Tau-functions à la Dubédat and cylindrical events in the double-dimer model

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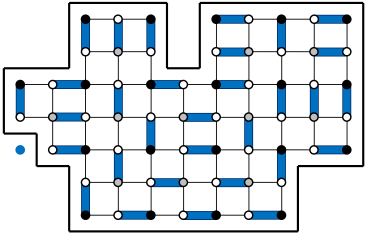
Setup: double-dimer loop ensembles in Temperley discretizations on $\mathbb{Z}^2$

• **Temperley discretizations Ω^δ on $\mathbb{Z}^2$:** simply connected domains s.t. all corners are of the same type out of four: B_0, B_1, W_0, W_1.



• **Dimer (= domino) model on Ω^δ:** perfect matchings, chosen uniformly at random.

• **Kasteleyn theorem:** $\mathcal{Z}^{\mathrm{dimers}} = \det K$,

where $K : \mathbb{C}^{\mathcal{B}} \to \mathbb{C}^{\mathcal{W}}$ is a weighted adjacency matrix (= discrete $\overline{\partial}$ operator on Ω^δ).
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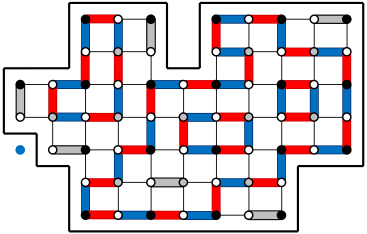

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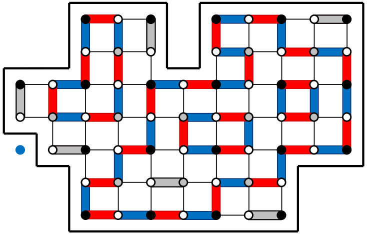
- **Double-dimer model:** two independent dimer configurations on the same domain. Configuration $\mathcal{L}^{\text{dbl-d}}$ is a fully-packed collection of loops and double-edges,

$$\mathcal{Z}^{\text{dbl-d}} = \sum_{\mathcal{L}^{\text{dbl-d}}} 2^{\#\text{loops}(\mathcal{L}^{\text{dbl-d}})} = \det \begin{pmatrix} 0 & K^\top \\ K & 0 \end{pmatrix} = \det \mathcal{K}, \qquad \mathcal{K} : (\mathbb{C}^2)^{\mathcal{B}} \to (\mathbb{C}^2)^{\mathcal{W}}.$$

Goal (cf. Kenyon'10, Dubédat'14): conformal invariance, convergence to CLE_4



- **Random height functions and GFF:**

Choosing the orientation of loops $\gamma \in \mathcal{L}^{\text{dbl-d}}$ randomly, one gets a height function $h^{\text{dbl-d}}$.

Kenyon'00: $h^{\text{dbl-d}} \to \mathrm{GFF}(\Omega)$ as $\delta \to 0$.

- **Random loop ensembles and CLE_4:**

It is a famous prediction (supported by many strong results) that $\mathcal{L}^{\text{dbl-d}}$ converges to the nested conformal loop ensemble $\mathrm{CLE}_4(\Omega)$.

[!] The convergence of $h^{\text{dbl-d}}$ is not strong enough for the level lines $\mathcal{L}^{\text{dbl-d}}$ of $h^{\text{dbl-d}}$.

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Kenyon (2010): $\mathrm{SL}_2(\mathbb{C})$-monodromies and Q-determinants for double-dimers

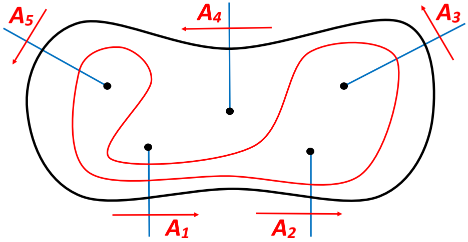
Let $\rho : \pi_1(\Omega \setminus \{\lambda_1, \ldots, \lambda_n\}) \to \mathrm{SL}_2(\mathbb{C})$.

Down-to-earth viewpoint: draw cuts from punctures λ_k to $\partial\Omega$ and choose $A_k \in \mathrm{SL}_2(\mathbb{C})$.

- Kasteleyn's theorem generalizes as follows:

$$\mathbb{E}\Big[\prod\nolimits_{\gamma \in \mathcal{L}^{\text{dbl-d}}} (\tfrac{1}{2}\,\mathrm{Tr}\,\rho(\gamma))\Big] = \frac{\mathrm{Qdet}\,\mathcal{K}^{(\rho)}}{\det \mathcal{K}},$$

where $\mathcal{K}^{(\rho)} : (\mathbb{C}^2)^{\mathcal{B}} \to (\mathbb{C}^2)^{\mathcal{W}}$ is obtained from $\mathcal{K}$ by putting the matrices $A_k^{\pm 1}$ on cuts.



$\rho(\gamma) = A_5 A_1^{-1} A_3 A_2 A_1$

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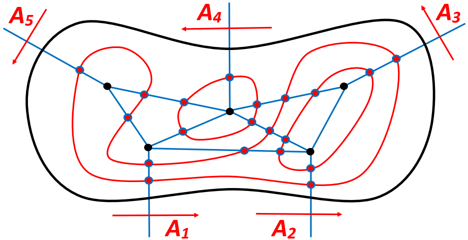

$n(L) = (2,2,2,1,1,1,2,0,1,3,3,1,2)_{e\in\mathcal{E}}$

Remark: A better viewpoint is to fix a triangulation of $\Omega \setminus \{\lambda_1, \ldots, \lambda_n\}$ and to consider discrete $\mathbb{C}^2$-vector bundles and flat $\mathrm{SL}_2(\mathbb{C})$-connections on them:

$$(\mathrm{Fun}(\pi_1(\Omega\setminus\{\lambda_1,\ldots,\lambda_n\}) \to \mathrm{SL}_2(\mathbb{C})))^{\mathrm{SL}_2(\mathbb{C})} \;\simeq\; (\mathrm{Fun}(\mathrm{SL}_2(\mathbb{C})^{\mathcal{E}}))^{\mathrm{SL}_2(\mathbb{C})^{\mathcal{F}}}.$$

Dubédat (2014): locally unipotent monodromies and convergence to the Jimbo–Miwa–Ueno isomonodronic τ-function

Let Ω^δ, $\delta \to 0$, be a sequence of Temperley approximations to a simply connected domain $\Omega \subset \mathbb{C}$. Fix a collection of (pairwise distinct) punctures $\lambda_1, \ldots, \lambda_n \in \Omega$.

Theorem (Dubédat, 2014): Let $\rho : \pi_1(\Omega \setminus \{\lambda_1, \ldots, \lambda_n\}) \to \mathrm{SL}_2(\mathbb{C})$ be such that $\mathbf{Tr}\, \boldsymbol{\rho}([\boldsymbol{\gamma_k}]) = \mathbf{2}$ for each of the loops $[\lambda_k]$ surrounding a single puncture λ_k.

(i) Then $\quad \mathbb{E}\big[\prod_{\gamma \in \mathcal{L}^{\text{dbl-d}}} (\tfrac{1}{2} \operatorname{Tr} \rho(\gamma))\big] \;=:\; \boldsymbol{\tau}^{\boldsymbol{\delta}}(\boldsymbol{\rho}) \;\to\; \boldsymbol{\tau}^{\mathbf{JMU}}(\boldsymbol{\rho})$ as $\delta \to 0$.

Remark: In fact, this convergence is uniform on compact subsets of

$$\boldsymbol{X}_{\mathbf{unip}} \subset \boldsymbol{X} := \{\rho : \pi_1(\Omega \setminus \{\lambda_1, \ldots, \lambda_n\}) \to \mathrm{SL}_2(\mathbb{C})\}.$$

(ii) Moreover, provided that $\rho \in X_{\text{unip}}$ is close enough to Id, one has

$$\boldsymbol{\tau}^{\mathbf{JMU}}(\boldsymbol{\rho}) \;=\; \boldsymbol{\tau}^{\mathbf{CLE_4}}(\boldsymbol{\rho}) \;:=\; \mathbb{E}\big[\prod_{\gamma \in \mathcal{L}^{\mathrm{CLE}_4}} (\tfrac{1}{2} \operatorname{Tr} \rho(\gamma))\big].$$

Dubédat (2014): locally unipotent monodromies and convergence to the Jimbo–Miwa–Ueno isomonodronic τ-function

Notation: *Lamination* L = collection of loops in $\Omega \setminus \{\lambda_1, \dots, \lambda_n\}$ up to homotopies.

$$p_L^\delta := 2^{-\#\mathrm{loops}(L)} \cdot \mathbb{P}[\mathcal{L}^{\mathrm{dbl\text{-}d}} \simeq_{\mathrm{macro}} \mathrm{L}], \qquad f_L(\rho) := \prod_{\gamma \in L} \mathrm{Tr}\, \rho(\gamma).$$

The results of Dubédat give $\tau^\delta(\rho) = \sum_{L-\mathrm{macro}} p_L^\delta f_L(\rho) \to \tau^{\mathrm{JMU}}(\rho),\ \rho \in X_{\mathrm{unip}}.$
The goal is to deduce the convergence of p_L^δ for each macroscopic lamination L.

Remark: The **isomonodronic τ-function** can be thought of as $:\det \overline{\partial}^{(\rho)}_{[\Omega;\lambda_1,\dots,\lambda_n]}:$, where $\overline{\partial}^{(\rho)}$ stands for the $\overline{\partial}$ operator acting on functions $\Omega \to \mathbb{C}^2$ with monodromy ρ.

- The function $\tau^{\mathrm{JMU}}(\rho)$ is defined for all $\rho \in X_{\mathrm{unip}}$ and is conformally invariant.
- The identity $\tau^{\mathrm{JMU}} = \tau^{\mathrm{CLE}_4}$ is a separate statement (also due to Dubédat'14).

Main result (joint w/ Mikhail Basok, 2018)

Let $\mathbb{D}_r$ denote the "ball of radius R" in $X = \{\rho : \pi_1(\Omega \setminus \{\lambda_1, \ldots, \lambda_n\}) \to \mathrm{SL}_2(\mathbb{C})\}$.
[normalization: $\|A\| := \mathrm{Tr}(AA^*)$, in particular $X \cap \mathbb{D}_r = \emptyset$ if $r \leqslant \sqrt{2}$]

Theorem: There exists an absolute constant $k_0 > 1$ such that the following holds:

(i) Let $r > \sqrt{2}$, $R := k_0 r$ and $F : X_{\mathrm{unip}} \cap \overline{\mathbb{D}}_R \to \mathbb{C}$ be a holomorphic function.
Then there exist coefficients $p_L = O\big(r^{-|n(L)|} \cdot \|F\|_{L^\infty(\overline{\mathbb{D}}_R)}\big)$ such that

$$F(\rho) \;=\; \sum\nolimits_{L-\mathrm{macro}} p_L f_L(\rho), \quad \rho \in X_{\mathrm{unip}} \cap \mathbb{D}_r\,.$$

(ii) Let $r > k_0\sqrt{2}$ and two sets of coefficients $p_L, \tilde{p}_L = O(r^{-|n(L)|})$ be such that

$$\sum\nolimits_{L-\mathrm{macro}} p_L f_L(\rho) \;=\; \sum\nolimits_{L-\mathrm{macro}} \tilde{p}_L f_L(\rho), \quad \rho \in X_{\mathrm{unip}} \cap \mathbb{D}_r.$$

Then $p_L = \tilde{p}_L$ for all macroscopic laminations L.

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Corollary: Since the isomonodromic tau-function is holomoprhic on the whole X_{unip}, there exist unique coefficients p_L^{JMU} s.t. $\tau^{\mathrm{JMU}}(\rho) = \sum_{L-\mathrm{macro}} p_L^{\mathrm{JMU}} f_L(\rho)$, $\rho \in X_{\mathrm{unip}}$.

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Corollary: **(a)** Uniform boundedness of topological correlators τ^δ on $\overline{\mathbb{D}}_R$ for all $R > 0$ implies the uniform (in δ) estimate $p_L^\delta = O(r^{-|n(L)|})$ for all $r > 0$.

(b) Convergence (as $\delta \to 0$) of topological correlators $\tau^\delta \to \tau^{\mathrm{JMU}}$ on $\overline{\mathbb{D}}_R$ implies convergence of coefficients: $p_L^\delta \to p_L^{\mathrm{JMU}}$ for all macroscopic laminations L.

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Warning: It is easy to see that $p_L^{\mathrm{CLE}_4} = O(r_0^{-|n(L)|})$ for some $r_0 > \sqrt{2}$ and Dubédat proved that $\tau^{\mathrm{CLE}_4}(\rho) = \tau^{\mathrm{JMU}}(\rho)$ for $\rho \in X_{\mathrm{unip}} \cap \mathbb{D}_{r_0}$ $(= \text{near } \mathrm{Id})$.

Unfortunately, this does not directly imply $p_L^{\mathrm{CLE}_4} = p_L^{\mathrm{JMU}}$ for all laminations L: we also need a superexponential (in fact, $r_0 > \sqrt{2}k_0$ is enough) decay of $p_L^{\mathrm{CLE}_4}$.

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Some comments on the proof:

Recall that we are interested in the existence and uniqueness of expansions of holomorphic functions living on the (algebraic) manifold

$$X_{\mathrm{unip}} \subset X = \{\rho : \pi_1(\Omega \setminus \{\lambda_1, \dots, \lambda_n\}) \to \mathrm{SL}_2(\mathbb{C})\}$$

in the basis $f_L(\rho) := \prod_{\gamma \in L} \mathrm{Tr}(\rho(\gamma))$. Two problems arise:

- Even on the whole manifold X, *the functions f_L form a bad basis.*
- Passage from $\mathrm{Fun}_{\mathrm{hol}}(X)$ to $\mathrm{Fun}_{\mathrm{hol}}(X_{\mathrm{unip}})$ is not trivial.

Some comments: f_L is a bad basis (estimate of Fock–Goncharov coefficients)

Theorem (Fock–Goncharov, 2006): There exists another "good" (e.g., orthogonal on $(\mathrm{SU}_2(\mathbb{C})^{\mathcal{E}})^{\mathrm{SU}_2(\mathbb{C})^{\mathcal{F}}}$) basis g_L on X such that the change between these bases is given by lower-triangular (with respect to the natural partial order on $n(L)$) matrices.

Consider the following naive example:
$$(g_n(z))_{n\geqslant 0} := (1\,,\, z\,,\, z^2\,,\, z^3\,,\ldots)$$
$$(f_n(z))_{n\geqslant 0} := (1\,,\, z-2\,,\, z^2-2z\,,\, z^3-2z^2\,,\ldots)$$

Then $\sum_{n\geqslant 0} p_n f_n(z) \equiv 0$ near $z=0 \implies p_n = 0$ provided that $p_n = O((\frac{1}{2}-\varepsilon)^n)$ but

$$f_0(z) + \tfrac{1}{2} f_1(z) + \tfrac{1}{4} f_2(z) + \cdots + \mathbf{2^{-n}} f_n(z) + \cdots = 0 \quad \text{for} \quad |z| < 2.$$

Warning: This can be even worse: for $(f_n(z))_{n\geqslant 0} := (1\,,\, z-2\,,\, z^2-4z\,,\, z^3-8z^2\,,\ldots)$,

$$f_0(z) + \tfrac{1}{2} f_1(z) + \tfrac{1}{8} f_2(z) + \cdots + \mathbf{2^{-\frac{1}{2}n(n+1)}} f_n(z) + \cdots = 0 \quad \text{for all} \quad z.$$

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Proposition: Let $g_L = \sum_{L': n(L') \leqslant n(L)} c_{LL'} f_{L'}$. Then $|c_{LL'}| \leqslant 4^{|n(L)|}$.

Key ingredients: We would like to thank Vladimir Fock for a very helpful discussion.

- existence of monodromies $\rho \in X$ s.t. $\mathrm{Tr}(\rho(\gamma)) \leqslant -2$ for all nontrivial simple loops γ, which can be constructed via Thurston's *shear coordinates of hyperbolic structures* on $\Omega \setminus \{\lambda_1, \ldots, \lambda_n\}$ (see Chekhov–Fock(1997+) and Bonahon–Wong(2011+));
- D. Thurston's theorem (2014) on the positivity of structure constants of the *bracelets basis in the Kauffman skein algebra* $\mathrm{Sk}(\Omega \setminus \{\lambda_1, \ldots, \lambda_n\}, 1)$.

Some comments: from $\mathrm{Fun}_{\mathrm{hol}}(X)$ to $\mathrm{Fun}_{\mathrm{hol}}(X_{\mathrm{unip}})$

Intuition behind the uniqueness: Let $F(\rho) := \sum_{L-\mathrm{macro}} p_L f_L(\rho) = 0$ on X_{unip}.

– Recall that X can be parameterized by collections of matrices $A_1, \ldots, A_n \in \mathrm{SL}_2(\mathbb{C})$ and the subvariety $X_{\mathrm{unip}} \subset X$ is cut of by the conditions $\mathrm{Tr}\, A_k = 2,\ k = 1, \ldots, n$.

– Replacing A_k^{-1} by $A_k^{\vee}$, one can extend the functions $\mathrm{Tr}\, \rho_{A_1,\ldots,A_n}(\gamma)$ to $A_k \in \mathbb{C}^{2\times 2}$.

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– If F were a *finite* linear combination of f_L, then (due to *Hilbert's Nullstellensatz*):

$$F(\rho_{A_1,\ldots,A_n}) = \sum_{k=1}^{n} F_k(A_1, \ldots, A_n)(\mathrm{Tr}\, A_k - 2) + \sum_{k=1}^{n} G_k(A_1, \ldots, A_n)(\det A_k - 1).$$

and hence $$\sum_{L-\mathrm{macro}} p_L f_L(\rho) \ = \ \sum_{k=1}^{n} F_k(\rho)(\mathrm{Tr}\, \rho([\lambda_k]) - 2) \quad \text{on} \ \ X.$$

– Since each of F_k can be expanded as $\sum_L c_L^{(k)} f_L$ and $f_L(\rho)\,\mathrm{Tr}\, \rho([\lambda_k]) = f_{L\sqcup[\lambda_k]}(\rho)$ this implies $p_L = 0$ for all L due to the uniqueness of such decompositions on X.

Some comments: from $\mathrm{Fun}_{\mathrm{hol}}(X)$ to $\mathrm{Fun}_{\mathrm{hol}}(X_{\mathrm{unip}})$

Key ingredients:

- A version of the Nullstellensatz for $\mathrm{Fun}_{\mathrm{hol}}(X)$ instead of $\mathrm{Fun}_{\mathrm{alg}}(X)$.
- A theorem due to Manivel (1993), which allows one to extend holomorphic functions from X_{unip} to X while controlling the L^2-norms of such extensions.

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Conclusions: double-dimer loop ensembles in Temperley domains

- The results of Dubédat (uniform convergence $\tau^{\delta}(\rho) \to \tau^{\mathrm{JMU}}(\rho)$ on big compact subsets of X_{unip}) do imply the convergence of probabilities of cylindrical events:

$$p_L^{\delta} \to p_L^{\mathrm{JMU}} \text{ as } \delta \to 0 \text{ for all macroscopic laminations } L.$$

  The limits p_L^{JMU} are conformally invariant ($\sum_{L-\mathrm{macro}} p_L^{\mathrm{JMU}} f_L = \tau^{\mathrm{JMU}}$ on X_{unip}).

- This statement does not require any RSW theory for double-dimers:
  a uniform (super)exponential decay of p_L^{δ} as $|n(L)| \to \infty$ follows from the uniform boundedness of topological correlators $\tau^{\delta}(\rho)$ on big compact subsets of X_{unip}.

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- To conclude that $p_L^{\mathrm{JMU}} = p_L^{\mathrm{CLE}_4}$ one needs $p_L^{\mathrm{CLE}_4} = O(r^{-|n(L)|})$ for all $r > 0$.

- To claim the convergence of double-dimer loop ensembles to CLE_4 (in any reasonable topology) it is enough to prove the tightness of those ($\sim$ RSW).

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a uniform (super)exponential decay of p_L^δ as $|n(L)| \to \infty$ follows from the uniform boundedness of topological correlators $\tau^\delta(\rho)$ on big compact subsets of X_{unip}.

- Question: *Is there a natural interpretation of* $\tau(\rho) := \mathbb{E}\big[\prod_{\gamma\in\mathcal{L}^{\mathrm{CLE}_4}}(\frac{1}{2}\operatorname{Tr}\rho(\gamma))\big]$ *with* Tr *replaced by a quantum trace and* CLE_4 *replaced by* CLE_κ, $\kappa\neq 4$?

Thank you for your attention!