Random Geometry and Statistical Physics Seminar

s-/t- embeddings and Ising/bipartite dimer model on planar graphs

November 12, 2020

ROADMAP: "abstract" graphs → embedding → (massive) holomorphicity

- Weighted planar graphs (maps) carrying [Ising / bipartite dimers]
  → fermionic observables satisfy a simple local ($\mathbb{R}$-linear) relation ⊛

- Choice of a "distinguished" ($\mathbb{C}$-valued) solution(s) of ⊛ → $\mathcal{S}$-/$\mathcal{T}$-embedding of G (or G^*) into $\mathbb{C}$ or, better, into Minkowski $\mathbb{R}^{2+1}$ or $\mathbb{R}^{2+2}$

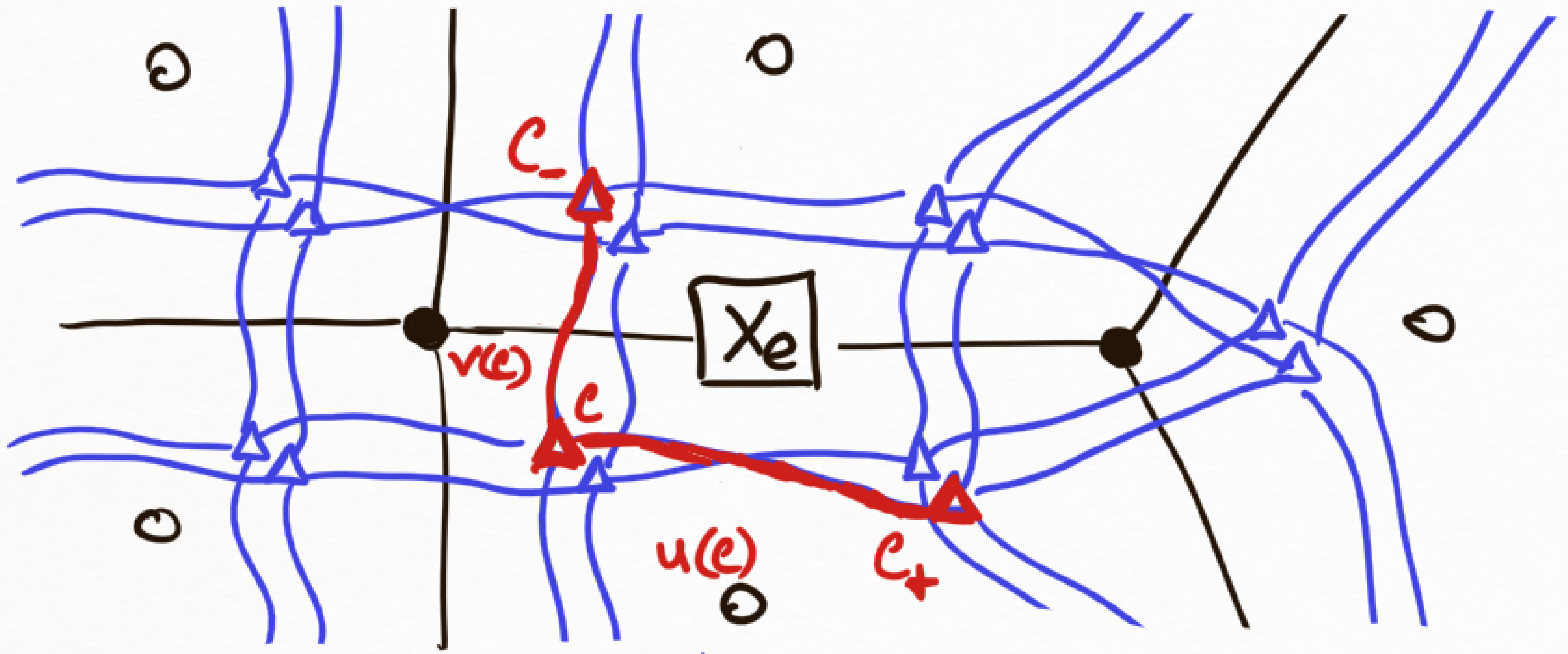
- Re-interpretation of ⊛:
  in discrete: $F d\mathcal{T} + \bar{F} d\mathcal{O}$ - closed
  ↓ "small mesh size limit"
  in continuum: $\partial_{\bar{z}} \psi = m \bar{\psi}$

- ζ - conformal parametrization of a surface in $\mathbb{R}^{2+1(2)}$
- m - mean curvature (× metric)
- minimal surfaces ⇝ conformal invariance

Ising vs BIPARTITE DIMERS : s-embeddings <—> t-embeddings

Fermionic observables (Ising)

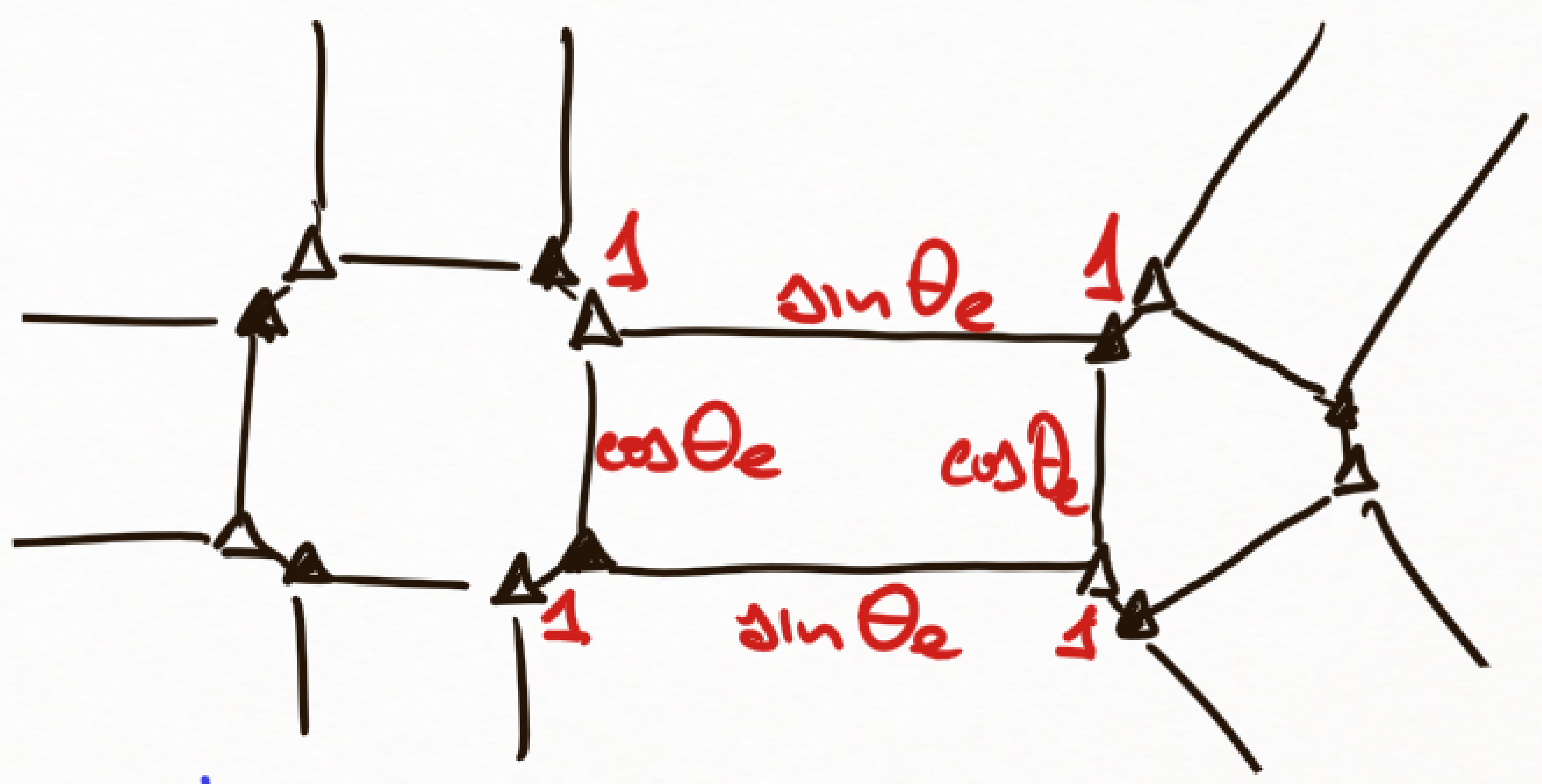
(parametrization of interactions $x_e = \tan \frac{1}{2}\theta_e$)



$$X(c) := \langle \sigma_{u(e)} \mu_{v(e)} \dots \rangle$$

[Kadanoff - Ceva spin-disorder does not require an embedding]

Inverse Kasteleyn matrix (dimers)



"Propagation equation" :

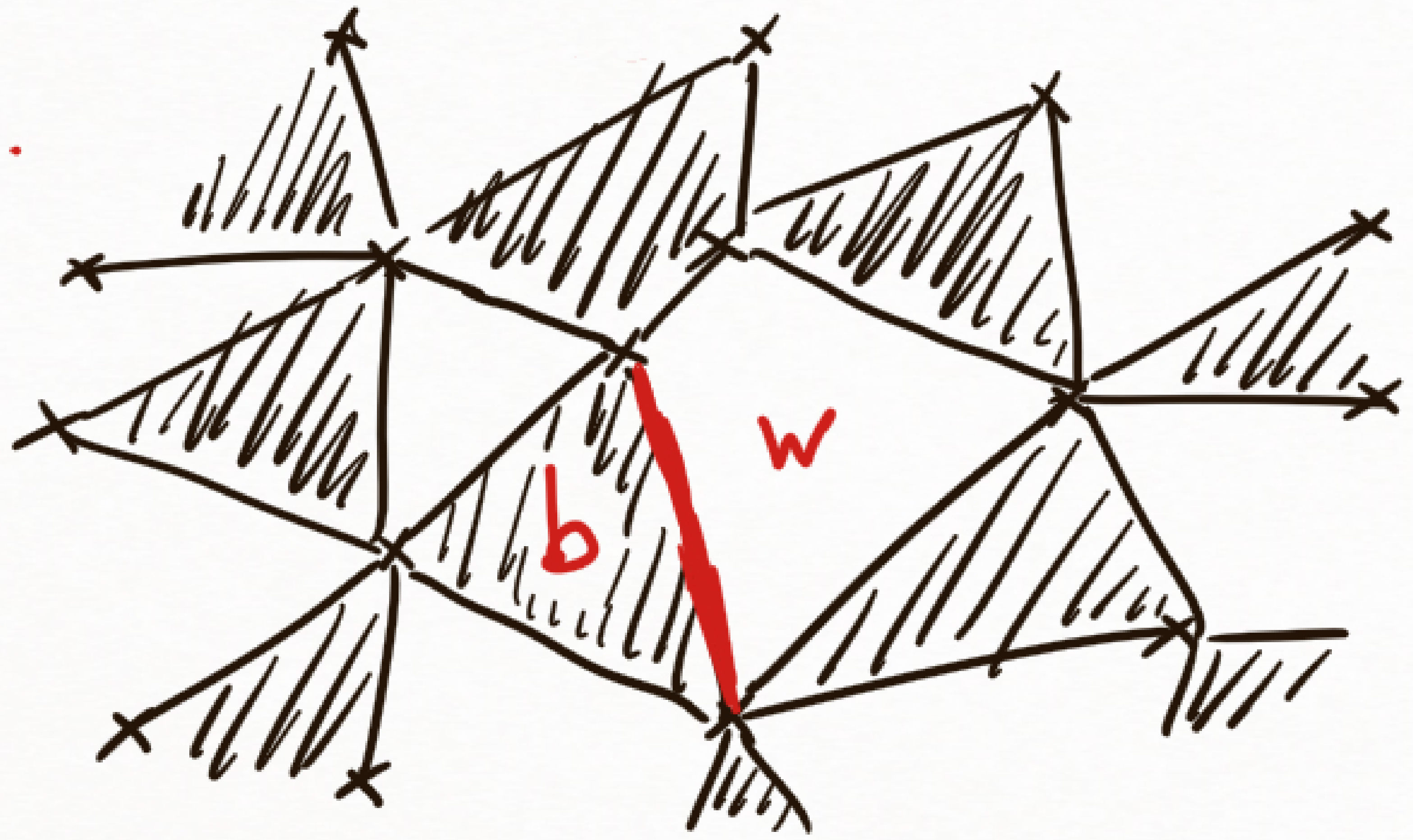
$$X(c) = X(c_-)\cos\theta_e + X(c_+)\sin\theta_e$$

$$\Longleftrightarrow \quad KX = 0 \quad (\text{locally})$$

[choice of a section of the double cover <=> choice of ± Kasteleyn signs]

DIMERS: t-embeddings [CLR] <=> Coulomb gauges [KLRR]

$\mathcal{T}$ = embedding of G^* into $\mathbb{C}$ s.t.



① lengths are gauge equiv. to (given) dimer weights

② angles at (inner) vertices are balanced : $\sum \text{white} = \pi = \sum \text{black}$

G - weighted bipartite graph

$K(b,w)$ - real-signed Kasteleyn

Let $g^\circ \in \mathbb{C}^W : Kg^\circ = 0$
$g^\bullet \in \mathbb{C}^B : g^\bullet K = 0$ (locally)

Define $\mathcal{K}(b,w) := g^\bullet(b) K(b,w) g^\circ(w)$

(gauge equivalent weights)

and $d\mathcal{T}((bw)^*) := \mathcal{K}(b,w)$

(well-def. on G^* if $Kg^\circ = 0 = g^\bullet K$)

Rem: ② <=> Kasteleyn sign condition

ISING : s-embeddings

$\mathbb{Z}^2 \subset$ isoradial (rhombi)

$\cap$

Lis' circle patterns (kites)

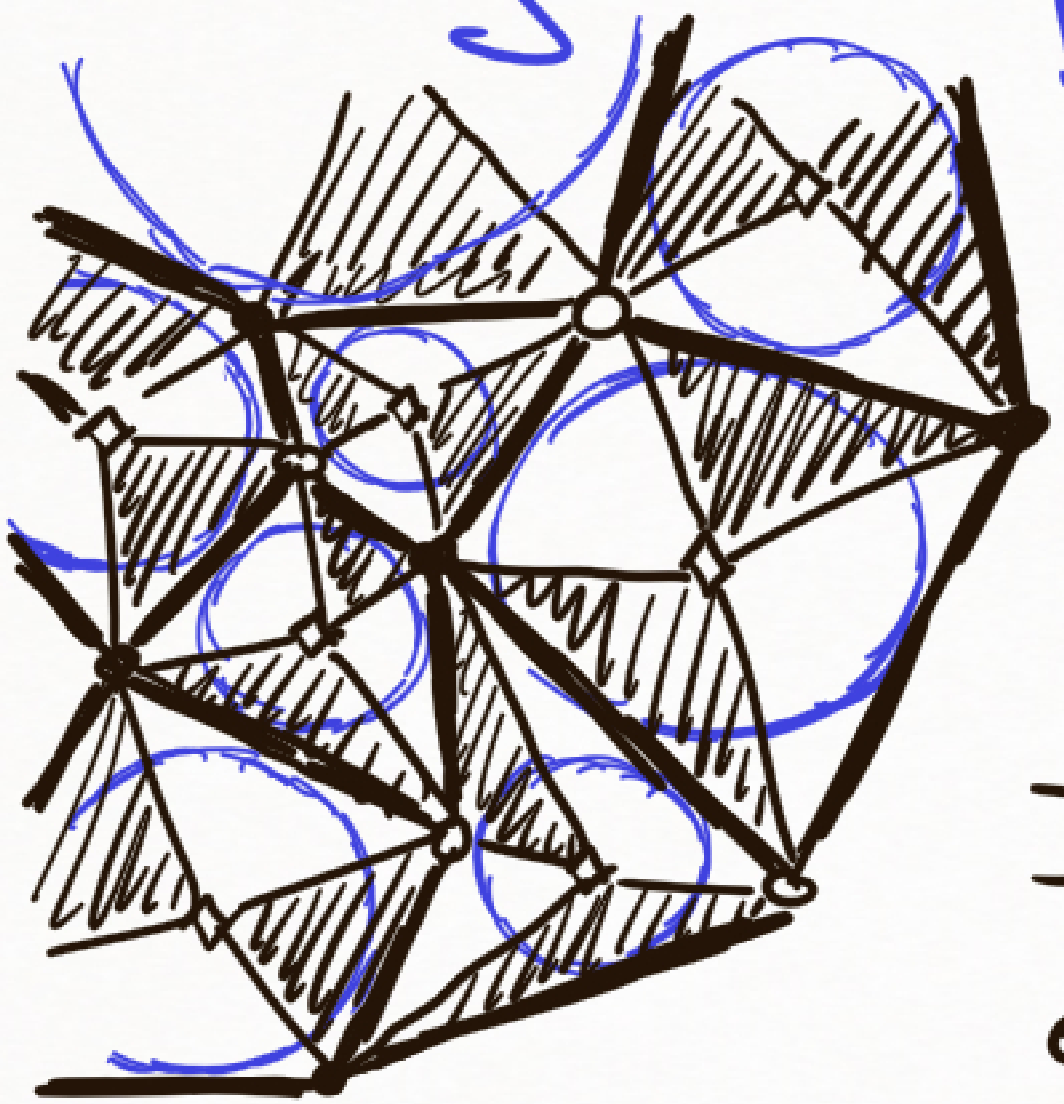
$\cap$

tangential quads [non-convex OK]

$\cap$

t-embeddings

[coherent w/ Ising $\leftrightarrow$ dimers correspondence]



ORIGAMI MAP : "definition"

"fold $\mathbb{C}$ along each of the edges"

[angle condition $\sum$ black $= \pi = \sum$ white $\Rightarrow$ local consistency]

t-embeddings : $(\mathcal{T}, \mathcal{O}) \in \mathbb{R}^{2+2}$

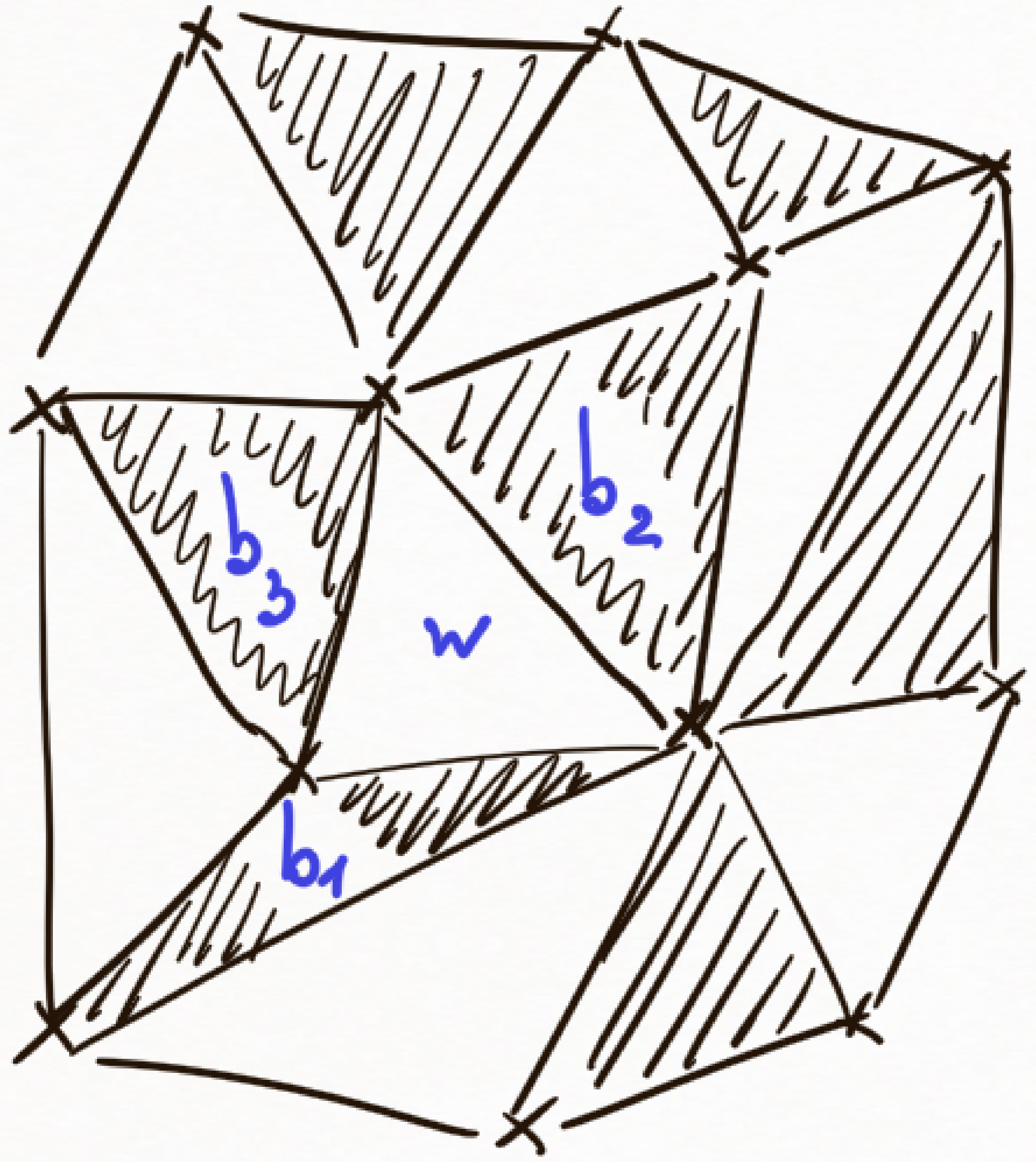
s-embeddings : $(\mathcal{S}, \mathcal{Q}) \in \mathbb{R}^{2+1}$

[isoradial $\Rightarrow$ $\mathcal{Q} = \pm \delta/2$ on $\bullet \cup \circ$]

$$|\mathcal{O}(z) - \mathcal{O}(z')| \le |\mathcal{T}(z) - \mathcal{T}(z')|$$

$\rightsquigarrow$ discrete space-like surfaces in Minkowski spaces $\mathbb{R}^{2+2(1)}$

RE-INTERPRETATION : $\mathbb{R}$-valued fermionic observables $\rightsquigarrow$ $\mathbb{C}$-valued



For simplicity, assume that T is a triangulation

Consider functions defined, say, on B s.t.

① $F^\bullet(b) \in \eta_b\mathbb{R} = \overline{g^\bullet(b)}\,\mathbb{R}$ $\quad$ [e.g., $F^\bullet_{w_0}(b) := g^\circ(w_0)\mathcal{K}^{-1}(w_0,b)$]

② $F^\bullet \mathcal{K} = 0$ (locally)

Lemma: On triangulations, ①+② $\iff$

$\exists\, F^\circ(w) \in \mathbb{C}$: $F^\bullet(b_k) = \mathrm{Pr}\,[F^\circ(w);\, \eta_{b_k}\mathbb{R}]$, $k = 1,2,3$

Rem: In the Ising context, this generalizes Smirnov's definition from isoradial graphs to s-embeddings

RE-INTERPRETATION OF FERMIONIC OBSERVABLES (continued)

$\mathbb{R}$-valued fcts on an "abstract" graph [K^{-1} or spin-disorders] $\xrightarrow[\mathcal{T} \text{ or } \mathcal{S}]{\text{choice of}}$ $\mathbb{C}$-valued (piece-wise constant) functions defined in $\mathbb{C}$ s.t.

$$\boxed{\begin{array}{l} F_w \, d\mathcal{T} + \overline{F_w} \, d\mathcal{O} \\ F_b \, d\mathcal{T} + \overline{F_b} \, d\overline{\mathcal{O}} \end{array} \text{ are closed forms}}$$

Dimers: height correlations are linear combinations of $\int \mathrm{Re}\left[F_w F_b \, d\mathcal{T} + \overline{F_w} F_b \, d\mathcal{O} \right]$

[$\int$ in each of the variables]

Ising: following Smirnov,

$H_F := \int \mathrm{Re}\left[F^2 d\mathcal{T} + |F|^2 d\mathcal{O} \right]$ is a crucial tool to work with b.v.p.'s

A PRIORI REGULARITY of t-holomorphic functions [CLR]

① **Assumption** $\mathrm{Lip}(k,\delta)$, $k<1$:

$$|\sigma^\delta(x)-\sigma^\delta(y)| \le k\cdot|\tau^\delta(x)-\tau^\delta(y)|$$
provided that $\underbrace{|\tau^\delta(x)-\tau^\delta(y)|}_{\geq\delta}$

$\Rightarrow$ t-hol. fcts on τ^δ are Hölder above scale δ

② **Assumption** $\mathrm{Exp\text{-}Fat}(\delta)$, $\delta\to0$:

[triangulations] $\forall\beta>0$, if we remove all $\exp(-\beta\delta^{-1})$-fat triangles from τ^δ, then the size of remaining vertex-connected components $\xrightarrow[\delta\to0]{} 0$

$\Rightarrow$ Harnack-type control of t-hol. fcts via primitives (or via H_F for Ising

Under ①+②, if primitives of F^δ's are bounded on compacts, then $\exists$ subseq. limits f_w, f_b

PROVIDED $\{(\tau,\sigma)\}\to\{(z,\vartheta(z))\}$

$f_w dz+\overline{f_w}d\vartheta$ & $f_b dz+\overline{f_b}d\overline{\vartheta}$

are closed diff. forms

RE-WRITING "$f\,dz + \bar f\,d\theta$ - closed": massive holomorphicity

① **Ising** ($\theta \in \mathbb{R}$)

$f\,dz + \bar f\,d\theta$ - closed $\iff$ $\boxed{\partial_{\bar\varsigma}\psi = m\,\overline{\psi}}$

m = mean curvature ($\times$ metric element)

ς - conformal parametrization of the surface $\{(z,\theta(z))\} \subset \mathbb{R}^{2+1}$

$\psi := (z_\varsigma)^{1/2} f + (\bar z_\varsigma)^{1/2} \bar f$

② **Dimers** ($\theta \in \mathbb{C}$):

similar w/ $\begin{array}{l} \partial_{\bar\varsigma}\psi_w = m\,\overline{\psi_w} \\ \partial_{\bar\varsigma}\psi_b = \bar m\,\overline{\psi_b} \end{array}$ $\rightsquigarrow$ height correlations ("bosonization") of the form $\int \mathrm{Re}\,[\psi_w \psi_b\, d\varsigma]$

$$\boxed{\text{Minimal surfaces } (m=0) \iff \text{holomorphic jets in } \varsigma}$$

RESULTS (available as for now)

Ising: lengths $\asymp\delta$, angles $\asymp 1$, $Q^\delta = O(\delta)$ $\Rightarrow$ RSW, convergence of basic F^δ's, conv. to $SLE(^{16}/_3)$

[this is very unsatisfactory though already includes, e.g., all periodic models]

UNDER CONSTRUCTION:

- relaxing assumptions
- correlations
- massive SLEs

Dimers:

EXISTENCE REMAINS AN OPEN QUESTION

Thm: (CLR) $\mathcal{T}^\delta$ – 'perfect' ⊕ EXP-FAT(δ)

$\{(\mathcal{T}^\delta, \mathcal{O}^\delta)\} \xrightarrow[\delta\to 0]{} \mathcal{S} \subset \mathbb{R}^{2+1} \subset \mathbb{R}^{2+2}$

$\mathcal{S}$-minimal $\Rightarrow$ height fluctuations conv. to the GFF (conf. param. of $\mathcal{S}$)

Rem: (w/ Ramassamy) This setup is relevant, e.g., for Aztec diamonds [Lorentz-minimal surfaces do arise in this way in concrete examples]

DREAMS on critical planar maps ⊕ Ising

It seems that critical planar maps weighted by Ising could/should (via s-embeddings) lead to "canonical" fluctuating space-like surfaces in $\mathbb{R}^{2+1}$

- Can we "guess" the law?
- Is there a way to speak about Dirac $\begin{pmatrix} im & \bar\partial \\ \partial & -im \end{pmatrix}$ on such (rough!) surfaces, where m = "mean curvature"?
- Could $(\det D)^{1/4}$ actually lead to a proper definition?

COULD ALL THAT (S-HOL. FUNCTIONS ON S-EMBEDDINGS) ACTUALLY BE A BEGINNING OF ANOTHER BEAUTIFUL STORY?!

Thank you for attention!