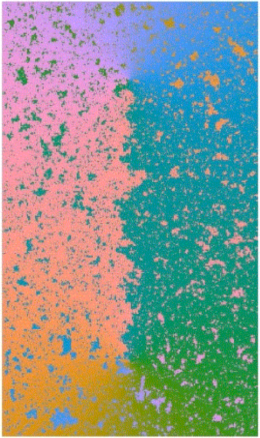
<u>Outline</u>:

- Nearest-neighbor Ising model in 2D:
  - definition, phase transition
  - fermionic observables
  - local relations: s-holomorphicity
  - dimers and Kac–Ward matrices
- Conformal invariance at criticality:
  - s-holomorphic observables
  - spin correlations and other fields
  - interfaces and loop ensembles
- Research routes



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Nearest-neighbor Ising or Lenz-Ising model in 2D

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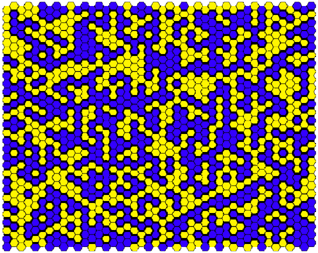

[sample of a honeycomb percolation]

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$$\mathbb{P}\left[\text{conf. } \sigma \in \{\pm 1\}^{V(G^*)}\right] \propto \exp\left[\beta \sum\nolimits_{e=\langle uv \rangle} J_{uv}\sigma_u\sigma_v\right] \propto \prod\nolimits_{e=\langle uv\rangle : \sigma_u \neq \sigma_v} x_{uv} ,$$

where $J_{uv} > 0$ are interaction constants assigned to edges $\langle uv \rangle$, $\beta = 1/kT$ is the inverse temperature, and $x_{uv} = \exp[-2\beta J_{uv}]$.

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Disclaimer:

no external magnetic field.

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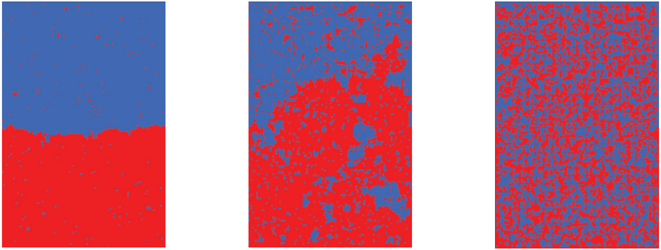
Phase transition (e.g., on $\mathbb{Z}^2$)

- Dobrushin boundary conditions: $+1$ on (ab) and -1 on (ba)

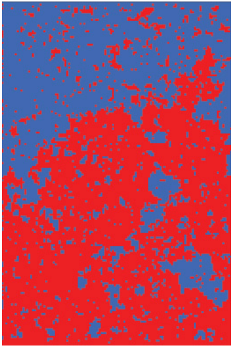


$x < x_{\mathrm{crit}}$ $\quad$ $x = x_{\mathrm{crit}}$ $\quad$ $x > x_{\mathrm{crit}}$

- Ising (1925): no phase transition in 1D $\rightsquigarrow$ doubts about 2+D;
- Peierls (1936): existence of the phase transition in 2D;
- Kramers-Wannier (1941): $x_{\text{self-dual}} = \sqrt{2} - 1 = \tan(\frac{1}{2} \cdot \frac{\pi}{4})$;
- Onsager (1944): sharp phase transition at $x = \sqrt{2} - 1$.

At criticality (e.g., on $\mathbb{Z}^2$):

- Kaufman-Onsager(1948-49), Yang(1952): scaling exponent $\frac{1}{8}$ for the magnetization (some spin correlations in $\mathbb{Z}^2$ at $x \uparrow x_{\mathrm{crit}}$).
- In particular, for $\Omega_\delta \to \Omega$ and $u_\delta \to u \in \Omega$, it should be $\mathbb{E}_{\Omega_\delta}[\sigma_{u_\delta}] \asymp \delta^{\frac{1}{8}}$ as $\delta \to 0$.



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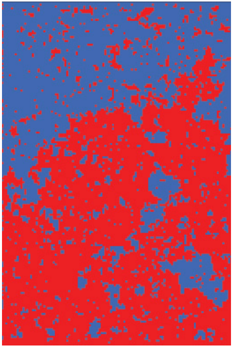

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Questions for the part #2:

- Convergence of correlations, e.g.

$$\delta^{-\frac{n}{8}} \cdot \mathbb{E}_{\Omega_\delta}[\sigma_{u_{1,\delta}} \ldots \sigma_{u_{n,\delta}}] \underset{\delta \to 0}{\longrightarrow} \langle \sigma_{u_1} \ldots \sigma_{u_n} \rangle_\Omega \ ?$$

- Convergence of curves: interfaces (e.g. generated by Dobrushin boundary conditions) to SLE_3's, loop ensembles to CLE_3's?

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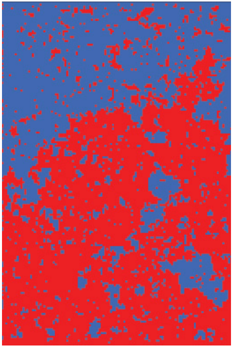

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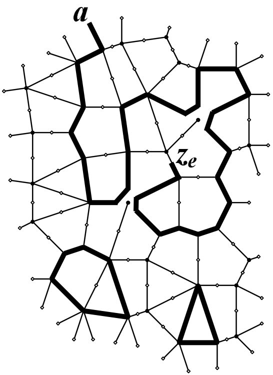
Q: Why these limits are conformally invariant (covariant)?

Fermionic observables: combinatorial definition [Smirnov '00s]

For an oriented edge a of G and a midpoint z_e of another edge e,

$$F_G(a, z_e) \ := \ \overline{\eta}_a \sum_{\omega \in \mathrm{Conf}_G(a, z_e)} \Big[e^{-\frac{i}{2}\mathrm{wind}(a \rightsquigarrow z_e)} \prod_{\langle uv \rangle \in \omega} x_{uv} \Big],$$

where η_a denotes the (once and forever fixed) square root of the direction of a.



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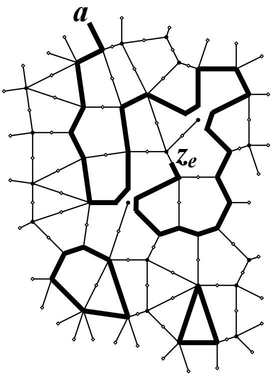

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- When both a and e are "boundary" edges, the factor $\overline{\eta}_a e^{-\frac{i}{2}\mathrm{wind}(a\rightsquigarrow z_e)} = \pm\overline{\eta}_e$ is fixed and $F_G(a,z_e)$ becomes the partition function of the Ising model (on G^*) with Dobrushin boundary conditions.

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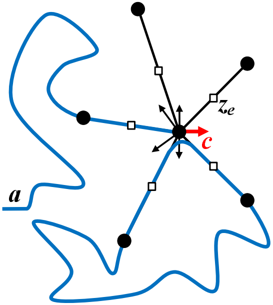

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• **Local relations:** if we similarly define $F_G(a,\cdot)$ on "corners" of G, then for any $c \sim z_e \neq z_a$ one has

$$F_G(a,c) = e^{\pm\frac{i}{2}(\theta_e - \boldsymbol{\alpha}(\mathbf{c},\mathbf{e}))}\mathrm{Proj}[\, F_G(a,z_e)\,;\; e^{\mp\frac{i}{2}\theta_e}\overline{\eta}_e\,]\,.$$

Fermionic observables: local relations

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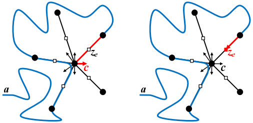
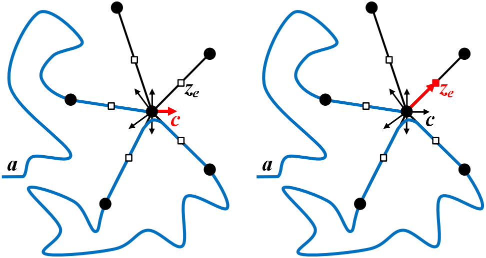
- **Claim:** $F_G(a,c) \;=\; e^{\pm\frac{i}{2}(\theta_e-\alpha(c,e))}\cdot \mathrm{Proj}[\,F_G(a,z_e)\,;\, e^{\mp\frac{i}{2}\theta_e}\overline{\eta}_e\,]\,.$

- **Proof:** a bijection between $\mathrm{Conf}_G(a,c)$ and $\mathrm{Conf}(a,z_e)$. Case A: Case B:





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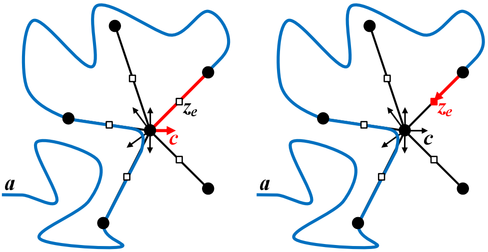

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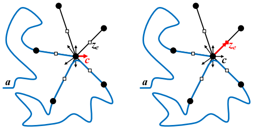

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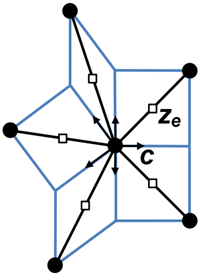

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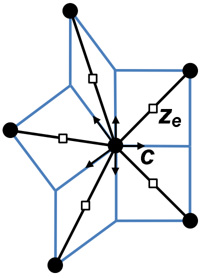

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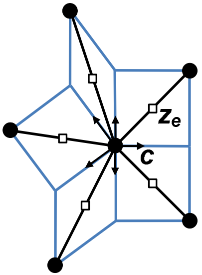
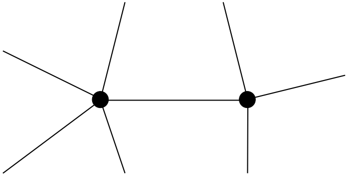


- $\Rightarrow$ critical weights on regular grids:
  - square: $x_{\mathrm{crit}}=\tan\frac{\pi}{8}=\sqrt{2}-1$,
  - honeycomb: $x_{\mathrm{crit}}=\tan\frac{\pi}{6}=1/\sqrt{3}$, ...

2D Ising model as a dimer model on a non-bipartite graph (..., Fisher, Kasteleyn, ..., Kenyon, Dubedat, ...)

• There exist **several representations** of the 2D Ising model via dimers on an auxiliary graph



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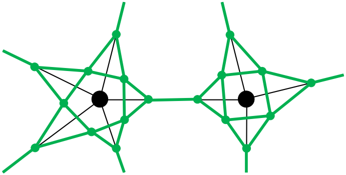

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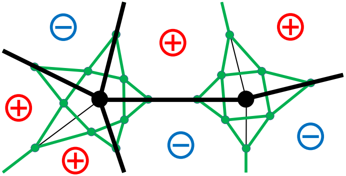

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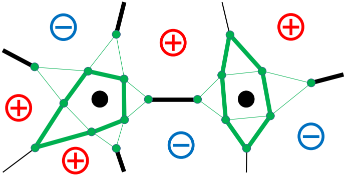

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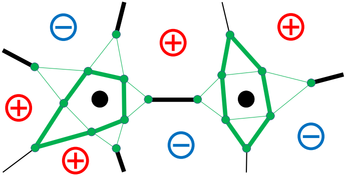

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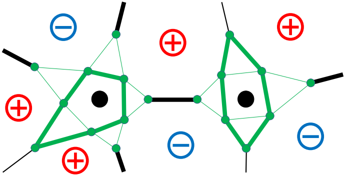

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- **Definition of fermionic observables via dimers on $\mathbf{G_F}$:**

$$F_G(a,c) = \overline{\eta}_c \mathrm{K}^{-1}_{c,a} \qquad \text{and} \qquad F_G(a,z_e) = \overline{\eta}_e \mathrm{K}^{-1}_{e,a} + \overline{\eta}_{\overline{e}} \mathrm{K}^{-1}_{\overline{e},a}\,.$$

- **Local relations:** an equivalent form of the identity $\mathbf{K \cdot K^{-1} = Id}$

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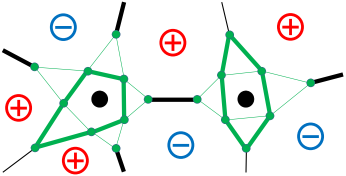

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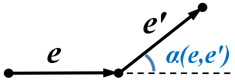
• **Kac–Ward formula (1952–..., 1999–...):** $\mathcal{Z}^2 = \det[\mathrm{Id} - \mathrm{T}]$,

$$\mathrm{T}_{e,e'} = \begin{cases} \exp[\frac{i}{2}\alpha(e,e')] \cdot (x_e x_{e'})^{1/2} \\ 0 \end{cases}$$



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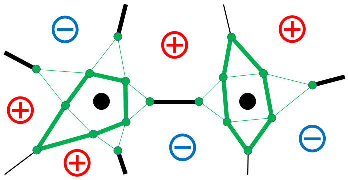

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$$\mathrm{T}_{e,e'} = \begin{cases} \exp[\frac{i}{2}\alpha(e,e')] \cdot (x_e x_{e'})^{1/2} & \text{if } e' \text{ prolongs } e \text{ but } e' \neq \bar{e}; \\ 0 & \text{otherwise.} \end{cases}$$

[**is equivalent** to the **Kasteleyn theorem for dimers on** $\mathbf{G_F}$]

2D Ising model as a dimer model on a non-bipartite graph (..., Fisher, Kasteleyn, ..., Kenyon, Dubedat, ...)

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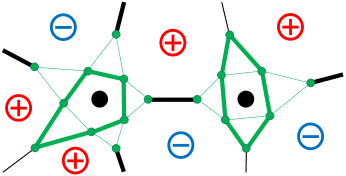

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- **More information:** `arXiv:1507.08242` *[Ch., Cimasoni, Kassel]*

Part II: conformal invariance at criticality *[Smirnov '06]*
[Ch.,Duminil-Copin,Hongler,Izyurov,Kemppainen,Kytölä,... '09−...]

Main tool: discrete (s-)holomorphic functions

• (A fair amount of) work is needed to understand how to use them for the rigorous analysis when $\Omega_\delta \to \Omega$, especially in rough domains formed by fractal interfaces.

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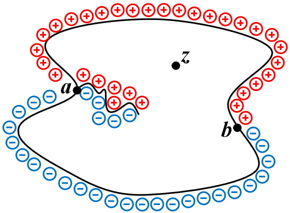

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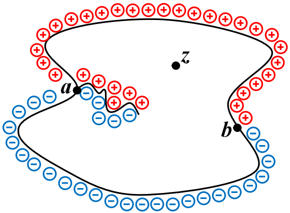

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- prove uniform convergence of the (re-scaled) quantities as $\delta \to 0$ [the one above (done in 2012) is not an optimal choice, there are others that are easier to handle (first done in 2006–2009)];
- prove the convergence $\gamma_\delta \to \gamma$ and recover the law of γ using this family of martingales [some probabilistic techniques are needed].

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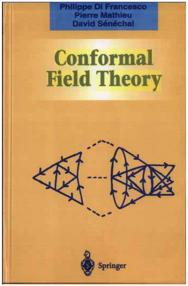
Conformal covariance of correlation functions at criticality



- Three primary fields:
  $\mathbf{1}$, $\boldsymbol{\sigma}$ (spin), $\boldsymbol{\varepsilon}$ (energy density);
  Scaling exponents: 0, $\frac{1}{8}$, 1.

- **Energy density:** for an edge e of Ω, let

$$\begin{aligned} \varepsilon_e &:= \sigma_{e^\sharp}\sigma_{e^\flat} - \varepsilon_{\text{inf.vol.}} \\ &= (1 - \varepsilon_{\text{inf.vol.}}) - 2 \cdot \chi[e \in \omega] \end{aligned}$$

where $e^\sharp$ and $e^\flat$ are two faces adjacent to e.

[$\varepsilon_{\text{inf.vol.}}$ is lattice-dependent: $= 2^{-\frac{1}{2}}$(square), $= \frac{2}{3}$(honeycomb), ...]

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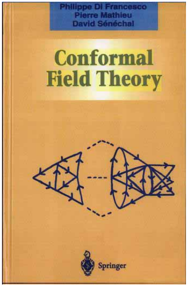

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  If $\Omega_\delta \to \Omega$ and $e_{k,\delta} \to e_k$ as $\delta \to 0$, then

$$\boldsymbol{\delta^{-n}} \cdot \mathbb{E}^{+}_{\Omega_\delta}[\varepsilon_{u_{1,\delta}} \dots \varepsilon_{u_{n,\delta}}] \underset{\delta\to 0}{\to} \mathcal{C}^n_\varepsilon \cdot \langle \boldsymbol{\varepsilon_{e_1} \dots \varepsilon_{e_n}} \rangle^{+}_{\boldsymbol{\Omega}},$$

where $\mathcal{C}_\varepsilon$ is a lattice-dependent constant,

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for any conformal mapping $\varphi : \Omega \to \Omega'$, and

$$\langle \boldsymbol{\varepsilon_{z_1} \dots \varepsilon_{z_n}} \rangle^{+}_{\mathbb{H}} = (\pi i)^{-n} \cdot \mathrm{Pf}\left[(z_s - z_m)^{-1}\right]^{2n}_{s,m=1}, \quad z_s = \overline{z}_{2n+1-s}.$$

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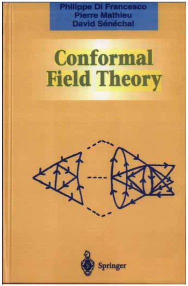

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- **Theorem:** *[Hongler–Smirnov, Hongler]*
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- **Ingredients: convergence of $\mathbf{K^{-1}_{e,a}}$** and **Pfaffian formalism**

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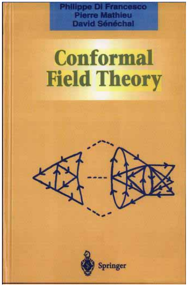

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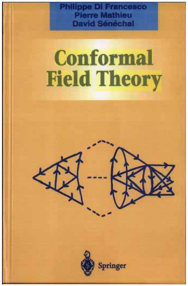

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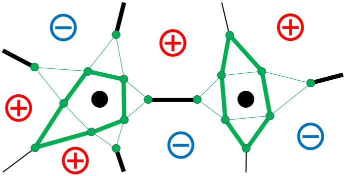

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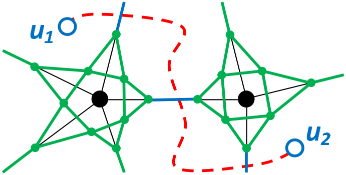

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$$\mathbb{E}[\sigma_{u_1} \ldots \sigma_{u_n}] \;=\; \frac{\mathrm{Pf}[\,\mathrm{K}_{[u_1,\ldots,u_n]}\,]}{\mathrm{Pf}[\,\mathrm{K}\,]},$$

where $\mathrm{K}_{[u_1,\ldots,u_n]}$ is obtained from K by changing the sign of its entries on **slits linking $u_1, \ldots, u_n$** (and, possibly, u_{out}) pairwise.

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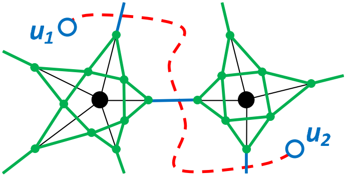

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- **More invariant way** to think about entries of $\mathbf{K}^{-1}_{[u_1,\ldots,u_n]}$:

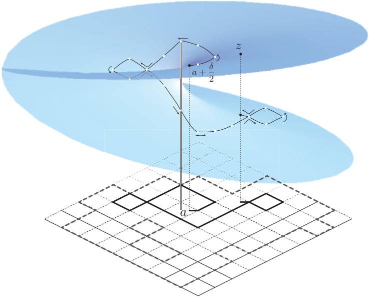
double-covers of G branching over $u_1,\ldots,u_n$

Conformal covariance of spin correlations at criticality

Main tool: spinors on the double cover $[\Omega_\delta; u_1, \ldots, u_n]$.

$$F_{\Omega_\delta}(z) := \left[\mathcal{Z}^+_{\Omega_\delta}[\sigma_{u_1} \ldots \sigma_{u_n}]\right]^{-1} \cdot \sum_{\omega \in \mathrm{Conf}_{\Omega_\delta}(u_1^{\rightarrow}, z)} \phi_{u_1,\ldots,u_n}(\omega, z) \cdot x_{\mathrm{crit}}^{\#\mathrm{edges}(\omega)},$$

$$\phi_{u_1,\ldots,u_n}(\omega, z) := e^{-\frac{i}{2}\mathrm{wind}(\mathrm{p}(\omega))} \cdot (-1)^{\#\mathrm{loops}(\omega \setminus \mathrm{p}(\omega))} \cdot \mathrm{sheet}(\mathrm{p}(\omega), z).$$



Conformal covariance of spin correlations at criticality

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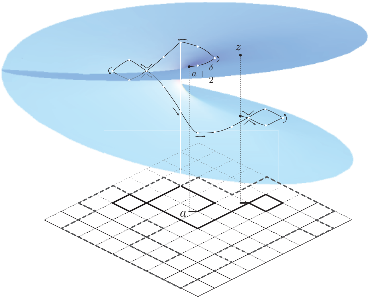

- $\mathrm{wind}(\mathrm{p}(\gamma))$ is the winding of the path $\mathrm{p}(\gamma) : u_1^{\to} = u_1 + \frac{\delta}{2} \rightsquigarrow z$;
- $\#\mathrm{loops}$ – those containing an odd number of $u_1, \ldots, u_n$ inside;
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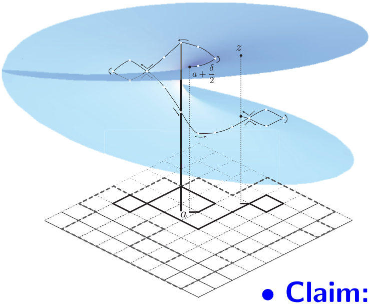

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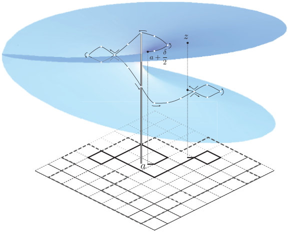
- **Claim:** $F_{\Omega_\delta}(u_1 + \frac{3\delta}{2}) = \dfrac{\mathbb{E}^+_{\Omega_\delta}[\sigma_{u_1+2\delta} \ldots \sigma_{u_n}]}{\mathbb{E}^+_{\Omega_\delta}[\sigma_{u_1} \ldots \sigma_{u_n}]}$

Conformal covariance of spin correlations at criticality

Example: to handle $\mathbb{E}^{+}_{\Omega_\delta}[\sigma_u]$, one should consider the following b.v.p.:



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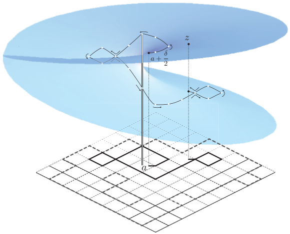

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Claim: For $\Omega_\delta \to \Omega$ as $\delta \to 0$,

- $(2\delta)^{-1} \log \left[\mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_\delta+2\delta}] \,/\, \mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_\delta}]\right] \to \mathrm{Re}[\mathcal{A}_\Omega(u)]\,;$
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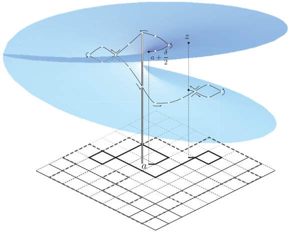

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Conformal covariance $\frac{1}{8}$: for any conformal map $\phi : \Omega \to \Omega'$,

- $f_{[\Omega,a]}(w) = f_{[\Omega',\phi(a)]}(\phi(w)) \cdot (\phi'(w))^{1/2}\,;$
- $\mathcal{A}_\Omega(z) = \mathcal{A}_{\Omega'}(\phi(z)) \cdot \phi'(z) + \frac{1}{8} \cdot \phi''(z)/\phi'(z)\,.$

Conformal covariance of spin correlations at criticality

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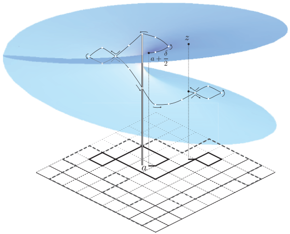

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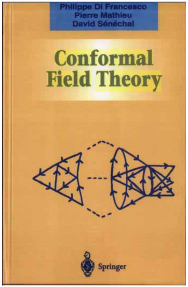
Work to be done:

- to handle tricky boundary conditions (Dirichlet for $\int \mathrm{Re}[f^2 dz]$);
- to prove convergence, incl. near singularities [complex analysis];
- to recover the normalization of $\mathbb{E}^+_{\Omega_\delta}[\sigma_u]$ [probabilistic techniques].

Some research routes

• Better understanding of the
CFT description at criticality:

more fields, Virasoro algebra at the lattice level, “geometric” observables, height functions, Riemann surfaces etc.



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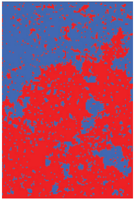
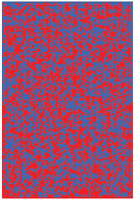


$x = x_{\mathrm{crit}}$



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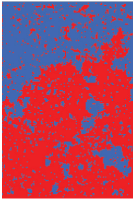

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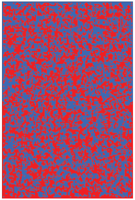
• **Renormalization**

fixed $x > x_{\mathrm{crit}}$, $\delta \to 0$

$$\longrightarrow$$

$(x - x_{\mathrm{crit}}) \cdot \delta^{-1} \to \infty$



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