Universality and conformal invariance in the 2D critical Ising model

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joint work with Stanislav Smirnov (Geneva)

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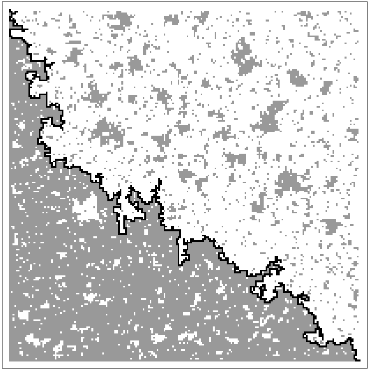
Critical Ising model on the square grid:

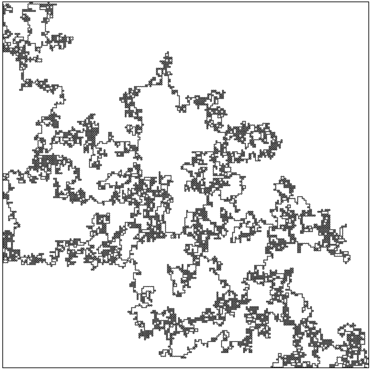
[S. Smirnov. Towards conformal invariance of 2D lattice models. Proceedings of the international congress of mathematicians (ICM), Madrid, Spain, August 22–30, 2006.]

spin-Ising model



Interface $\to \mathrm{SLE}_3$ as mesh $\to 0$.

FK-Ising model



Interface $\to \mathrm{SLE}_{16/3}$ as mesh $\to 0$.

Main steps:

I. *"Combinatorics"*: Construction of the martingale observable ("holomorphic fermion") $F^{\delta}_{(\Omega^{\delta};a^{\delta},b^{\delta})}(z^{\delta})$, $z^{\delta}\in\Omega^{\delta}$, solving some discrete boundary value problem such that

- F^{δ} is *discrete holomorphic* (w.r.t. z^{δ}) for all $(\Omega^{\delta};a^{\delta},b^{\delta})$;
- $F^{\delta}_{(\Omega^{\delta}\setminus\gamma^{\delta}[0,n];\gamma^{\delta}(n),b^{\delta})}$ is a *martingale* (for any fixed z^{δ}) w.r.t. the (discrete) interface γ^{δ} growing from a^{δ}.

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II. *"Complex analysis"*: F^δ is *uniformly close* (w.r.t. all possible simply-connected domains, including those with rough boundaries) *to* its continuous *(conformally covariant)* counterpart $f_{(\Omega^\delta;a^\delta,b^\delta)}$ [solving the continuous version of the same boundary value problem]

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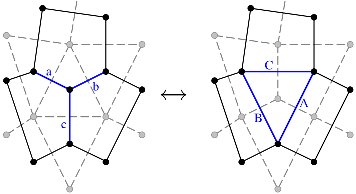
III. *"Probability"*: $\Rightarrow$ discrete interfaces converge to $SLE(\kappa)$, where $\kappa:\quad f_{(\mathbb{C}_+\setminus SLE_\kappa[0,t];SLE_\kappa(t),\infty)}(z)$ is a martingale for all $z\in\mathbb{C}_+$.

More general lattices.
$Y - \Delta$ invariance.



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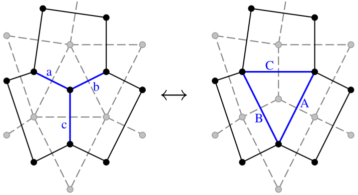

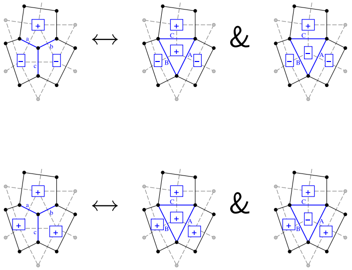

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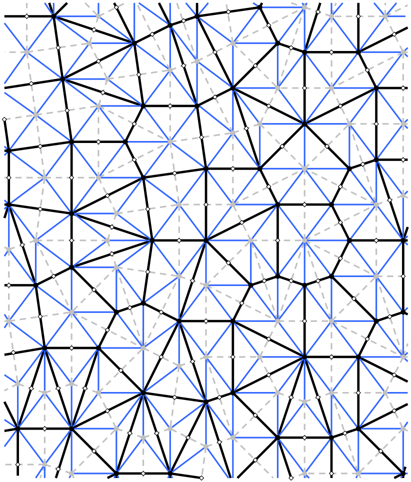
[R. Costa-Santos '06] Local weights satisfying $Y - \Delta$ relation naturally lead to the isoradial embedding of the graph.

Isoradial embedding means that all faces can be inscribed into circles of equal radii δ (the mesh of the "lattice").

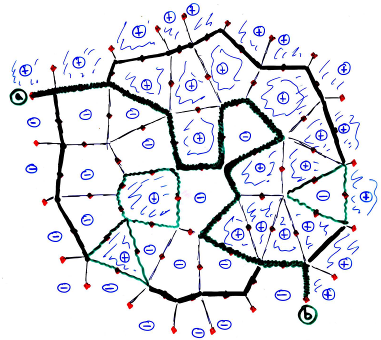
Isoradial graphs. Notations.



- *isoradial graph* Γ (black vertices),
- dual isoradial graph Γ^* (gray vertices);
- *rhombic lattice* ($\Lambda = \Gamma \cup \Gamma^*$, blue edges)
- and the set $\diamondsuit = \Lambda^*$ (white "diamonds").

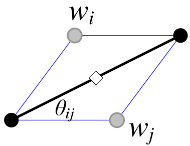
(♠): we assume that rhombi angles are uniformly bounded away from 0 and π.

Critical Ising model on isoradial graphs.



[C. Mercat '01; V. Riva, J. Cardy '06; C. Boutillier, B. de Tilière '09; ...]

$$Z = \sum_{\text{config.}} \prod_{w_i \neq w_j} \tan \frac{\theta_{ij}}{2}$$

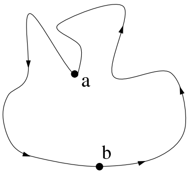


Observable (discrete holomorphic martingale):

$$F^{\delta}(z) := \frac{Z_{config.:a\rightsquigarrow z} \cdot e^{-\frac{i}{2}\text{winding}(a\rightsquigarrow z)}}{Z_{config.:a\rightsquigarrow b} \cdot e^{-\frac{i}{2}\text{winding}(a\rightsquigarrow b)}}, \quad z \in \diamondsuit.$$

Riemann-Hilbert boundary value problem.

- $F(z)$ is holomorphic in Ω;
- $\mathrm{Im}[F(\zeta)(\tau(\zeta))^{\frac{1}{2}}] = 0$ on the boundary $\partial\Omega \setminus \{a\}$;
- proper normalization at b:



- $\tau(b)^{\frac{1}{2}} = +1$;
- $\frac{\partial H}{\partial y}\Big|_b = F^2(b) = 1$;
- H is nonnegative everywhere in Ω.

Remark. F is well defined in rough domains via $H = \mathrm{Im} \int F^2 dz$ which is the imaginary part of the conformal mapping from $(\Omega; a, b)$ onto the upper half-plane $(\mathbb{C}_+; \infty, 0)$ normalized at b.

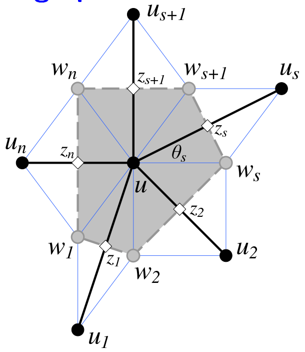
Discrete complex analysis on isoradial graphs.

[R.J. Duffin '60s; C. Mercat '01; R. Kenyon '02;

A. Bobenko, C. Mercat, Yu. Suris '05 ...]

Difference operators $\Delta^\delta, \partial^\delta, \overline{\partial}^\delta$:

$H : \Lambda \to \mathbb{C}$;



$$\partial^\delta H(z_s) := \frac{1}{2}\left[\frac{H(u_s)-H(u)}{u_s-u}+\frac{H(w_{s+1})-H(w_s)}{w_{s+1}-w_s}\right];$$

$$\overline{\partial}^\delta H(z_s) := \frac{1}{2}\left[\frac{H(u_s)-H(u)}{\overline{u}_s-\overline{u}}+\frac{H(w_{s+1})-H(w_s)}{\overline{w}_{s+1}-\overline{w}_s}\right];$$

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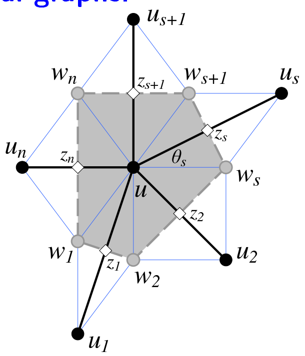

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$$H : \Lambda \to \mathbb{C}; \qquad F : \Diamond \to \mathbb{C}.$$

$$\overline{\partial}^\delta F(u) \quad (= (\partial^\delta)^* F(u)) :=$$

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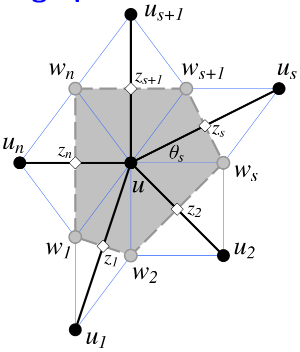

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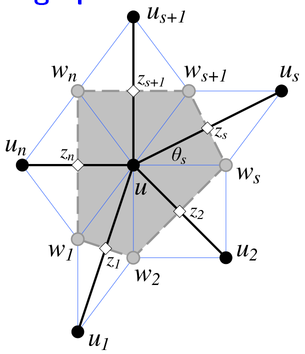

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$$\Delta^\delta H(u) := 4\overline{\partial}^\delta \partial^\delta H(u) = \frac{1}{\mu_\Gamma^\delta(u)} \sum_{u_s \sim u} \tan\theta_s \cdot [H(u_s) - H(u)].$$

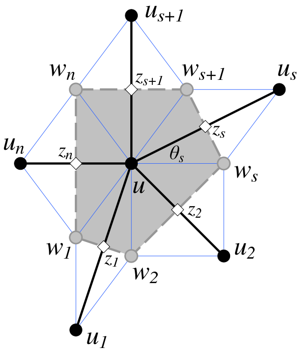
Discrete complex analysis on isoradial graphs.

Corresponding random walk on Γ*:*

$$\mathrm{RW}(t+1) = \mathrm{RW}(t) + \xi^{(t)}_{\mathrm{RW}(t)},$$

where $\xi^{(t)}$ are independent and

$$\mathbb{P}(\xi_u = u_k - u) = \frac{\tan\theta_k}{\sum_{s=1}^n \tan\theta_s}.$$



Then:

$$\mathbb{E}[\mathrm{Re}\,\xi_u] = \mathbb{E}[\mathrm{Im}\,\xi_u] = 0,$$

$$\mathbb{E}[\mathrm{Re}\,\xi_u\,\mathrm{Im}\,\xi_u] = 0, \qquad \mathbb{E}[(\mathrm{Re}\,\xi_u)^2] = \mathbb{E}[(\mathrm{Im}\,\xi_u)^2] = \delta^2 \cdot T_u$$

(where $T_u = \sum_{s=1}^n \sin 2\theta_s / \sum_{s=1}^n \tan\theta_s$).

Discrete complex analysis on isoradial graphs.

Convergence for discrete harmonic functions:

- The *uniform* (w.r.t. (a) shape of the simply-connected domain Ω^δ_Γ and (b) structure of the underlying isoradial graph) *C^1-convergence in the bulk* of the basic objects of the discrete potential theory to their continuous counterparts holds true.

(i) harmonic measure (exit probability) $\omega^\delta(\,\cdot\,; a^\delta b^\delta; \Omega^\delta_\Gamma)$ of boundary arcs $a^\delta b^\delta \subset \partial\Omega^\delta_\Gamma$;

(ii) Green function $G_{\Omega^\delta_\Gamma}^\delta(\,\cdot\,; v^\delta)$, $v^\delta \in \mathrm{Int}\,\Omega^\delta_\Gamma$;

(iii) Poisson kernel $P^\delta(\,\cdot\,; v^\delta; a^\delta; \Omega^\delta_\Gamma) = \dfrac{\omega^\delta(\,\cdot\,; \{a^\delta\}; \Omega^\delta_\Gamma)}{\omega^\delta(v^\delta; \{a^\delta\}; \Omega^\delta_\Gamma)}$, $a^\delta \in \partial\Omega^\delta_\Gamma$ normalized at the inner point $v^\delta \in \mathrm{Int}\,\Omega^\delta_\Gamma$;

(iv) Poisson kernel $P^\delta_{o^\delta}(\,\cdot\,; a^\delta; \Omega^\delta_\Gamma)$, $a^\delta, o^\delta \in \partial\Omega^\delta_\Gamma$, normalized at the boundary by the discrete analogue of the condition $\frac{\partial}{\partial n} P|_{o^\delta} = -1$.

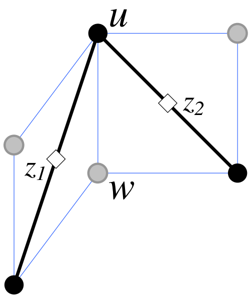
Discrete complex analysis on isoradial graphs.

S-holomorphic functions:

We call F (defined on some subset of $\diamondsuit$) **s-holomorphic**, if

$$\Pr[F(z_1)\,;[i(w-u)]^{-\frac{1}{2}}] = \Pr[F(z_2)\,;[i(w-u)]^{-\frac{1}{2}}]$$

for any two neighbors $z_0 \sim z_1$.



- implies standard discrete holomorphicity (i.e., $\overline{\partial}^{\delta} F = 0$);
- *holds for observables in the critical Ising model*;
- can be reformulated as "propagation equation" (or Dotsenko-Dotsenko equation) for some *discrete spinor* defined on the (double covering of) edges uw [cf. C.Mercat '01]

Discrete complex analysis on isoradial graphs.

Convergence for the "spin-Ising observable":

(A) S-holomorphicity: $F^\delta(z)$ *is s-holomorphic inside* $\Omega^\delta_\diamondsuit$.
(B) Boundary conditions: $\mathrm{Im}[F^\delta(\zeta)(\tau(\zeta))^{\frac{1}{2}}]=0$ *for all* $\zeta\in\partial\Omega^\delta_\diamondsuit$ *except* a^δ, *where* $\tau(\zeta)$ *is the tangent vector at* ζ *oriented in the counterclockwise direction (and* $\tau(b^\delta)^{\frac{1}{2}}=+1$*).*
(C) Normalization at the target point: $F^\delta(b^\delta)=1$.

Theorem (Ch.-Smirnov): *After some re-normalization by constants* $K^\delta\asymp 1$ *(which depend on the structure of* $\diamondsuit^\delta$ *but don't depend on the shape of* Ω^δ*), the solution of the discrete boundary value problem (A)&(B)&(C) is uniformly close in the bulk to its continuous counterpart* $f_{(\Omega^\delta;a^\delta,b^\delta)}$.

Namely, there exists $\varepsilon(\delta) = \varepsilon(\delta, r, R, s, t)$ *such that for all simply-connected discrete domains* $(\Omega^\delta_\diamondsuit; a^\delta, b^\delta)$ *having "straight" boundary near* b^δ *and* $z^\delta \in \Omega^\delta_\diamondsuit$ *the following holds true:*

$$\text{if } B(z^\delta, r) \subset \Omega^\delta \subset B(z^\delta, R), \text{ then}$$
$$|K^\delta \cdot F^\delta(z^\delta) - f_{(\Omega^\delta; a^\delta, b^\delta)}(z^\delta)| \leqslant \varepsilon(\delta) \to 0 \text{ as } \delta \to 0$$

(uniformly w.r.t. the shape of Ω^δ *and the structure of* $\diamondsuit^\delta$*).*

Technical remark: we assume that discrete domains Ω^δ contain some fixed rectangle $[-s, s] \times [0, t]$ and their boundaries near target points $b^\delta \approx 0$ approximate the straight segment $[-s, s]$;

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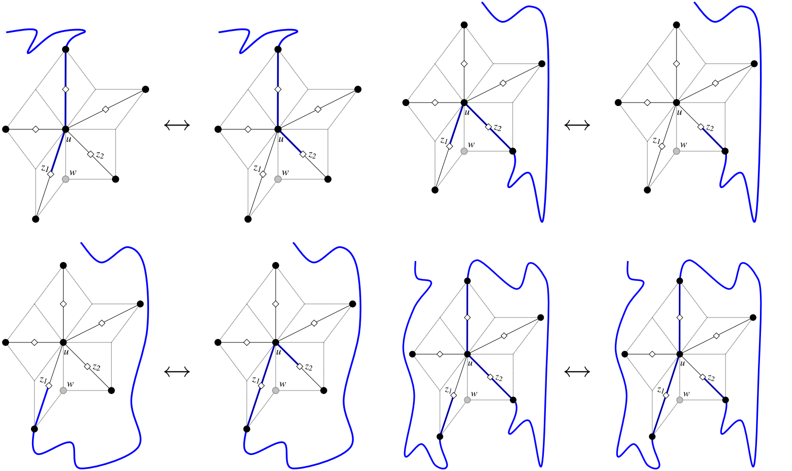
Corollary (universality of the critical Ising model):
The convergence of interfaces of the critical spin-Ising (FK-Ising) model to SLE_3 *(*$\mathrm{SLE}_{16/3}$*, respectively) holds true on isoradial graphs independently on their particular structure.*

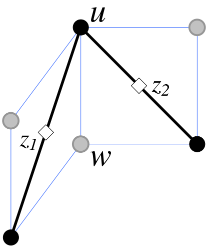
S-holomorphicity of the observable in the spin-Ising model:

[bijection between pictures with interfaces ending at $z_1 \leftrightarrow$ *at* z_2*]*



Two tricks:

I. Integration of F^2 (as on the square grid): If F is **s-holomorphic**, then one can correctly define (up to an additive constant) the function



$$H = \mathrm{Im} \int^{\delta} (F(z))^2 d^{\delta} z \qquad \text{by}$$

$$H(u) - H(w) := 2\delta \cdot \left| \Pr \left[F(z_j) \, ; [i(w-u)]^{-\frac{1}{2}} \right] \right|^2 .$$

(i) for any neighboring $v_1, v_2 \in \Gamma$ or $v_1, v_2 \in \Gamma^*$ one has

$$H(v_2) - H(v_1) = \mathrm{Im}[(v_2 - v_1)(F(\tfrac{1}{2}(v_1 + v_2)))^2].$$

(ii) *H is (discrete) subharmonic on Γ and superharmonic on Γ^*.*

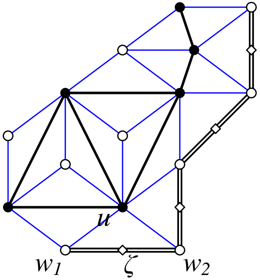
Two tricks:

II. "Boundary modification":

Let $\zeta \in \partial\Omega^\delta_\diamondsuit \subset \diamondsuit$ be a boundary vertex and $\mathrm{Im}[F(\zeta)\tau(\zeta)^{\frac{1}{2}}] = 0$, where $\tau(\zeta) = w_2 - w_1$.

Then $H(w_2) = H(w_1)$ (and so $H\big|_{\Gamma^*} \equiv c$ on this part of $\partial\Omega^\delta_{\Gamma^*}$).

How to deal with $H\big|_{\Gamma}$?



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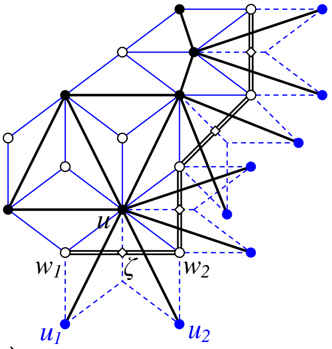

Trick: *Set formally $H(u_{1,2}) := H(w_{1,2})$.*

Then $H\big|_{\Gamma}$ is still *discrete subharmonic* on the new graph (which still has an isoradial structure) and $H\big|_{\Gamma} \equiv c$ *on the "modified" boundary.*

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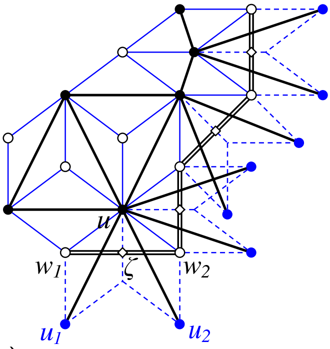

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Remark. *This trick allows us to avoid the using of Onsager's magnetization estimate (as it was in the original Smirnov's proof).*

Convergence of the observable (spin-case):

I. Define $H^\delta = \mathrm{Im} \int^\delta (F^\delta(z))^2 d^\delta z$. Note that
— $+\infty > H^\delta\big|_\Gamma \geqslant H^\delta\big|_{\Gamma^*} \geqslant 0$;
— $H^\delta\big|_\Gamma$ is subharmonic, $H^\delta\big|_{\Gamma^*}$ is superharmonic;
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III. Let $\Omega^\delta \to \Omega$ as $\delta \to 0$. Deduce that both $\{F^\delta\}$ and $\{H^\delta\}$ are normal families on each compact subset of Ω
($\Rightarrow F^\delta \rightrightarrows f$, $H^\delta \rightrightarrows h = \mathrm{Im}\int f^2 dz$ along some subsequence δ_k).

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— $+\infty > H^\delta\big|_\Gamma \geqslant H^\delta\big|_{\Gamma^*} \geqslant 0$;

— $H^\delta\big|_\Gamma$ is subharmonic, $H^\delta\big|_{\Gamma^*}$ is superharmonic;

— both $H^\delta\big|_\Gamma = 0$ and $H^\delta\big|_{\Gamma^*} = 0$ on the boundary.

II. Prove that H^δ are uniformly bounded away from a (Hint: normalization at $b \Rightarrow$ boundedness in the bulk).

III. Let $\Omega^\delta \to \Omega$ as $\delta \to 0$. Deduce that both $\{F^\delta\}$ and $\{H^\delta\}$ are normal families on each compact subset of Ω ($\Rightarrow F^\delta \rightrightarrows f$, $H^\delta \rightrightarrows h = \mathrm{Im} \int f^2 dz$ along some subsequence δ_k).

IV. Keep track that $h \geqslant 0$ in Ω; $h\big|_{\partial\Omega\setminus\{a\}} = 0$ and $\frac{\partial h}{\partial y}(b) = 1$.

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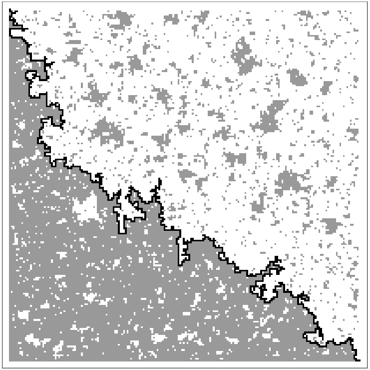
V. Obtain the *uniform* convergence using compactness arguments (Carathéodory topology on the set of simply-connected domains).

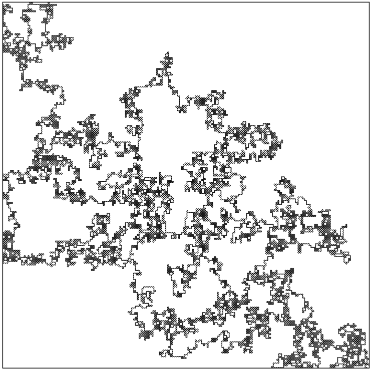
Critical Ising model on isoradial graphs:

spin-Ising model



Interface → SLE_3.

FK-Ising model



Interface → $\mathrm{SLE}_{16/3}$.

THANK YOU!