Convergence of discrete harmonic functions and the conformal invariance in (critical) lattice models on isoradial graphs

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Geometry and Integrability

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- VERY SHORT INTRODUCTION: CONFORMALLY INVARIANT RANDOM CURVES
  - Examples: loop-erased random walk, percolation
  - Schramm-Lowner Evolution (SLE), Martingale principle

- DISCRETE HARMONIC FUNCTIONS ON ISORADIAL GRAPHS
  - Basic definitions
  - Convergence theorems (harmonic measure, Green function, Poisson kernel)
    D. Chelkak, S. Smirnov: Discrete complex analysis on isoradial graphs. `arXiv:0810.2188`
  - Key ideas of the proofs

- (SPIN- AND FK-)ISING MODEL ON ISORADIAL GRAPHS
  - Definition, martingale observables
  - S-holomorphic functions
  - Convergence results, universality of the model
  - (?) Star-triangle transform: connection to the 4D-consistency

Very short introduction: Conformally invariant random curves

S. Smirnov. **Towards conformal invariance of 2D lattice models.** Proceedings of the international congress of mathematicians (ICM), Madrid, Spain, August 22–30, 2006. Vol. II: Invited lectures, 1421-1451. Zürich: European Mathematical Society (EMS), 2006.

Example 1: Loop-erased Random Walk.

G. F. Lawler, O. Schramm, W. Werner, Conformal invariance of planar loop-erased random walks and uniform spanning trees, *Ann. Probab.* **32** (2004), 939–995.

I. Sample the random walk (say, on $(\delta\mathbb{Z})^2$) starting from 0 till the first time it hits the boundary of the unit disc $\mathbb{D}$. **II.** Erase all loops starting from the beginning.

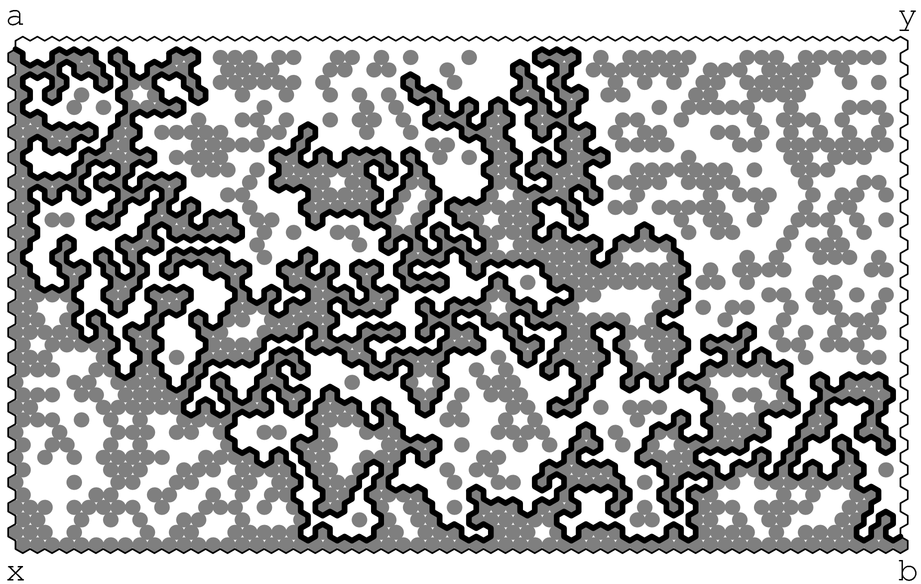
The result: simple curve going from 0 to $\partial\mathbb{D}$.

Question: How to describe its scaling limit as $\delta \to 0$?
(should be conformally invariant since the Brownian motion (scaling limit of random walks) is conformally invariant and the loop-erasure procedure is pure topological)

Example 2: Percolation interfaces (site percolation on the triangular lattice).

S. Smirnov, Critical percolation in the plane: Conformal invariance, Cardy's formula, scaling limits, *C. R. Acad. Sci. Paris* **333**, 239–244 (2001).

Take some simple-connected discrete domain Ω^δ. For each site toss the (fair) coin and paint the site black or white.



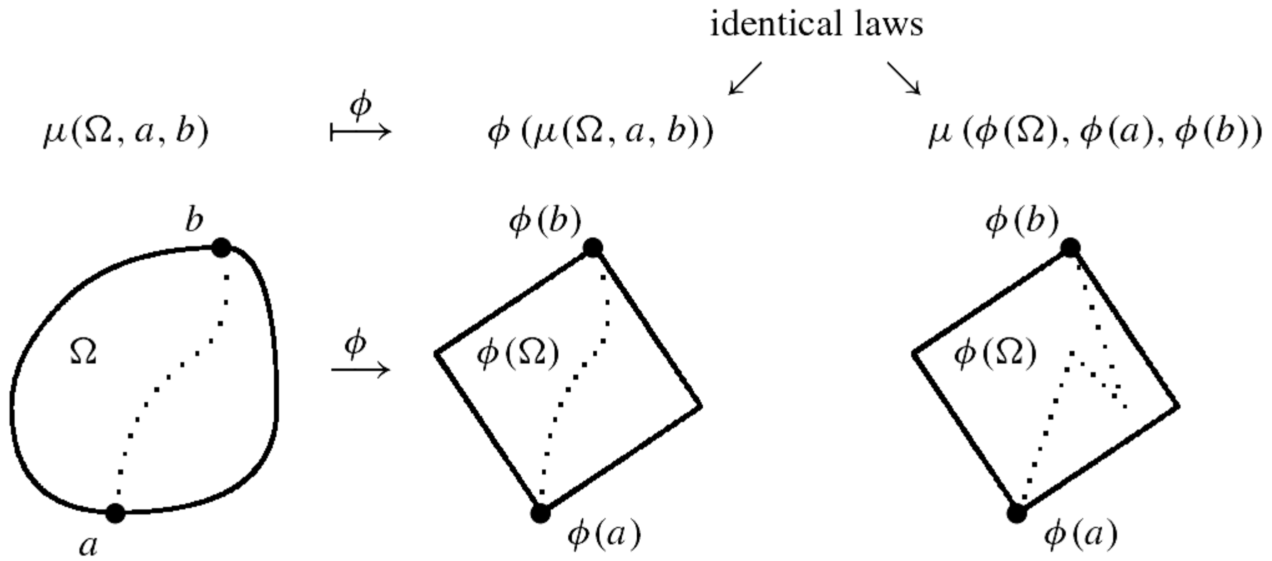
Boundary conditions: black on the boundary arc ab; white on the complementary arc ba, $a, b \in \partial\Omega^\delta$.

Question: What is the scaling limit of the interface (random curve) going from a to b as $\delta \to 0$? (conformal invariance was predicted by physicists)

Oded Schramm's principle:

(A) Conformal invariance. *For a conformal map of the domain* Ω *one has*

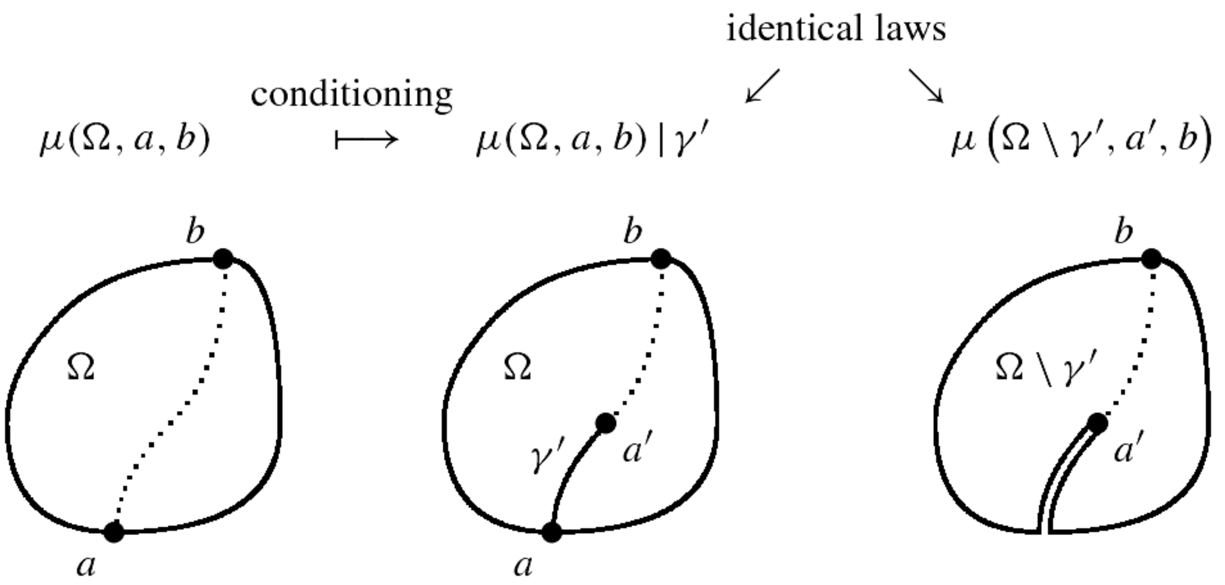
$$\phi(\mu(\Omega, a, b)) = \mu(\phi(\Omega), \phi(a), \phi(b)).$$



ODED SCHRAMM'S PRINCIPLE:

(B) Domain Markov Property. *The law conditioned on the interface already drawn is the same as the law in the slit domain:*

$$\mu(\Omega, a, b)|\gamma' = \mu(\Omega\gamma', a', b).$$



Oded Schramm's principle:

O. Schramm, Scaling limits of loop-erased random walks and uniform spanning trees, *Israel J. Math.* **118**, 221–288 (2000).

(A) Conformal invariance & (B) Domain Markov Property $\Rightarrow$ μ is $\mathrm{SLE}(\kappa)$: Schramm's Stochastic-Lowner Evolution for some real parameter $\kappa \geqslant 0$.

Remark: SLE is constructed dynamically via the Lowner equation in $\mathbb{C}_+$

Remark: Nowadays a lot is known about the SLE. For instance, the Hausdorff dimension of $\mathrm{SLE}(\kappa)$ is $\min(1+\frac{\kappa}{8}, 2)$ almost surely (V. Beffara).

Universality: The conformally invariant scaling limit should **not** depend on the structure of the underlying graph.

How to prove the convergence to SLE?

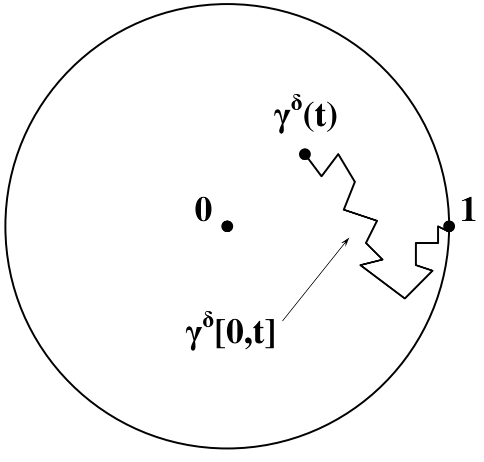
(in an appropriate weak-$*$ topology)

Martingale principle: *If a random curve γ admits a (non-trivial) conformal martingale $F_t(z) = F(z; \Omega \setminus \gamma[0,t], \gamma(t), b)$, then γ is given by* SLE *(with the parameter κ derived from F).*

Discrete example (combinatorial statement for the time-reversed LERW in $\mathbb{D}$):

the *discrete martingale* is

$P^\delta(z) :=$ Poisson kernel in $\mathbb{D}^\delta \setminus \gamma^\delta[0,t]$
(mass at the single point $\gamma^\delta(t)$)
normalized by $P^\delta(0) = 1$.



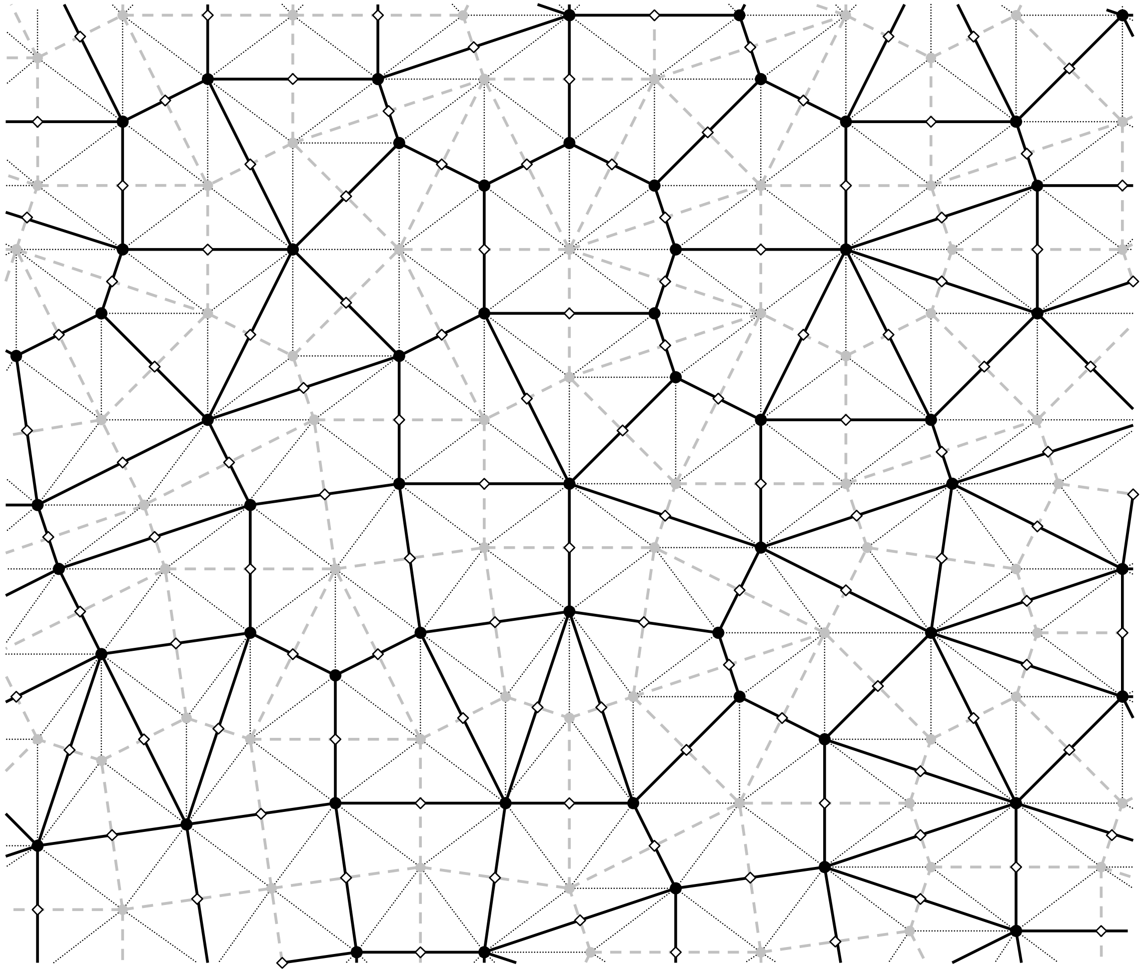
Convergence results are important:

One needs to know that the solutions of various discrete boundary value problems converge to their continuous counterparts as the mesh of the lattice goes to 0.

Remark: (i) Without any regularity assumptions about the boundary;

(ii) Universally on different lattices (planar graphs).

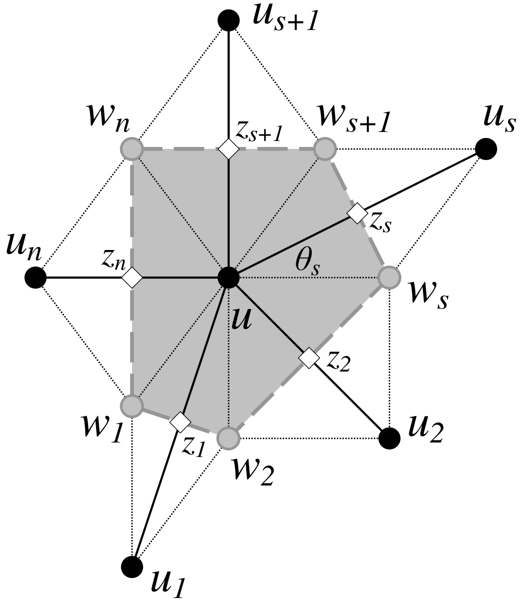
Isoradial graphs



An *isoradial graph* Γ (black vertices, solid lines), its dual isoradial graph Γ^* (gray vertices, dashed lines), the corresponding *rhombic lattice* (or *quad-graph*) (vertices $\Lambda = \Gamma \cup \Gamma^*$, thin lines) and the set $\diamondsuit = \Lambda^*$ (white diamond-shaped vertices).

The rhombi angles are uniformly bounded from 0 and π (i.e., belong to $[\eta, \pi - \eta]$ for some $\eta > 0$).

Discrete Laplacian:



Let Ω^δ_Γ be some connected discrete domain and $H : \Omega^\delta_\Gamma \to \mathbb{R}$. The **discrete Laplacian** of H at $u \in \operatorname{Int} \Omega^\delta_\Gamma$ is

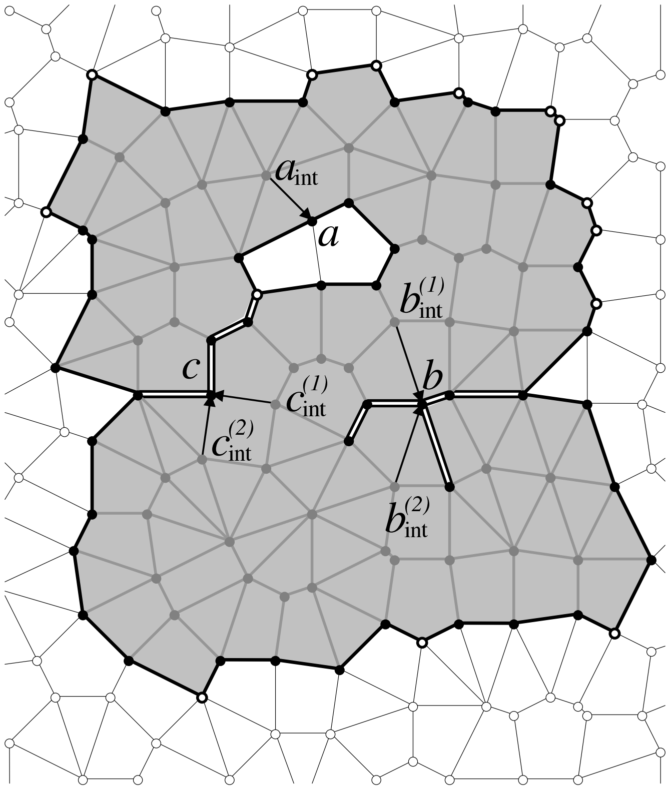
$$[\Delta^\delta H](u) :=$$

$$\frac{1}{\mu^\delta_\Gamma(u)} \sum_{u_s \sim u} \tan \theta_s \cdot [H(u_s) - H(u)],$$

where $\mu^\delta_\Gamma(u) = \frac{\delta^2}{2} \sum_{u_s \sim u} \sin 2\theta_s$.

H is **discrete harmonic** in Ω^δ_Γ iff $[\Delta^\delta H](u) = 0$ at all $u \in \operatorname{Int} \Omega^\delta_\Gamma$.

Discrete domain:



The interior vertices are gray, the boundary vertices are black and the outer vertices are white. $b^{(1)} = (b; b^{(1)}_{\text{int}})$ and $b^{(2)} = (b; b^{(2)}_{\text{int}})$ are different elements of $\partial\Omega^\delta_\Gamma$.

Maximum principle: For harmonic H,

$$\max_{u\in\Omega^\delta_\Gamma} H(u) = \max_{a\in\partial\Omega^\delta_\Gamma} H(a).$$

Discrete Green formula:

$$\sum_{u\in\mathrm{Int}\,\Omega^\delta_\Gamma} [G\Delta^\delta H - H\Delta^\delta G](u)\mu^\delta_\Gamma(u) =$$

$$\sum_{a\in\partial\Omega^\delta_\Gamma} \tan\theta_{aa_{\text{int}}} \cdot [H(a)G(a_{\text{int}}) - H(a_{\text{int}})G(a)]$$

TWO FEATURES OF THE LAPLACIAN ON ISORADIAL GRAPHS:

- **Approximation property:** Let $\phi^\delta = \phi\big|_\Gamma$. Then

  (i) $\Delta^\delta\phi^\delta \equiv \Delta\phi \equiv 2(a+c)$, if $\phi(x+iy) \equiv ax^2+bxy+cy^2+dx+ey+f$.

  (ii) $\big|\,[\Delta^\delta\phi^\delta](u) - [\Delta\phi](u)\,\big| \leqslant \mathrm{const}\cdot\delta\cdot\max_{W(u)}|D^3\phi|$.

- **Asymptotics of the (free) Green function** $H = G(\cdot\,;u_0)$:

  (i) $[\Delta^\delta H](u) = 0$ for all $u \neq u_0$ and $\mu_\Gamma^\delta(u_0)\cdot[\Delta^\delta H](u_0) = 1$;
  (ii) $H(u) = o(|u-u_0|)$ as $|u-u_0| \to \infty$;
  (iii) $H(u_0) = \frac{1}{2\pi}(\log\delta - \gamma_{\mathrm{Euler}} - \log 2)$, where γ_{Euler} is the Euler constant.

(Improved) Kenyon's theorem (see also Bobenko, Mercat, Suris):
There exists unique Green's function

$$G_\Gamma(u;u_0) = \frac{1}{2\pi}\log|u-u_0| + O\left(\frac{\delta^2}{|u-u_0|^2}\right).$$

Discrete harmonic measure:

For each $f : \partial\Omega^\delta_\Gamma \to \mathbb{R}$ there exists unique discrete harmonic in Ω^δ_Γ function H such that $H|_{\partial\Omega^\delta_\Gamma} = f$ (e.g., H minimizes the corresponding Dirichlet energy). Clearly, H depends on f linearly, so

$$H(u) = \sum_{a\in\partial\Omega^\delta_\Gamma} \omega^\delta(u;\{a\};\Omega^\delta_\Gamma)\cdot f(a)$$

for all $u \in \Omega^\delta_\Gamma$, where $\omega^\delta(u;\cdot;\Omega^\delta_\Gamma)$ is some probabilistic measure on $\partial\Omega^\delta_\Gamma$ which is called **harmonic measure at u**.

It is harmonic as a function of u and has the standard interpretation as the exit probability for the underlying random walk on Γ (i.e. the measure of a set $A \subset \partial\Omega^\delta_\Gamma$ is the probability that the random walk started from u exits Ω^δ_Γ through A).

D. Chelkak, S. Smirnov: Discrete complex analysis on isoradial graphs. arXiv:0810.2188

We prove *uniform* (with respect to the shape Ω^δ_Γ and the structure of the underlying isoradial graph) *convergence* of the basic objects of the discrete potential theory to their continuous counterparts. Namely, we consider

(i) harmonic measure $\omega^\delta(\cdot; a^\delta b^\delta; \Omega^\delta_\Gamma)$ of arcs $a^\delta b^\delta \subset \partial\Omega^\delta_\Gamma$;

(ii) Green function $G^\delta_{\Omega^\delta_\Gamma}(\cdot; v^\delta)$, $v^\delta \in \mathrm{Int}\,\Omega^\delta_\Gamma$;

(iii) Poisson kernel $P^\delta(\cdot; v^\delta; a^\delta; \Omega^\delta_\Gamma) = \dfrac{\omega^\delta(\,\cdot\,; \{a^\delta\}; \Omega^\delta_\Gamma)}{\omega^\delta(v^\delta; \{a^\delta\}; \Omega^\delta_\Gamma)}$, $a^\delta \in \partial\Omega^\delta_\Gamma$, $v^\delta \in \mathrm{Int}\,\Omega^\delta_\Gamma$;

(iv) Poisson kernel $P^\delta_{o^\delta}(\cdot; a^\delta; \Omega^\delta_\Gamma)$, $a^\delta \in \partial\Omega^\delta_\Gamma$, normalized at the boundary by the discrete analogue of the condition $\frac{\partial}{\partial n}P|_{o^\delta} = -1$.

Remark: We also prove uniform convergence for the *discrete gradients* of these functions (which are discrete holomorphic functions defined on subsets of $\diamondsuit = \Lambda^*$).

Setup for the convergence theorems:

Let $\Omega = (\Omega; v, ..; a, b, ..)$ be a simply connected bounded domain with several marked interior points $v, .. \in \operatorname{Int}\Omega$ and boundary points (prime ends) $a, b, .. \in \partial\Omega$.

Let for each $\Omega = (\Omega; v, ..; a, b, ..)$ some harmonic function

$$h(\,\cdot\,;\Omega) = h(\,\cdot\,, v, ..; a, b, ..; \Omega) : \Omega \to \mathbb{R}$$

be defined.

Let $\Omega^\delta_\Gamma = (\Omega^\delta_\Gamma; v^\delta, ..; a^\delta, b^\delta, ..)$ denote simply connected bounded discrete domain with several marked vertices $v^\delta, .. \in \operatorname{Int}\Omega^\delta_\Gamma$ and $a^\delta, b^\delta, .. \in \partial\Omega^\delta_\Gamma$ and

$$H^\delta(\,\cdot\,;\Omega^\delta_\Gamma) = H^\delta(\,\cdot\,, v^\delta, ..; a^\delta, b^\delta, ..; \Omega^\delta_\Gamma) : \Omega^\delta_\Gamma \to \mathbb{R}$$

be some discrete harmonic in Ω^δ_Γ function.

Definition: Let Ω be a simply connected bounded domain, $u, v, .. \in \Omega$. We say that $u, v, ..$ are **jointly r-inside Ω** iff $B(u,r), B(v,r), .. \subset \Omega$ and there are paths $L_{uv}, ..$ connecting these points r-inside Ω (i.e., $\mathrm{dist}(L_{uv}, \partial\Omega), .. \geqslant r$). In other words, $u, v, ..$ belong to the same connected component of the r-interior of Ω.

Definition: We say that **$\mathbf{H}^\delta$ are uniformly $\mathbf{C}^1$–close to $\mathbf{h}$ inside $\mathbf{\Omega}^\delta$**, iff for all $0 < r < R$ there exists $\varepsilon(\delta) = \varepsilon(\delta, r, R) \to 0$ as $\delta \to 0$ such that If $\Omega^\delta \subset B(0,R)$ and $u^\delta, v^\delta, ..$ are jointly r-inside Ω^δ, then

$$\left| H^\delta(u^\delta, v^\delta, ..; a^\delta, b^\delta, ..; \Omega^\delta_\Gamma) - h(u^\delta, v^\delta, ..; a^\delta, b^\delta, ..; \Omega^\delta) \right| \leqslant \varepsilon(\delta)$$

and, for all $u^\delta \sim u_1^\delta \in \Omega^\delta_\Gamma$,

$$\left| \frac{H^\delta(u_1^\delta; \Omega^\delta_\Gamma) - H^\delta(u^\delta; \Omega^\delta_\Gamma)}{|u_1^\delta - u^\delta|} - \mathrm{Re}\left[2\partial h(u^\delta; \Omega^\delta) \cdot \frac{u_1^\delta - u^\delta}{|u_1^\delta - u^\delta|} \right] \right| \leqslant \varepsilon(\delta),$$

where $2\partial h = h'_x - ih'_y$. Here Ω^δ denotes the corresponding *polygonal* domain.

Key Ideas. Compactness argument – I:

Proposition: Let $H^{\delta_j} : \Omega_\Gamma^{\delta_j} \to \mathbb{R}$ be discrete harmonic in $\Omega_\Gamma^{\delta_j}$ with $\delta_j \to 0$.
Let $\Omega \subset \bigcup_{n=1}^{+\infty} \bigcap_{j=n}^{+\infty} \Omega^{\delta_j} \subset \mathbb{C}$ be some continuous domain.
If H^{δ_j} are uniformly bounded on Ω, then there exists a subsequence $\delta_{j_k} \to 0$ (which we denote δ_k for short) and two functions $h : \Omega \to \mathbb{R}$, $f : \Omega \to \mathbb{C}$ such that

$$H^{\delta_k} \rightrightarrows h \quad \text{uniformly on compact subsets } K \subset \Omega$$

and

$$\frac{H^{\delta_k}(u_2^k) - H^{\delta_k}(u_1^k)}{|u_2^k - u_1^k|} \rightrightarrows \mathrm{Re}\Big[f(u) \cdot \frac{u_2^k - u_1^k}{|u_2^k - u_1^k|} \Big],$$

if $u_1^k, u_2^k \in \Gamma^{\delta_k}$, $u_2^k \sim u_1^k$ and $u_1^k, u_2^k \to u \in K \subset \Omega$.
The limit function h is harmonic in Ω and $f = h'_x - ih'_y = 2\partial h$ is analytic in Ω.

Remark: Looking at the edge $(u_1 u_2)$ one (immediately) sees only the discrete derivative of H^δ along $\tau = (u_2 - u_1)/|u_2 - u_1|$ which converge to $\langle \nabla h(u), \tau \rangle$.

Key Ideas. Compactness argument – II:

The set of all simply-connected domains $\Omega : B(u,r) \subset \Omega \subset B(0,R)$ is **compact** in the Carathéodory topology (see the next slide).

Proposition: Let (a) h be Carathéodory-stable, i.e.,

$$h(u_k;\Omega_k) \to h(u;\Omega), \quad \text{if} \quad (\Omega_k;u_k) \xrightarrow{\text{Cara}} (\Omega;u) \quad \text{as} \quad k \to \infty;$$

and (b) $H^\delta \to h$ pointwise as $\delta \to 0$, i.e.,

$$H^\delta(u^\delta;\Omega^\delta_\Gamma) \to h(u;\Omega), \quad \text{if} \quad (\Omega^\delta;u^\delta) \xrightarrow{\text{Cara}} (\Omega;u) \quad \text{as} \quad \delta \to 0.$$

Then H^δ are uniformly C^1–close to h inside Ω^δ.

The Carathéodory convergence is the uniform convergence of the Riemann uniformization maps ϕ^δ on the compact subsets of $\mathbb{D}$.

It is equivalent to say that
(i) some neighborhood of each $u \in \Omega$ lies in Ω^δ, if δ is small enough;
(ii) for every $a \in \partial\Omega$ there exist $a^\delta \in \partial\Omega^\delta$ such that $a^\delta \to a$ as $\delta \to 0$.

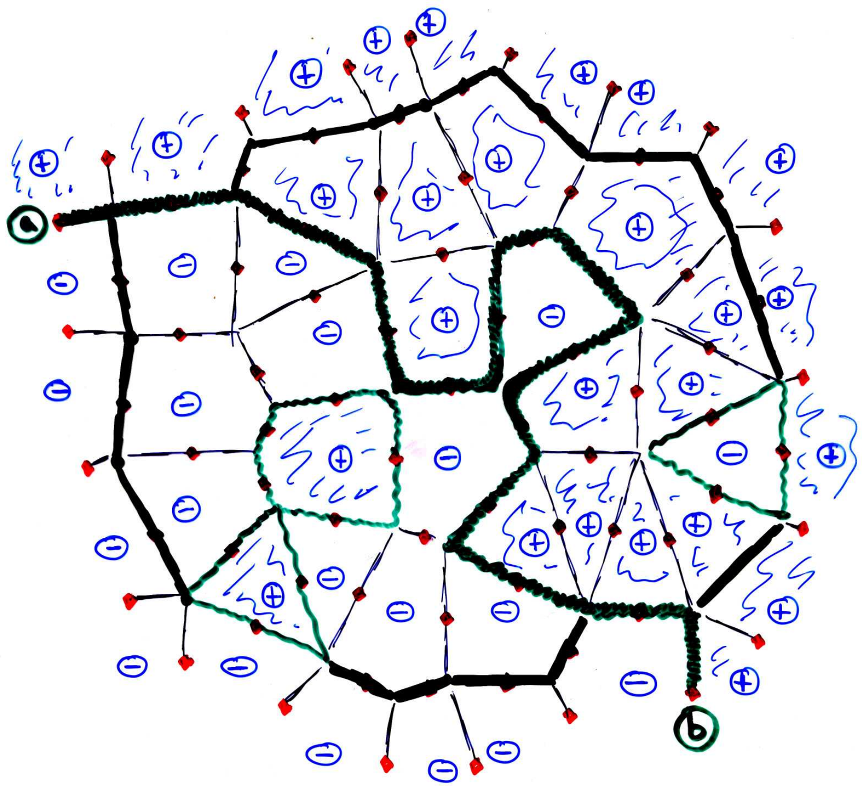
Scheme of the proofs:

- It is sufficient to prove the pointwise convergence $H^\delta(u^\delta) \to h(u)$ (compactness argument – II).

- Prove the uniform boundedness of the discrete functions. Then there is a subsequence that converges (uniformly on compact subsets) to some harmonic function H (compactness argument – I).

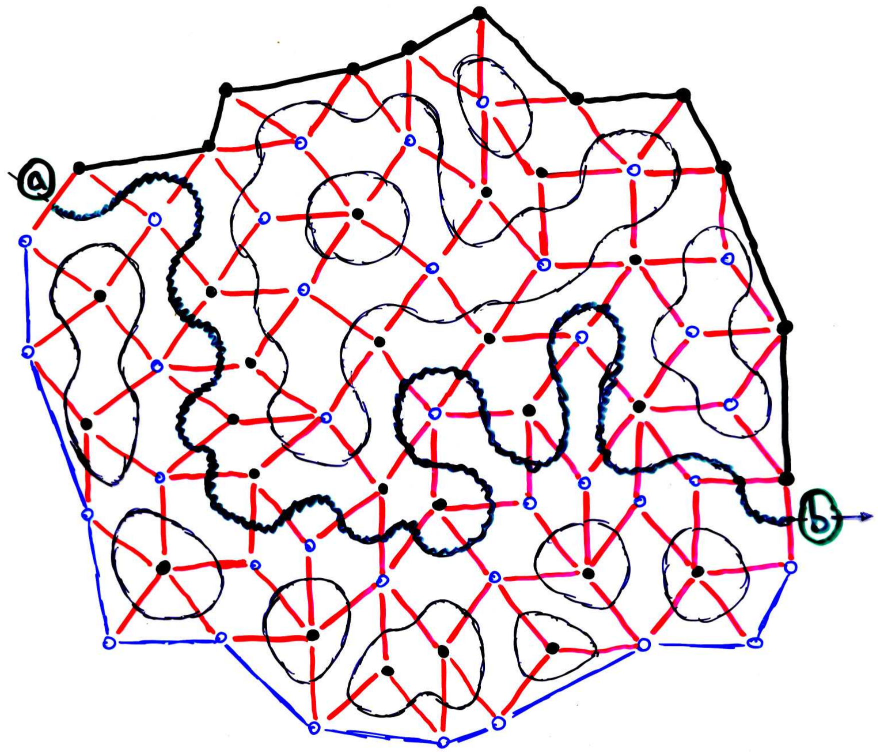
- Identify the boundary values of H with those of h. Then $H = h$ for each subsequential limit, and so for the whole sequence.

Critical spin-Ising model



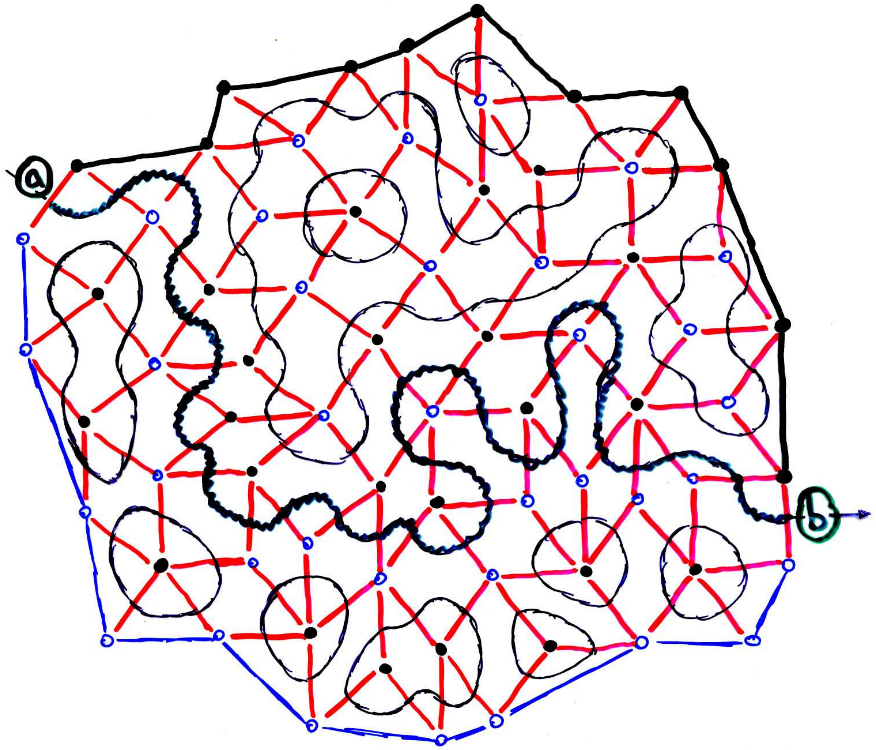
$$Z = \sum_{\text{config.}} \prod_{z:\oplus \leftrightarrow \ominus} \tan \frac{\theta(z)}{2}$$

Critical FK-Ising model



$$Z = \sum_{\text{config.}} \sqrt{2}^{\,\#\text{loops}} \prod_{z} \sin \frac{\theta(z)}{2}$$

Critical FK-Ising model

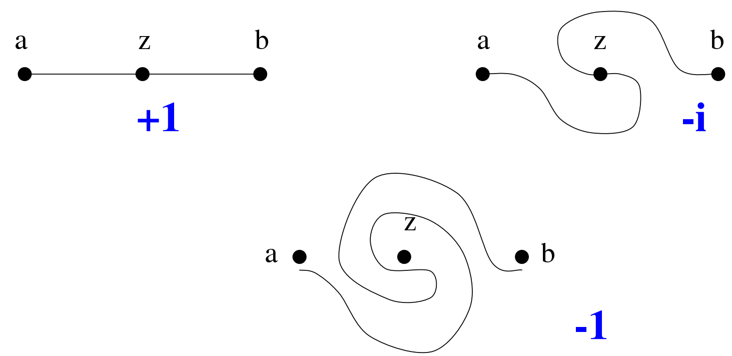


$$Z = \sum_{\text{config.}} \sqrt{2}^{\#\text{loops}} \prod_z \sin\frac{\theta(z)}{2}$$

The discrete holomorphic observable having the martingale property:

$$\mathbb{E}\, \chi[z \in \gamma] \cdot \exp[-\tfrac{i}{2} \cdot \text{wind}(\gamma, b \to z)],$$

where $z \in \diamondsuit$.



More information (from physicists):

V. Riva and J. Cardy. Holomorphic parafermions in the Potts model and stochastic Loewner evolution. *J. Stat. Mech. Theory Exp.*, (12): P12001, 19 pp. (electronic), 2006.

Convergence of the observable:

<ins>Square lattice case:</ins> S. Smirnov. Conformal invariance in random cluster models. I. Holomorphic fermions in the Ising model. `arXiv:0708.0039` *Annals Math.*, to appear.

Remark: The convergence of the observable provides the conformal invariance of interfaces for the scaling limit of the critical Ising model on the square lattice via the Martingale Principle.

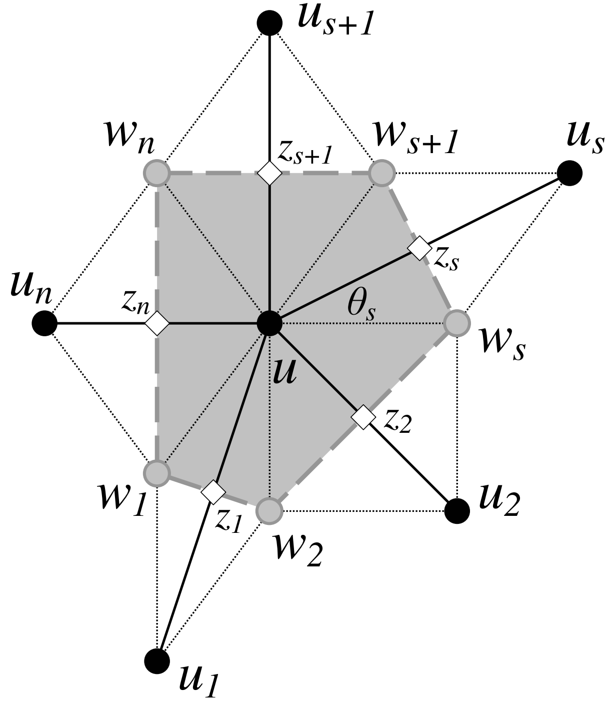
<ins>General isoradial graphs:</ins> D. Chelkak, S. Smirnov. In preparation.

Remark: The convergence of the observable provides the proof of the universality for the scaling limit of the critical Ising model on isoradial graphs.

<u>Special discrete holomorphic functions:</u> Discrete analyticity follows from the local rearrangements. Moreover, the stronger property holds:

For any two neighboring rhombi z_s, z_{s+1}
$F(z_s) - F(z_{s+1})$ is proportional to $\pm[i(w_{s+1}-u)]^{-\frac{1}{2}}$.
This is equivalent to

$$F(z_s)-F(z_{s+1}) = -i\delta^{-1}(\overline{w_{s+1}}-\overline{u})(\overline{F(z_s)}-\overline{F(z_{s+1})})$$



Remark: The standard definition of discrete holomorphic functions on $\diamondsuit$ is

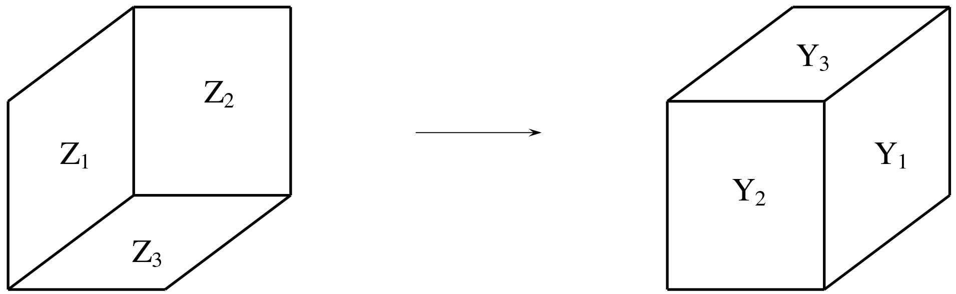
$$\sum_{s=1}^{n} F(z_s) \cdot (w_{s+1} - w_s) = 0.$$

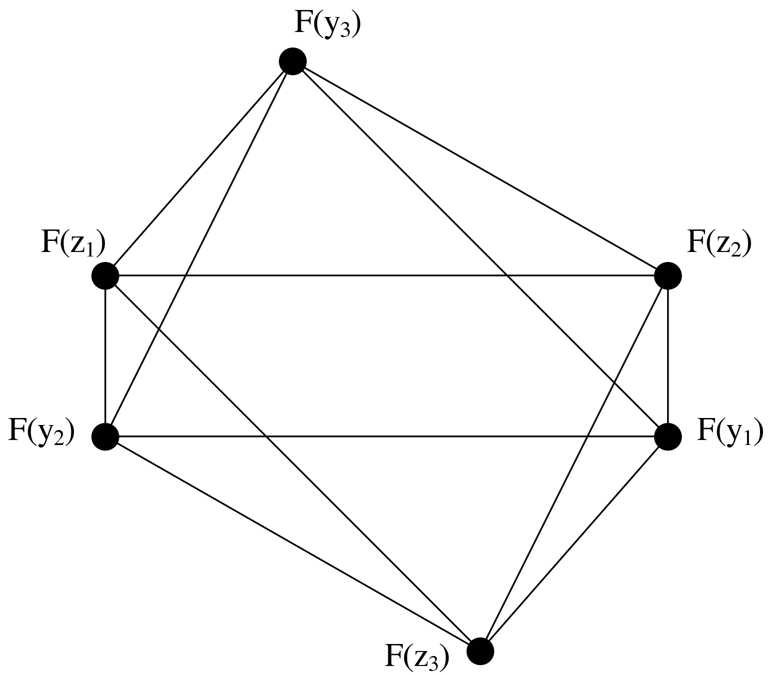
(more details concerning discrete holomorphic functions/forms in the Christian Mercat talk)

Some speculations: (result of discussions at Obergurgl)

special holomorphic functions and the 4D-consistency

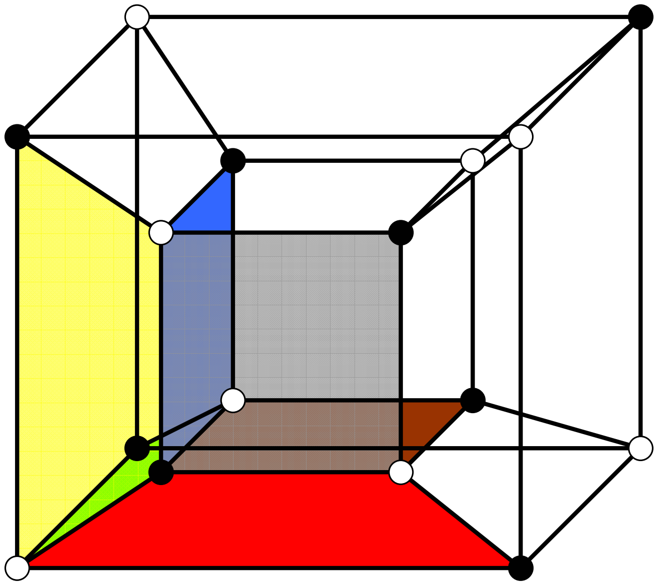
Star-triangle transform (flip)

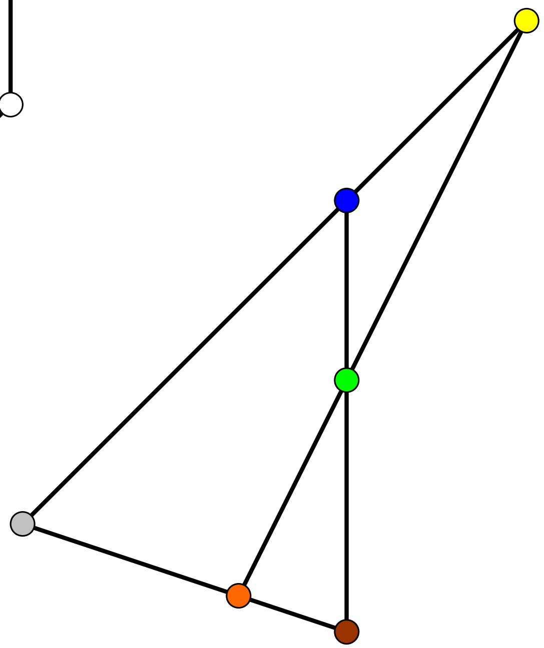




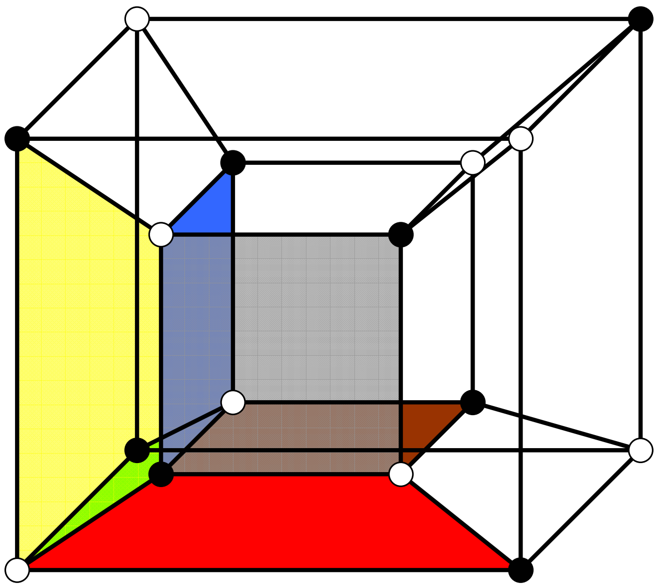
does not change the values of F outside this "cube". The values $F(z_1)$, $F(z_2)$, $F(z_3)$ and $F(y_1)$, $F(y_2)$, $F(y_3)$ are related in an elementary (real-linear) way:

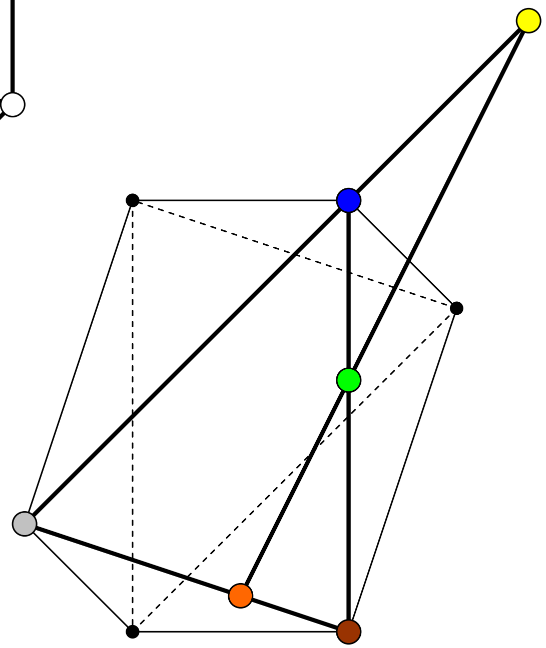
Special holomorphic functions and the 4D-consistency



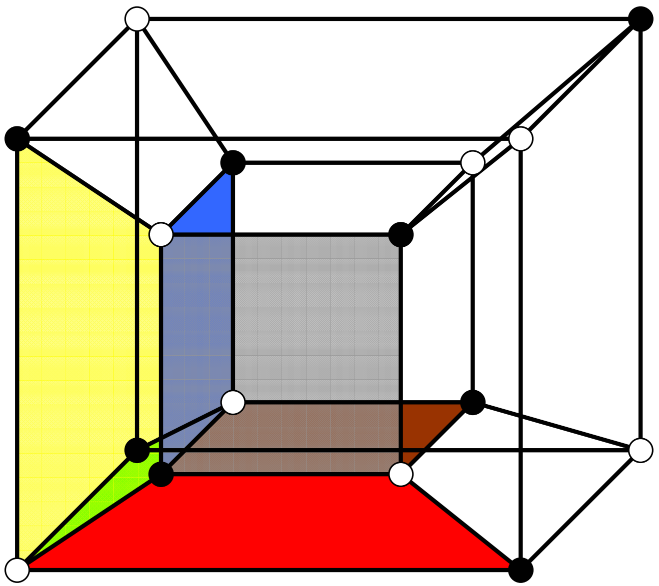


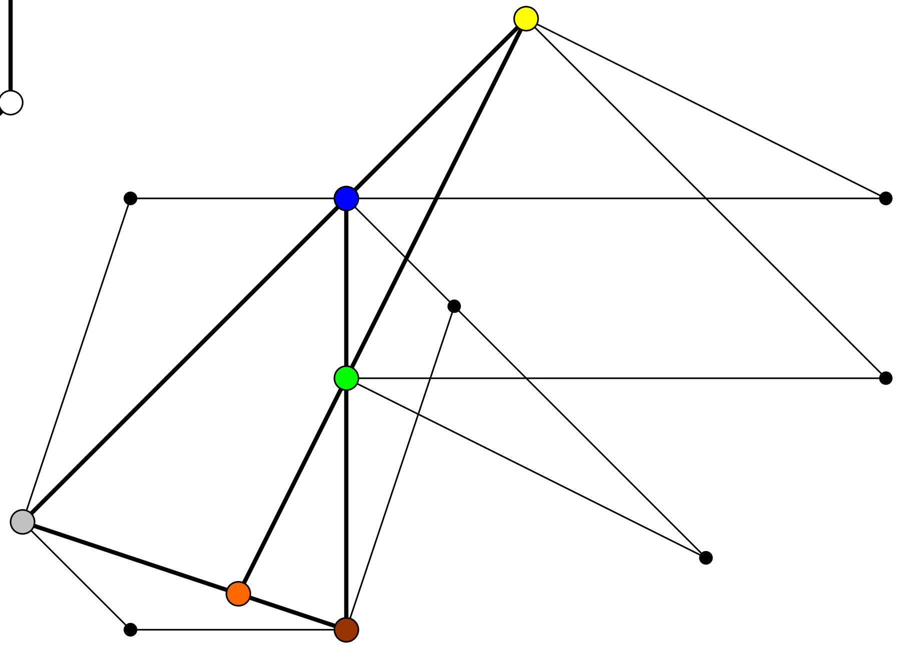
Special holomorphic functions and the 4D-consistency



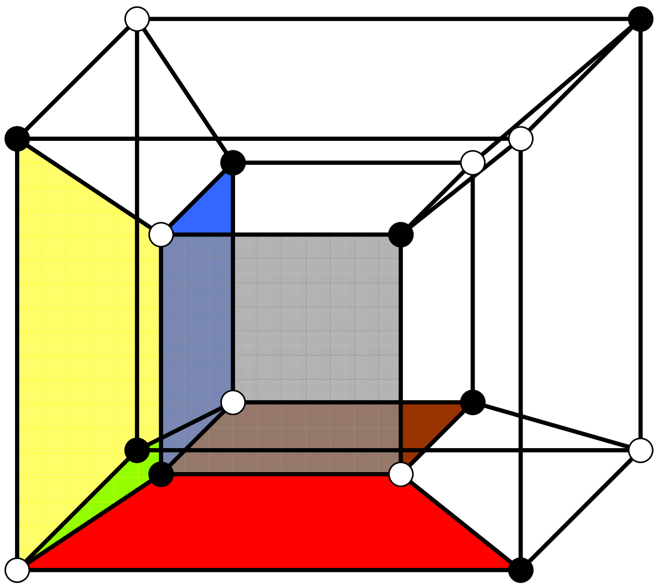


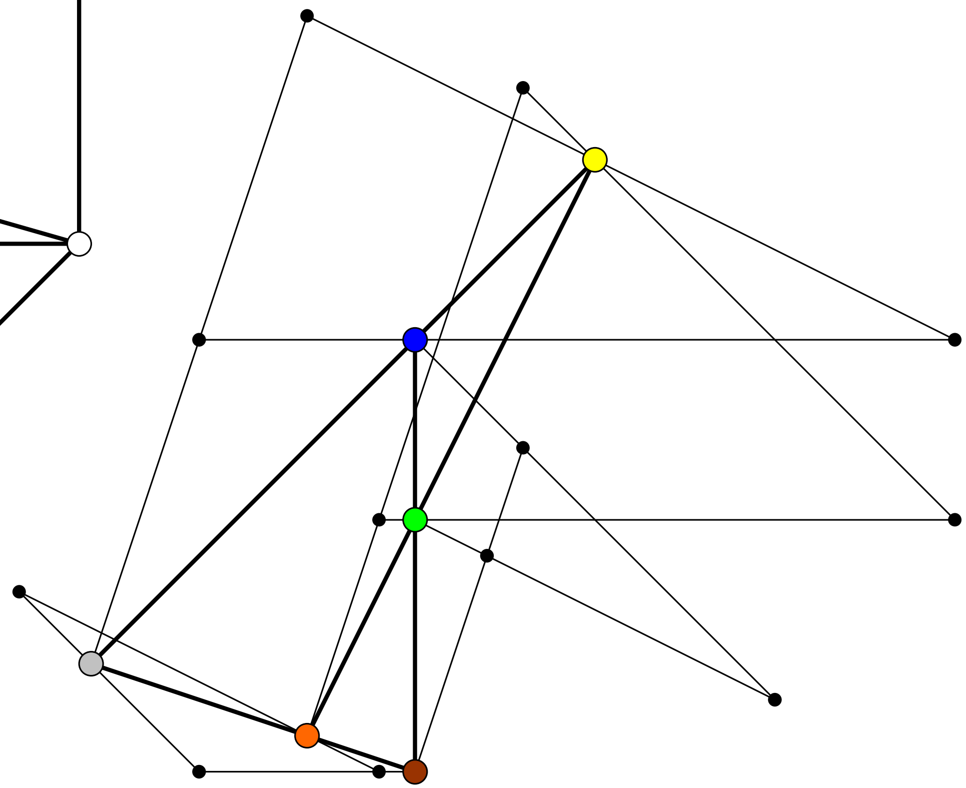
Special holomorphic functions and the 4D-consistency



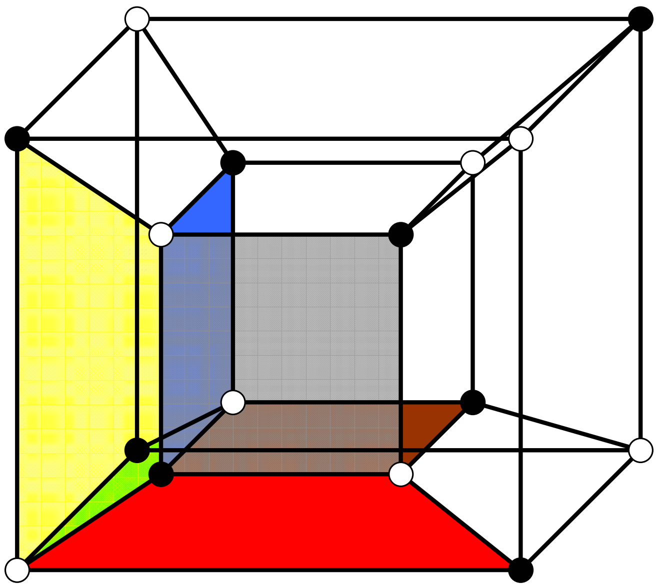


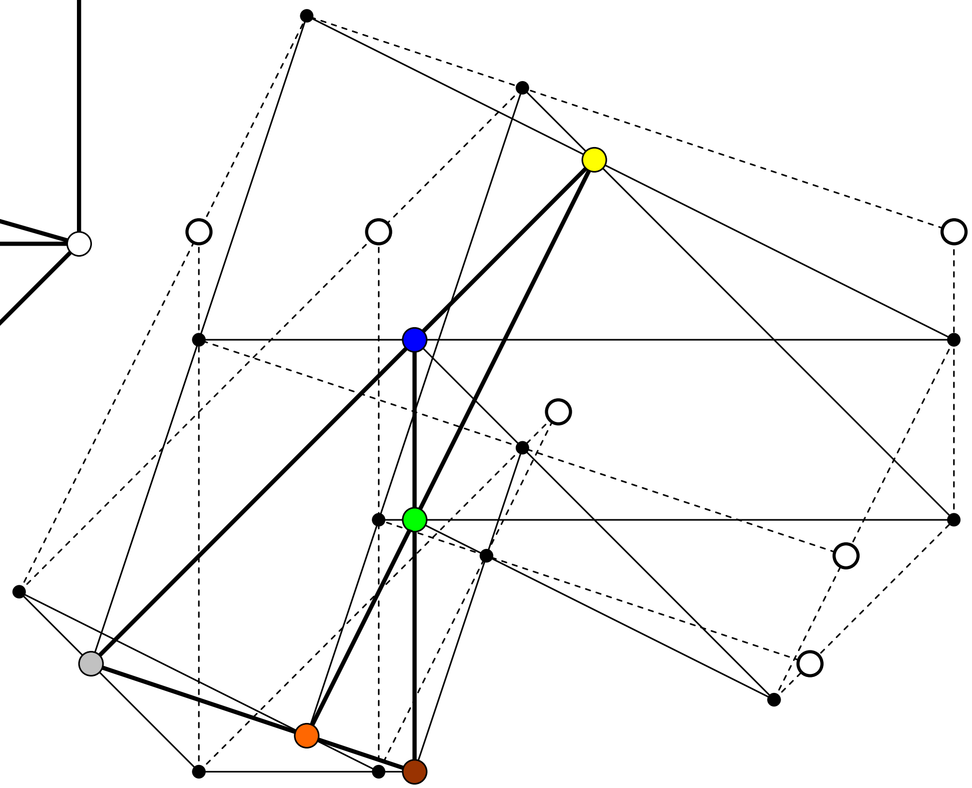
Special holomorphic functions and the 4D-consistency





Special holomorphic functions and the 4D-consistency





Special holomorphic functions and the 4D-consistency