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DAYS ON DIFFRACTION 2025

ABSTRACTS

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FOREWORD

"Days on Diffraction" is an annual conference taking place in May–June in St. Petersburg since 1968. The present event is organized by St. Petersburg Department of the Steklov Mathematical Institute and St. Petersburg Leonhard Euler International Mathematical Institute.

The conference is supported by the Ministry of Science and Higher Education of the Russian Federation (agreement 075-15-2025-344 dated 29/04/2025 for St. Petersburg Leonhard Euler International Mathematical Institute).

The abstracts of 76 talks, presented during 5 days of the conference, form the contents of this booklet. The author index is located on the last pages.

Full-length texts of selected talks will be published in the Conference Proceedings. Format file and instructions can be found at `http://www.pdmi.ras.ru/~dd/proceedings.php`. The final judgement on accepting the paper for the Proceedings will be made by editorial board after peer reviewing.

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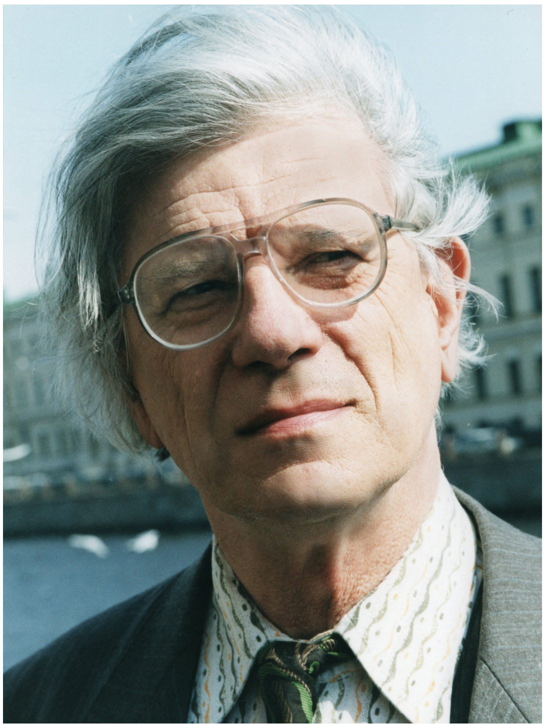

95 years to V. M. Babich

The Organizing Committee of "Days on Diffraction 2025" warmly congratulates Vassily Mikhailovich Babich on the occasion of his 95th birthday celebrated on June 13, 2025.

V. M. Babich is a distinguished representative of the Leningrad/St. Petersburg school of mathematical physics, renowned for his outstanding contributions to the mathematical theory of diffraction and wave propagation, as well as his profound personal influence.

From 1954 Vassily Mikhailovich taught in the Department of Mathematics and Mechanics and in the Department of Physics of Leningrad State University. From 1967 he was the head of the Laboratory for Mathematical Problems of Geophysics in the Leningrad branch of the Steklov Mathematical Institute. His weekly seminar at PDMI has an undisputed reputation, upheld by the exceptional quality of the talks.

V. M. Babich's work laid the foundation for modern methods in mathematical physics, and his teaching shaped generations of scientists. For his fundamental work on the ray method applied to seismic wave propagation, Vassily Mikhailovich was awarded the USSR State Prize in 1982. In 1998, he received the V. A. Fock Prize for his groundbreaking advancements in asymptotic methods in diffraction theory.

This year marks V. M. Babich's 58th conference "Days on Diffraction" as its chairperson. The Organizing Committee expresses its deepest gratitude and profound respect to Vassily Mikhailovich, wishing him good health and further inspiration in his scientific work.

List of talks

Short-wave solutions of the wave equation with localized velocity perturbations whose wavelength is not comparable to the scale of localized inhomogeneity. One-dimensional case

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In this work studies a wave equation whose velocity has a localized perturbation at some point x_0. The initial condition has the form of a rapidly oscillating wave packet whose wavelength is not comparable with the scale of the inhomogeneity. In this case, the length of the initial wave is of the order of ε, and the width of the localized inhomogeneity is of the order of $\sqrt{\varepsilon}$, where ε is a small parameter that tends to 0.

Quantum reservoir computing on decoherence-free subspaces

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We present results of numerical simulations of a quantum reservoir network used for classification of two-qubit states between entangled and separable classes. The output of the network, consisting of 4 randomly connected qubits, is projected onto 2 singlet states — decoherence-free subspaces — as proposed in [1] for decoherence-free quantum computing. The simulation, performed in Lindblad approximation, showcases different dynamical behaviour of quantum reservoir network [2] for entangled and separable states. The differences between these two repertoires are easy to catch with a usual perceptron classifier. The example of the reservoir output dynamics projected onto two-singlet phase space is shown below in Fig. 1 for an entangled input state.

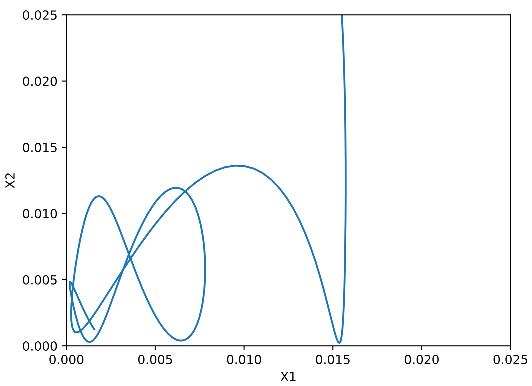

Fig. 1: Two decoherence-free subspaces for a 4-qubit reservoir are given by

$$|\Psi_I\rangle = \frac{1}{2}\left(|\uparrow\downarrow\uparrow\downarrow\rangle - |\uparrow\downarrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle\right),$$
$$|\Psi_{II}\rangle = \frac{1}{2}\left(|\uparrow\downarrow\uparrow\downarrow\rangle - |\uparrow\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\downarrow\rangle - |\downarrow\uparrow\downarrow\uparrow\rangle\right).$$

The phase space dynamics of the 4-qubit reservoir is shown in the coordinates $X_1 = \langle\Psi_I|\rho|\Psi_I\rangle$, $X_2 = \langle\Psi_{II}|\rho|\Psi_{II}\rangle$, where ρ is the reservoir density matrix. The dynamics is shown for a network fed by entangled state.

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Diffraction of H-polarized plane electromagnetic wave by a VO_2 ring: transition from dielectric scattering to plasmons modes

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A rigorous numerical method (MDS) [1, 2] is used to solve the problem of diffraction of a plane H-polarized electromagnetic wave by a vanadium dioxide (VO_2) 2D ring in the visible to near-infrared wavelength range (0.3 µm to 2.5 µm). The material under study exhibits a thermally induced phase transition from a dielectric to a metallic/plasmon-like state within a narrow temperature range. This transition is reversible and stable over multiple heating–cooling cycles. Temperature variation leads to significant changes in the complex dielectric permittivity of VO_2, including a sign change of its real part [3–6]. We investigate the frequency dependencies of the normalized scattering and absorption cross-sections, the Hz field component at a point on the surface of the structure, the near-field distribution of the Hz component, and the far-field radiation pattern at different temperatures, both with and without accounting for material losses. It is shown that, at the same frequency but different temperatures, the field distribution undergoes a qualitative transformation: in the dielectric phase, the response is characterized by typical diffraction patterns, whereas in the metallic phase, strongly localized surface-bound modes emerge, consistent with plasmon-like behavior, in agreement with previously reported thermal phase-dependent scattering phenomena in VO_2-based structures.

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Interaction on shock waves in non-ideal magnetogasdynamics

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This study employs the principles of weakly nonlinear geometrical acoustics and the method of multiple time scales to analyze high-frequency, small-amplitude wave propagation in a non-ideal magnetogasdynamic medium. The governing equations, formulated as a one-dimensional quasilinear hyperbolic system, characterize compressible, unsteady flow with generalized, spherically, and cylindrically symmetric geometries. Transport equations for resonantly interacting wave amplitudes are derived, and the nonlinear evolution of non-resonant wave modes is examined. The study investigates how non-ideality and magnetic fields influence wave dynamics, ultimately leading to shock wave formation. These findings provide insights into the complex interplay between gas non-ideality, magnetohydrodynamic effects, and shock evolution in high frequency wave propagation.

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On spherical trigonometry in one-qubit gates theory and Gauss shoelace formula for sphere

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In certain quantum systems, one-qubit gates can be performed by means of a controlling radio-frequency field or by means of Raman excitation. In the mathematical model, this corresponds to rotating the qubit by an angle proportional to the irradiation time around certain axes. Often, in order to reduce thermal stresses, a single large pulse is tried to be replaced by a series of short pulses.

Using the local isomorphism of the groups $SU(2)$ and $SO(3)$, we transfer the consideration to the surface of the sphere. We give an important example of an effective π-angle rotation. In this case, a series of small rotations with respect to two orthogonal axes is displayed on the surface of the sphere in the form of a polygonal path with the orthogonal angles.

Using the methods of spherical geometry and generalizing the classical "shoelace formula" to the spherical case, we can use elementary methods to determine the parameters of this series of turns.

Evaluation of elastic guided wave dispersion curves in multilayered composite materials with NVIDIA CUDA technology

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In many practical problems arising in ultrasonic-based non-destructive testing and structural health monitoring of thin-walled elongated structures (e.g., damage localization, identification of elastic properties, etc.), prior evaluation of elastic guided wave (EGW) dispersion properties are among the key supplementary tasks. For modern composite materials, which are typically multilayered structures with anisotropic elastic properties of their constituent sublayers, numerical calculation of EGW dispersion curves (DC) especially in the "frequency-propagation direction" domain is still a time-consuming problem. Regardless of the numerical approach used to calculate DCs in layered waveguides (i.e., analytical or mesh-based), this will lead to numerous matrix operations, which can be accelerated using modern parallel programming techniques.

In this talk, the application of general-purpose computing on graphics processing units for the optimization of DC evaluation with semi-analytical integral approach for multilayered anisotropic composites is discussed. As a basis for software implementation, the algorithm for constructing the Fourier-symbol of the Green's matrix for the corresponding media described in [1] is employed and revised for its realization with NVIDIA CUDA technology. The results of comparison of calculations for composite samples with various lay-ups of unidirectional transversally isotropic prepregs and number of layers using NVIDIA CUDA and single-threaded Fortran solution are given, illustrating both the correctness of the proposed CUDA-realization and the multiple acceleration of the computation process when using this technology.

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Towards full numerical construction of modern Mid-IR optical sensors

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Infrared sensors are increasingly being used in a wide range of applications, from environmental monitoring for hazardous substance detection to medical diagnostics for identifying disease biomarkers in exhaled breath. Quantum cascade detectors (QCD) and quantum well infrared photodetectors (QWIP) are of particular interest in this field [1], though their fabrication remains a challenging and time-consuming process, requiring either multiple cycles of experimental structure growth or computationally intensive modeling.

In this work, we propose a comprehensive approach to the design of MIR optical sensors based on QC and superlattice structures. Our approach includes a complete simulation cycle: calculation of the structure of the active region based on the self-consistent solution of the system of Schrödinger–Poisson equations [2], and simulation of the efficiency of the radiation input through special diffraction gratings or plasmonic waveguides by the numerical methods for solving Maxwell's equations, such as the Fourier modal method [3], the modified integral method, the transmission line method, or the finite element method. The proposed approach was developed to be used in conjunction with optimization algorithms. The methodology used is based on the simultaneous work of Bayesian optimization method and Genetic algorithm, achieving near-optimal output characteristics for the given structural design.

This method was employed to achieve a nearly 10% improvement in responsivity of one of the existing designs, providing quantitative and qualitative insights concerning the impact of design parameters on device performance. The presented technique allows us to obtain optimal configurations for different application scenarios — from narrow-band detectors for gas analysis to broadband sensors for spectroscopic applications — at low computational and time cost.

The work was supported by the Ministry of Education and Science of the Russian Federation (government assignment "youth laboratory") and the Russian Science Foundation (No. RSF 23-29-00216).

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BC-method as a look at Inverse Problems

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About 40 years of experience in the development of the boundary control method (BCM) can be reduced to a few basic ideas and theses.

1. By the general system theory, the observer that investigates a system S from its input/output correspondence (Inverse Data) R is, in principle, not able to recover the system itself. The only relevant and properly formalized understanding of "to recover a system" is to create its *copy* (model) S', whose input/output map R' coincides with the original one, i.e., provides $R' = R$.

2. The main source of the copies are the model theorems of functional analysis, which provide 'material' from which the copy can be constructed. In the algebraic version of the BCM for 2d electric impedance tomography, the copy of the surface under reconstruction is constructed from the Gelfand spectrum of the holomorphic function algebra, with the spectrum being extracted from the Dirichlet-to-Neumann map (as ID). In time-domain IPs, by the ID the observer constructs a canonical functional model (the so-called *wave model*) of the operator which governs the evolution of the system S.

The concrete applications, which certify the relevance and productivity of the above theses, are demonstrated on examples of the Riemann manifold reconstruction.

A model and characterization of a class of symmetric semibounded operators

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Let $\mathscr{G}$ be a Hilbert space and $\mathfrak{B}(\mathscr{G})$ be the algebra of bounded operators, let $\mathscr{H} = L_2([0,\infty);\mathscr{G})$. An operator-valued function $Q \in L_{\infty,\mathrm{loc}}([0,\infty);\mathfrak{B}(\mathscr{G}))$ determines a multiplication operator in $\mathscr{H}$

by $(Qy)(x) = Q(x)y(x)$, $x \geqslant 0$. We say that an operator L_0 in a Hilbert space is a Schrödinger type operator if it is unitarily equivalent to $-\frac{d^2}{dx^2} + Q(x)$ on a relevant domain. We give a characterization of a class of such operators in terms of properties of an evolutionary dynamical system associated with L_0, [1, 2]. It provides a way to construct a functional Schrödinger model of L_0.

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Development of metamaterials in Russia and former USSR

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The artificial media with electromagnetic properties not available in natural materials were studied for more than 80 years. These days such substances are called as metamaterials. The discovery of left-handed metamaterials is usually associated with work of Viktor Georgievich Veselago (1929–2018) [1]. However, the media with simultaneously negative permittivity and permeability supporting backward waves and demonstrating negative refraction were discussed in the works of academician Leonid Isaakovich Mandelshtam (1879–1944) [2–5] and Dmitry Vasilievich Sivukhin (1914–1988) [6] much earlier. More details about the early history of metamaterials are available in [7] and [8].

The negative refraction and backward waves are known to be observed in photonic crystals and the publications of Masaya Notomi about this effects are dated by 2000 year [9]. It is interesting that the phenomena were known in USSR since 1970s. The key person who was studying these effects was Robert Andreevich Silin (1927–2018): he published papers [10, 11] and even books [12–14], the last one was a table book for microwave engineers in USSR.

The special kinds of metamaterials are known as bianisotropic media. These structures were actively studied within development of STEALTH technologies. The major branch of studies of bianisotropic media in USSR was done by the school of academician Fedor Ivanovich Fedorov (1911–1994) [15] in Belorus. This contribution was well appreciated by international bianisotropics and metamaterials community [16].

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Uniform stability of the inverse problem for the non-self-adjoint Sturm–Liouville operator

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Consider the non-self-adjoint Sturm–Liouville problem $L = L(q, h, H)$:

$$-y'' + q(x)y = \lambda y, \quad x \in (0, \pi),$$
$$y'(0) - hy(0) = 0, \quad y'(\pi) + Hy(\pi) = 0,$$

where $q \in L_2(0, \pi)$ is a complex-valued function, h and H are complex constants, and λ is the spectral parameter. We focus on the inverse spectral problem of [1] that consists in the recovery of q, h, and H from the eigenvalues $\{\lambda_n\}_{n\ge1}$ of L and the generalized weight numbers $\{\alpha_n\}_{n\ge1}$ related to the singularities of the Weyl function

$$M(\lambda) = \sum_{n\in I} \sum_{\nu=0}^{m_n-1} \frac{\alpha_{n+\nu}}{(\lambda - \lambda_n)^{\nu+1}},$$

where I is the index set of all the distinct eigenvalues among $\{\lambda_n\}_{n\ge1}$ and m_n is the multiplicity of λ_n. For definiteness, assume that $h + H + \frac{1}{2}\int_0^\pi q(x)\,dx = 0$.

Our goal is to find relevant constraints on the spectral data and/or on the problem parameters (q, h, H) that guarantee the uniform stability estimate

$$\|q^{(1)} - q^{(2)}\|_{L_2} + |h^{(1)} - h^{(2)}| + |H^{(1)} - H^{(2)}| \le C \left(\sum_{n\ge1} n^2 \bigl(|\rho_n^{(1)} - \rho_n^{(2)}|^2 + |\alpha_n^{(1)} - \alpha_n^{(2)}|^2\bigr) \right)^{1/2} \quad (1)$$

for any two eigenvalue problems $L^{(i)} = L(q^{(i)}, h^{(i)}, H^{(i)})$ $(i = 1, 2)$ of the considered class and their spectral data $\bigl\{\lambda_n^{(i)}, \alpha_n^{(i)}\bigr\}_{n\ge1}$ $(i = 1, 2)$, $\rho_n^{(i)} := \sqrt{\lambda_n^{(i)}}$. Moreover, we generalize (1) to the case of two problems with different eigenvalue multiplicities.

Our results are presented in [2]. They generalize the uniform stability theorem by Savchuk and Shkalikov [3] for L_2-potentials to the non-self-adjoint case. It is worth mentioning that a local version of estimate (1) is given in [4].

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From Fano resonances to Fano combs

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Fano resonance is one of the most universal phenomena in physics, observed in all areas where wave propagation and interference are possible [1]. Fano resonance arises from the interference of broad and narrow spectral signatures and is an important tool in fundamental and applied research [2]. The simple Fano formula describes four types of contours, including a symmetric Lorentzian, a symmetric inverted contour with zero amplitude at the center of the line, and two asymmetric contours, with mirror-symmetric shapes describing the transition from the Lorentzian to the inverted contour. Probably almost everything is known about Fano resonance itself, numerous reviews have been written [1–7], however, there is an amazing manifestation of Fano resonance, namely Fano combs, which are less common, but contain much more information than the traditional solo Fano resonance.

The well-known case of Fano combs is experimentally observed in the infinite overlapping Rydberg series of autoionizing states. In each comb, the Fano resonances are not equidistant in either frequency or wavelength and are described by the Rydberg formulas [4].

Another experimental observation of Fano combs has recently been reported [[6]. In an elegant experiment, the authors observed Mie scattering from evaporating water droplets. By selecting only the TM modes, the authors demonstrate a scattering spectrum consisting of three separate overlapping combs, within which the resonances are equidistant. Each comb consists of individual Fano resonances whose line shapes evolve from the symmetric Lorentz shape to the asymmetric shape of the classical Fano resonance.

We demonstrate the generality of this effect by demonstrating experimentally and numerically overlapping Fano combs in the scattering spectra of classical dielectric resonators, namely a ring and a split ring with a rectangular cross-section, as well as a rectangular parallelepiped. An unambiguous interpretation of the series of overlapping resonant Fano combs is obtained using the theory of quasi-normal modes [7]. In the case of a parallelepiped, equidistant Fano combs were found in the scattering spectra, which are formed by spectrally narrow photonic modes of the Fabry–Pérot type of different symmetries, interfering with broadband scattering from the entire parallelepiped.

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Waves in round resonator with transparent walls, generated by a delta-force, and their relation to quasistationary (Gamow) states

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We consider the two-dimensional Helmholtz equation with step-constant coefficient that models a round wave resonator with transparent walls. The solution is constructed for inhomogeneous equation with a delta-force with the Sommerfeld condition at infinity and the existence of the smooth first derivative on the wall. This solution is related to quasistationary states [1] (aka Gamow states [2]), that are similar to whispering-gallery waves inside the resonator. In the case, of an arbitrary convex resonator or varying coefficients an effective asymptotics for such states can be constructed as in [3].

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Asymptotics of solutions to the characteristic problem for the ultrahyperbolic equation along null geodesics

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We deal with the following problem in the Euclidean space

$$\begin{gathered}\left(\partial_{ts}^2 + \partial_{x_1}^2 + \ldots + \partial_{x_d}^2 - \partial_{y_1}^2 - \ldots - \partial_{y_n}^2\right) U = 0, \\ U|_{t=0} = U_0.\end{gathered} \tag{1}$$

The set $\{t = 0\}$, on which the initial data U_0 are given, is a characteristic hyperplane for the ultrahyperbolic equation (1). The well-posedness of this problem in a certain set of functions was established by A. S. Blagoveshchensky [1]. In particular, the following conservation law was obtained

$$\|U(t)\|_{L_2} = \|U_0\|_{L_2}, \quad t \in \mathbb{R}.$$

These facts make reasonable the study of the corresponding nonstationary scattering problem for equation (1) considering t as the time variable. One of crucial issues in such an investigation would be the asymptotic behavior of the solutions for large t, which is the subject of the present talk. Analogously to *hyperbolic* problems, the result is given in terms of asymptotic of the solution along null geodesics, transversal to the initial plane $\{t = 0\}$.

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Asymptotics of long waves in basins with shallow shores generated by spatially localized harmonic in time sources

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We consider the problem of short-wave asymptotic solutions of linear and nonlinear systems of shallow water equations in a basin with an uneven bottom and shallow shores describing waves excited by a spatially localized harmonic source. In the linear approximation, such asymptotic solutions are essentially expressed in terms of solutions of the Helmholtz equation, and the problem of constructing them is close to the problem of the asymptotics of the Green's function. We use recently developed approaches based on the canonical Maslov operator, which makes it possible to find a global asymptotic solution to the considered problem in any given domain, taking into account the presence of caustics and focal points, as well as Fermat's variational principle, which allows us to find an asymptotic solution locally, that is, in the vicinity of a given observation point. The presence of the shore leads to the appearance of a 'non-standard' caustics, in the vicinity of which the asymptotic solution of a system of linearized equations is expressed in terms of a modified canonical operator. This asymptotic solution does not determine the solution of the original nonlinear system, for which the problem with a free boundary is essentially considered. However, according to a recently developed approach based on the modified Carrier–Greenspane transformation, the asymptotic solution of a nonlinear system is expressed in terms of the solution of a linearized system in the form of parametrically defined functions. The obtained formulas describe, among other things, the effects of the waves raiding the shore.

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Propagation of coupled TE-TE waves in an open spatially inhomogeneous nonlinear slab

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We consider the propagation of an electromagnetic wave

$$\mathbf{E}_\omega = \mathbf{E}_1 \exp(-i\omega_1 t) + \mathbf{E}_2 \exp(-i\omega_2 t), \qquad \mathbf{H}_\omega = \mathbf{H}_1 \exp(-i\omega_1 t) + \mathbf{H}_2 \exp(-i\omega_2 t), \tag{1}$$

with complex amplitudes $\mathbf{E}_j$, $\mathbf{H}_j$ (everywhere below index j takes values 1, 2) having the form

$$\begin{aligned} \mathbf{E}_1 &= \big(0, e_y(x), 0\big)\exp(i\gamma_1 z), \qquad \mathbf{H}_1 = \big(h_x^{(1)}(x), 0, h_z(x)\big)\exp(i\gamma_1 z), \\ \mathbf{E}_2 &= \big(0, 0, e_z(x)\big)\exp(i\gamma_2 y), \qquad \mathbf{H}_2 = \big(h_x^{(2)}(x), h_y(x), 0\big)\exp(i\gamma_2 y), \end{aligned} \tag{2}$$

where $\gamma_j \in \mathbb{R}$ are unknown wave numbers, in an open slab $\Omega = \{(x,y,z) \in \mathbb{R}^3 : 0 \leqslant x \leqslant h, (y,z) \in \mathbb{R}^2\}$, sandwiched between two half-spaces. The permittivity of the media inside Ω have the form

$$\varepsilon_l = \begin{pmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{pmatrix}, \tag{3}$$

where ϵ_{xx} is arbitrary (it does not affect the field), and

$$\begin{aligned} \epsilon_{yy} &= \varepsilon_1(x) + c_1\epsilon_1(x) + (\alpha_1 + c_3 a(x))\,|(\mathbf{E}_1, \mathbf{e}_y)|^2 + c_4\beta(x)|(\mathbf{E}_2, \mathbf{e}_z)|^2, \\ \epsilon_{zz} &= \varepsilon_2(x) + c_2\epsilon_2(x) + (\alpha_2 + c_3 a(x))\,|(\mathbf{E}_2, \mathbf{e}_z)|^2 + c_4\beta(x)|(\mathbf{E}_1, \mathbf{e}_y)|^2; \end{aligned}$$

here $\alpha_j > 0$, c_j, c_{j+2} are real constants, ϵ_j, ε_j, a, β are real-valued continuous functions for $x \in [0,h]$, in addition, $\varepsilon_j(x) > 0$, $\varepsilon_j'(x) \geqslant 0$. The half-spaces surrounding the slab are filled with homogeneous linear dielectric media. Field (1) is called a (nonlinear) *coupled TE-TE* wave [1].

The problem is to find the values of γ_1, γ_2 such that there exist vector functions (1) satisfying Maxwell's equations, continuity condition for tangential components of the electric field at both boundaries of the slab, an additional condition

$$|(\mathbf{E}_1, \mathbf{e}_y)|^2\big|_{x=0} = A_1^2 = \text{const}, \qquad |(\mathbf{E}_2, \mathbf{e}_z)|^2\big|_{x=0} = A_2^2 = \text{const}, \tag{4}$$

and decaying for $|x| \to \infty$. These γ_j are called *propagation constants* of the waveguide Ω.

To study this problem we suggest a nonstandard perturbation method based on the usage of solutions to a simpler nonlinear problem in order to find solutions to the original one [2]. It allow us to prove existence of solutions to the problem (propagation constants of Ω) including nonlinearizable ones which correspond to novel guided modes of Ω.

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Energy distribution of guided waves in fluid-loaded anisotropic laminate plates

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The ultrasonic sounding of a fluid-loaded anisotropic laminate plate is considered. The present work deals with the investigation of wave energy transfer from the source to the composite plate. The study is carried out within a semi-analytical model presented at the preceding DD conferences [1,2]. The model is based on the explicit integral representations of the excited waves by the Fourier symbol of the Green's matrix of the coupled system — distributed source – anisotropic laminate plate – acoustic fluid. The far-field asymptotic representations for the guided waves (GWs) are obtained from those path integrals using the stationary phase method and the residual technique. The source energy distribution among the GWs is evaluated using a closed-form representation for the time-averaged wave energy transfer through a cylindrical surface.

The present research is focused on the energy distribution among GWs excited in immersed laminate plates with anisotropic sublayers. For this purpose, a three-layer transversally isotropic plate immersed in water is considered. The numerical examples illustrate GW energy diagrams, which yield the total (integrated over the vertical cross-section) amount of wave energy propagating from the source to infinity in all directions, GW energy-flux profiles. Amplitude diagrams for the GWs and their eigenforms are also presented and discussed.

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Physics-informed neural networks and neural operators for a study of EUV electromagnetic wave diffraction from a lithography mask

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Extreme ultraviolet (EUV) lithography is a critical component of contemporary semiconductor manufacturing processes. This technology allows the creation of smaller and more advanced semiconductor chips by utilizing shorter wavelengths of light to etch intricate designs onto silicon wafers, which depend on the light pattern during exposure. The light pattern on a wafer is formed with an EUV electromagnetic beam reflecting from a mask. Due to diffraction and interference phenomena, the patterns of the mask do not exactly match the light patterns on the wafer. To obtain the desired patterns on the wafer, a multi-stage optical proximity correction technology is used to correct the masks. One of these stages is the calculation of electromagnetic fields in the area of the mask location, which has a huge computational cost when using standard electromagnetic (EM) solvers.

To accelerate the EM simulations, various approximation models have been proposed, including the domain decomposition method and the M3D filtering. However, these models are still very resource-intensive and do not take into account the nonlocality of electromagnetic interaction. Recently, many attempts have been made to simulate the 3D effects of masks using deep neural network such as convolutional neural network [1] or Unet [2]. These approaches are based on supervised learning, require a fairly extensive dataset, still have a significant training time, and often do not demonstrate the necessary accuracy of the solution and the degree of its generalization.

In this paper, we present physics-informed neural networks (PINNs) [3] and neural operators (NOs) [4] for solving the problem of diffraction of EUV electromagnetic waves from a mask. An architecture of the neural operator, which is most suitable for this diffraction problem, is proposed. The training of these neural systems is carried out using unsupervised learning. The accuracy and time required to find solutions to several problems by using modern numerical solvers are compared with those obtained by PINNs and NOs. The emphasis is placed on investigating the accuracy of solutions provided by these artificial neural systems for wavelengths of current and promising industrial lithographers (13.5 and 11.2 nm) [5]. Numerical experiments have demonstrated PINNs and NOs achieve competitive accuracy and have relatively short time for the predicted field distributions.

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Hybrid numerical-analytical scheme for the elastodynamic simulation of layered elastic structures with multiple inhomogeneities

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One of the most significant stages in the development of ultrasonic-based non-destructive testing (NDT) and structural health monitoring (SHM) systems is the development of computational models for the excitation, propagation, and diffraction of elastic guided waves (such as Lamb waves and SH waves) in thin-walled structures. With their help, it might possible, for example, to optimize the location of the nodes in the actuator-sensor network in order to maximize coverage of the area being diagnosed. They can also be used to identify specific features of guided wave scattering and to select optimal frequency ranges for generating and measuring ultrasonic oscillations in terms of susceptibility to these features.

In this talk, a generalization of a hybrid numerical-analytical scheme [1, 2] to the case of a mulitlayered waveguide with finite number of inhomogeneities is discussed. The considered approach relies on the conjugation of mesh-based solutions for a localized area of the waveguide which contains any inhomogeneity (oscillation source, defect, obstacle, sensor, etc.) with analytically-based representation for wavefields in the remaining parts of the structure. Local numerical solutions can be obtained using any finite element software that allow to evaluate the elastodynamic behavior of a domain with specified boundary conditions, and COMSOL Multiphysics package is used in the current work. The resulting set of numerical solutions is combined with an analytical representation of the countables set of normal modes propagating in homogeneous parts of the waveguide. Amplitude factors of these waves which are initially unknown could be determined based on the continuity of stresses and displacements at the interfaces between local regions with inhomogeneities and remaining parts of the waveguide.

To illustrate the performance of the developed computational approach, a multilayer isotropic waveguide with several surface-mounted and internal obstacles separated by extended homogeneous regions is considered which is a typical structure for NDT and SHM applications. To excite and sense elastic guided waves, piezoelectric wafer active sensors adhesively attached to the waveguide surface are employed, while delaminations, interlayer cracks and impact type cracks are considered as defects. The analysis of the interaction of specific normal modes with damage, as well as their contribution to the sensor response is presented.

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On the scattering of light by a spheroidal particle with a spherical core

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Recently, we have published a paper [1] where the next step in the development of the theory of light scattering by particles was made, namely an analytical solution to the light scattering problem was derived in the case of the two-layered spheroids with non-confocal core and envelope. The previous steps of the theory were the analytical solutions for homogeneous spheres (Mie theory), layered spheres, homogeneous spheroids, and layered spheroids with confocal internal surfaces.

In [1] the main problems occurred not for the large diffraction parameter (the accurate solutions were derived for its values well above 100, i.e. in the geometrical optics domain), but for the large particle cores sufficiently close in their shape to spheres. To understand the source of the problems and to extend our solution applicability, in this paper we consider the limiting case being the layered spheroids with a spherical core. As in [1] we apply the extended boundary condition method (EBCM) and expand the fields in terms of the spheroidal functions related to spheroidal boundaries and in the spherical functions related to the core surface. In the layer outside the core, the "spheroidal" expansions are transformed in the "spherical" expansions, in other layers, if any, the spheroidal expansions are related through their transformations in each other using spherical functions as in [1]. The solution is formulated in the form of the usual ("spherical") T-matrix. Numerical results show that the suggested solution is applicable in a wide parameter value range, and stably works even in the cases of the spheroidal layer boundaries with high aspect ratios.

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Resonant transverse transformation of incident wave power flux by edge plasmons in a rectangular graphene ribbon

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Graphene plasmonics is a vibrant and rapidly developing area of flatland optoelectronics. Plasmons in graphene structures are able to localize the electromagnetic field down to a sub-wavelength scale [1]. In addition to the widely studied two-dimensional (2D) plasmons, edge plasmons [2] can be excited in graphene structures, which are capable of localizing electromagnetic fields on an even smaller scale. In a long structures, in which the lateral dimensions differ by two orders of magnitude, plasmon resonances with negative dispersion can be excited [3].

In this work, scattered fields were investigated at normal incidence of a plane electromagnetic wave on a ribbon graphene structure. The structure is a graphene ribbon 130 µm long and 0.4 µm wide, located on a dielectric substrate. An incident electromagnetic wave of the terahertz frequency range with polarization of the electric field along the short side of the ribbon excites plasmon resonances. The problem was solved by the integral equation method via the Galerkin procedure. The induced fields and currents in the structure, as well as the absorption cross-sections, forward and backward scattering, and the cross-section of transverse scattering in the graphene plane were calculated (Fig. 1a). The specta of cross sections consists of the main 2D plasmon resonance at the frequency of $f = 7.5$ THz, the edge plasmon is at a 6.4 THz ($0.85f$), and there are a set of backward edge plasmons at lower frequencies. It was found that a dipole oscillation at the short edges of the graphene ribbon scatter the wave in the plane along the graphene. The backward edge plasmons resonantly scatter the electromagnetic wave exclusively in the direction transverse to the incident wave, which is caused by the structure of the excited currents (Fig. 1b) and magnetic fields. Since the polarization of the induced electric and magnetic fields of the backward edge plasmons corresponds to a transverse electromagnetic wave scattered in the transverse direction.

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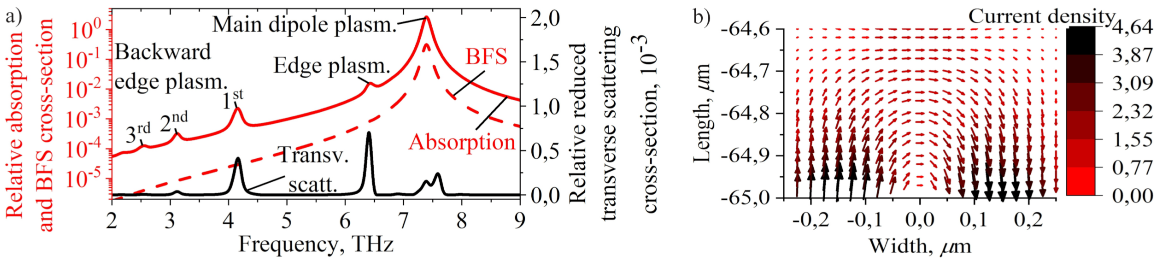

Fig. 1: (a) The spectra of absorption cross section, back and forward scattering cross section, and transverse scattering cross section divided by the extinction cross-section of an infinitely long ribbon. (b) Spatial distribution of current density in graphene for first backward edge plasmon resonance near the edge of the ribbon.

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Knotted polarization of short electromagnetic pulses

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For extremely short pulses, many concepts of traditional optics, including the state of radiation polarization, lose their meaning [1]. Here, we show that the polarization structure of such pulses is determined by the hodograph of the electric strength E, which can be a nontrivial node.

The initial are Maxwell's equations for a vacuum with positive and negative charges present and dynamically changing during the pulse. The figure shows a trefoil-shaped hodograph (b) oriented in accordance with the increase in time, generated by a localized distribution of the charge density ρ (a). The latter can be attributed to a distributed rotating dipole changing its magnitude with time. The other panels of the figure show the time dependences of the modulus (c) and the Cartesian components of E (d-f), which are similar throughout the entire field structure. We also note the possibility of a polarization band associated with the hodograph to represent a one-sided surface (the Mobius strip [2, 3]).

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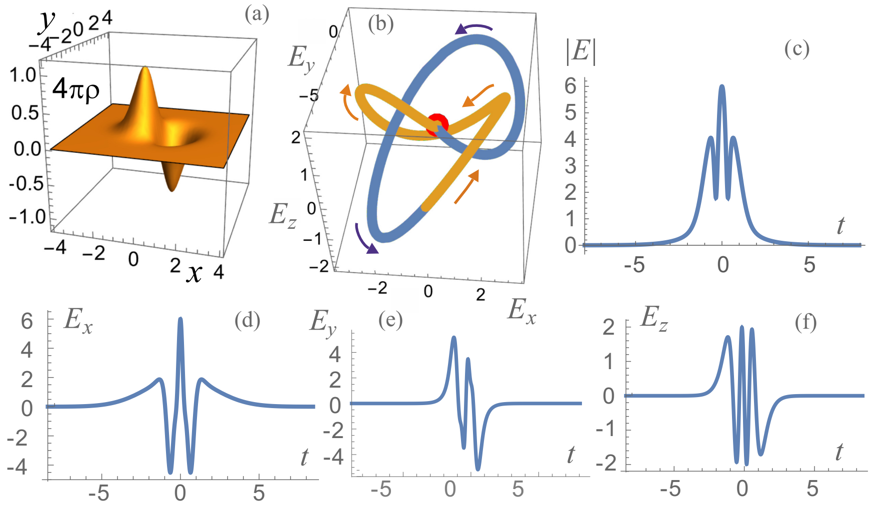

Fig. 1: Instantaneous electric charge profile in section $z = 0$ (a), hodograph of the electric strength E (b, trefoil, arrows show increase in time), its module (c) and Cartesian components (d-f).

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On the exact transparent boundary condition for the 1D Schrödinger equation with linear potential

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The boundary conditions for partial differential equations known as exact transparent boundary conditions (TBC) allow one to replace solution of spatially infinite problems with equivalent problems posed in finite computational domains. Such conditions are especially useful for truncation of finite-difference schemes making efficient numerical solution of various evolution and propagation problems possible. TBCs exist for many types of physical equations including the 1D/2D/3D Schrödinger equations [1]. In this work by transforming and splitting the 1D Schrödinger equation with a linear potential term into a couple of differential equations and using the known exact TBC for the free space 1D Schrödinger equation [2], an exact TBC for the one-dimensional quantum mechanical evolution problem with a varying linear potential is obtained. The derived TBC is then discretized for the use with an unconditionally stable Crank–Nicholson finite-difference scheme. For the periodically oscillating linear potential a numerical implementation is realized by a Matlab program code. A number of numerical experiments demonstrating transitions induced by an oscillating potential between bound states in an 1D potential well are conducted. They demonstrate that the new TBC enables one to successfully incorporate infinite time dependent linear potential terms into quantum mechanical problems with finite potentials. The obtained TBC can as well be used for solution of the 2D parabolic equation with an infinite linear term in dialectic permittivity, for instance, to describe propagation of the X-ray radiation in bent nano-waveguides and nano-pores [3].

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Modeling of guided wave excitation and propagation using experimentally determined effective elastic properties of layered waveguides

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Semi-analytical Green's matrix based modeling [1] of guided wave (GW) excitation and propagation in layered elastic structures has proven its efficiency. It gives the same quantitative results as a mesh-based numerical simulation (e.g., FEM) with an additional insight into the detailed wave structure with the wave characteristics of each excited mode, while extracting the latter from the total numerical solution requires a cumbersome additional post-processing. In order to numerically obtain the same wave pattern as observed in the experiment, it is necessary to determine the effective waveguide parameters (i.e., computer model inputs) that give the required computed amplitudes and velocities of the traveling waves excited by a given source.

The effective parameters, i.e., elastic modulus, thicknesses, density, etc., are determined by minimizing the discrepancy between the calculated and measured GW characteristics while varying the input parameters of the computer model. In addition to the development of the latter, here arises the independent problem of extracting wave characteristics from the array of measurement data. In the present talk we discuss the solution of these problems on the basis of explicit integral and asymptotic representations of wave fields, their excitation and registration by piezoelectric sensors and non-contact laser-based means, and the extraction of GW wavenumbers using the experimental B-scans, H-functions and the matrix pencil method (MPM). The general scheme is illustrated by examples of experimental measurements and calculations relating to both isotropic plates (steel, aluminum, glass) and anisotropic carbon fiber-reinforced composites.

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Resonance effects associated with mode repulsion and their use in ultrasonic inspection of layered materials

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As it is known, symmetric and antisymmetric modes of Lamb waves in a homogeneous elastic layer are determined from independent characteristic equations and therefore their dispersion curves can intersect. With an inhomogeneous waveguide, for example, in the presence of a thin adhesive interlayer connecting homogeneous layers, the problem becomes coupled, which prohibits the intersection of dispersion curves. In the areas of possible intersection, they sharply deviate from each other, forming characteristic bends that give zero or nearly zero group velocity (ZGV) of the corresponding traveling waves. The frequency ranges of mode repulsion depend on the elastic properties and thickness of the interlayer, which was proposed to be used for non-destructive diagnostics of the bond strength [1]. However, this requires the reconstruction of dispersion curves with good resolution from an array of surface measurement data, which is a challenging problem in itself.

On the other hand, ZGV and nearly ZGV modes manifest themselves in resonance peaks in the frequency response. These peaks are well visible in the frequency spectrum of the measured signals, i.e. it is much better to detect them then to track dispersion curves with a fine resolution. As the interlayer properties change, the peak frequencies shift together with the repulsion bands, so they can also be used as diagnostic indicators.

In the present work, we investigate the dependence of resonance manifestations on the waveguide properties, primarily resonances related to mode repulsion, ZGV, and backward wave effects. The numerical simulation is based on semi-analytical solutions in terms of Green's matrices of the laminate structures considered. The results for various combinations of aluminum, glass, steel, and CPFR composite plates bonded by thin interlayers as well as their experimental verification are discussed.

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In-plane elastic wave propagation through a damaged interface between dissimilar orthotropic media

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This study examines elastic wave propagation through an imperfect interface between dissimilar orthotropic materials, where the interfacial damage is represented by a random distribution of strip-like micro-cracks. The damage model is transformed into equivalent effective spring boundary conditions characterizing the imperfect interface. The scheme proposed in [1, 2] for isotropic media is extended here. An asymptotic solution in explicit form is first derived for plane wave scattering by a single interface crack [3]. The spring stiffness values for are obtained assuming the equivalence of two damage models. These derived explicit relationship for spring stiffnesses are expressed in terms of elastic properties of the contacting orthotropic materials, crack density parameters and micro-crack dimensions. The latter enable quantitative assessment of interfacial damage severity by evaluating the interface stiffness, which can be practically determined using ultrasonic measurement techniques.

The study was funded by the Ministry of Science and Higher Education of the Russian Federation (Project No. FZEN-2024-0003).

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Blazed gratings: effect of antiblaze angle on diffraction efficiency at normal and grazing light incidence

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The present communications report examines the effect of the antiblaze angle on the diffraction efficiency of electromagnetic radiation of TE and TM polarizations falling on a single-periodic (line) diffraction grating with a triangular groove profile (“with blaze”). Two general cases of diffraction are investigated: (1) resonant — at normal or near-normal incidence of light; (2) short-wave — at grazing incidence of light. Using the PCGrate^{TM} code [1], based on the rigorous boundary integral equations method [2], angular dependences are calculated for two cases of the direction of incidence radiation relatively to the geometry of the triangular groove (from the side of the blaze angle or from the side of the antiblaze angle), including with positive, right and negative antiblaze angles.

The computations were performed both for the case of perfect conductivity of the grating surface and for its finite conductivity and the corresponding refractive index of the selected grating material. Analysis of the obtained dependences shows a strong polarization dependence for resonant diffraction and a strong dependence on the incidence angle and triangle angles for short-wave diffraction. In the first case, the efficiency depends more on the groove depth than on angles of triangle shapes. In the second case, both angles play a decisive role in the grating blaze, depending on the side from which the radiation is incident. The obtained results can be useful both in the manufacture of master gratings obtained by liquid anisotropic etching of silicon wafers through a lithographic mask and also in producing obtaining copies by nanoreplication [3].

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Sharp estimates in singular perturbation applied to the first order partial differential equations in a bounded domain

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We discuss numerical and mathematical analysis of singular perturbation applied to the first order partial differential equations in a bounded domain and show their convergence rates. Our results contain the case of singular perturbation with the homogeneous Neumann condition. We focus on linear cases here as a fundamental study for cases of the first order quasi-linear equations, of which the historical approaches are found in Kružkov [2]. We remark convergence rates depend on boundary conditions posed in singular perturbation problem (e.g. [3]), and we deal with both the Dirichlet and the Neumann conditions.

Let Ω be a bounded domain in $\mathbb{R}^d$, β be a Lipschitz continuous vector field on Ω, and let $\mu \in L^\infty(\Omega)$. For a given real-valued function $f \in L^2(\Omega)$, we consider the following problem:

$$\beta \cdot \nabla u + \mu u = f \quad \text{in } \Omega, \tag{1}$$

$$u = 0 \quad \text{on } \Gamma_- = \{x \in \partial\Omega \mid \beta(x) \cdot n(x) < 0\}, \tag{2}$$

where n denotes the unit outward normal vector on $\partial\Omega$. Under the assumption $\inf_\Omega(\mu - \frac{1}{2}\nabla \cdot \beta) > 0$, the well-posedness of (1)–(2) is known (e.g., [1]). We introduce, for $\varepsilon > 0$, a singular perturbation term $-\varepsilon\Delta$ into (1), and we consider

$$-\varepsilon\Delta u^\varepsilon + \beta \cdot \nabla u^\varepsilon + \mu u^\varepsilon = f \quad \text{in } \Omega. \tag{3}$$

We show some sharp convergence rate $\|u^\varepsilon - u\|_{L^2(\Omega)}$ for the case that we impose the homogeneous Dirichlet and Neumann conditions on $\partial\Omega \setminus \Gamma_-$, and we also show some numerical results.

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Parabolic-equation approach to high-frequency grazing diffraction theory: from Leontovich and Fock to nowadays

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In the past 2024, we have celebrated the 80th anniversary of the parabolic-equation method in diffraction theory. An overview of the development of the theory of grazingly incident high-frequency wavefields on an obstacle is given, starting with pioneering research Leontovich and Fock on scattering by a smooth convex body [1, 2]. The main contribution to this theory is due to Soviet and Russian scientists. Diffraction by boundary inflection [3, 4], by a jump of curvature [5–7], and other problems approachable through the parabolic-equation method, are reviewed. We mention the results by I. V. Andronov, V. M. Babich, V. A. Fock, I. V. Kamotski, N. Ya. Kirpichnikova,

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Subresonance in the wave equation

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Consider the nonlinear wave equation with dispersion of the form:

$$\partial_t^2 u - L(\partial_x, u) = \varepsilon f(u, x, t), \quad 0 < \varepsilon \ll 1.$$

This equation has a rich set of solutions. Here, we consider solutions of the form:

$$u(x,t,\varepsilon) = \sum_{k=-\infty}^{\infty} A_k^+(t,\varepsilon) e^{i(\omega t + kx)} + A_k^-(t,\varepsilon) e^{i(-\omega t + kx)} + \text{c.c.}$$

It turns out that at super-large times $t \gg \varepsilon^{-1}$, the slow evolution of the amplitudes is determined by subresonant behavior. By subresonance, we mean solutions [1,2] whose evolution is governed by the appearance of small denominators of the form $\sum_{|l|=k}(\omega_{l_j} - \omega_k) \ll k^{-\alpha}$, $\alpha > 0$ in the perturbative corrections when constructing solutions as asymptotic series in the parameter ε. In contrast to resonances [3], the subresonant growth of the corrections is slower than linear, so they are detected when analyzing solutions on super-large timescales relative to the inverse perturbation parameter.

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Description of plates with Matrix Klein–Gordon equation

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Waveguide Finite Element Method [1], or Matrix Klein–Gordon equation (MKGE), is the most general model for the waveguides, which are homogeneous along their longitudinal coordinates.

MKGE can be obtained in the result of discretisation of the waveguide's cross-section. MKGE is formulated as follows:

$$\left(\mathbf{D}_2\frac{\partial^2}{\partial x^2}+\mathbf{D}_1\frac{\partial}{\partial x}+\mathbf{D}_0-\mathbf{M}\frac{\partial^2}{\partial t^2}\right)U(t,x)=F(t,x).$$

Here t is time, x is the longitudinal coordinate, $U(t,x)$ is a vector describing the cross-section state, vector $F(t,x)$ describes external forces. $\mathbf{D}_2$, $\mathbf{D}_1$, $\mathbf{D}_0$, $\mathbf{M}$ are the matrix coefficients, they are independent of x and t. One can see that MKGE is a matrix partial differential equation of the second order with respect to x and t.

Here we build MKGE for a plate and check if MKGE describes the flexural vibrations correctly even though such motion is obeyed to an equation of the fourth order with respect to the longitudinal coordinate. We show that approximation of the displacements field with linear functions with respect to the transversal coordinate leads to the shear locking, i.e. overestimation of the flexural rigidity. The effect is common to Mindlin plates [2] and a Finite Element model built for linear shape functions [3]. However, if we assume the dependence of the displacements on the transversal coordinate to be a polynomial of a higher order, the plates' vibrations are described correctly. This statement is proven algebraically.

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Determination of conformal invariants of surface with boundary via its DN map

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As is well-known, a surface M with a given boundary is determined by its Dirichlet-to-Neumann (DN) map up to a conformal diffeomorphism β that does not move the boundary points. The space $\mathcal{T}_{m,\Gamma}$ of conformal classes $[M]$ of surfaces M with given genus m and boundary $\partial M=\Gamma$ (diffeomorphic to a circle) is endowed with the canonical Teichmüller metric while the space $\mathcal{D}_{g,\Gamma}$ of the DN maps of such surfaces is endowed with the metric $d(\Lambda,\Lambda')=\|\Lambda-\Lambda'\|_{H^1(\Gamma)\to L_2(\Gamma)}$. In [1], it is proved that

that the map $\mathcal{R}: \mathcal{D}_{g,\Gamma} \to \mathcal{T}_{g,\Gamma}$ that relates a conformal class of a surface to its DN map is continuous (moreover, it is pointwise Lipschitz, see [2]).

The double $\tilde{M}$ of M is the surface obtained by gluing two copies of M along the boundaries. The conformal classes of doubles of a surfaces of genus m with boundaries diffeomorphic to a circle constitute the stratum of real dimension $6m-3$ in the moduli space $\mathcal{T}_g$ of surfaces of genus $g = 2m$. Due to the Torelli theorem, the b-period matrix $\mathbb{B}$ of a surface $X \in \mathcal{T}_g$ determines its conformal class; moreover, the period map $(X, l.) \mapsto \mathbb{B}$ (where $l.$ is the choice of a canonical homology basis on X) provides the local coordinates on $\mathcal{T}_g$. In the generic case, there is only one way to cut a double into two conformally equivalent surfaces. Thus, the period matrix $\mathbb{B}$ of the double $X = \tilde{M}$ determines all other conformal invariants of M.

In the talk, we propose the algorithm [3] for determining the b-period matrix of the double of $\tilde{M}$ of M via its DN-map. This algorithm is stable under small (in the operator norm) perturbations of the DN map Λ.

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Advanced software development for modeling transmission/reflection spectra and non-contact diagnostics

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A method for non-destructive diagnostics of semiconductor multilayer films is presented, based exclusively on the analysis of experimental transmission and reflection coefficient spectra (or, when necessary, just one of these spectra). Specialized software has been developed to implement this non-destructive diagnostic technique. For calculating transmission, reflection, and absorption coefficient spectra, the following methods were applied: the Transfer Matrix Method (TMM) for multilayer structures composed of coherent layers and the Random Phase Method (RPM) for multilayer structures consisting of alternating coherent and incoherent layers. Figure 1 shows a comparison of calculated transmission spectra in the 30–130 µm range obtained by both methods for an n-GaN/GaN buffer/sapphire structure (the sapphire substrate was considered as an incoherent layer).

To characterize the dielectric permittivity, we utilized both the Drude model and the Drude–Lorentz model, which provide adequate descriptions of dielectric permittivity for polar and nonpolar semiconductors as well as metals. Additionally, for certain substrates (such as various types of glass), experimental values of the real and imaginary parts of the complex refractive index were incorporated into the calculations. The non-destructive diagnostic technique involves solving the inverse problem by fitting theoretical spectra to experimental ones. This is accomplished by systematically varying film parameters (specifically free electron concentration and mobility) and layer thickness. We solved the inverse problem using three distinct optimization methods: The Nelder–Mead method (simplex method), Bayesian optimization and Genetic algorithm. A comprehensive comparison of these methods was conducted on a standardized set of test problems. All calculations were implemented using our custom-developed software.

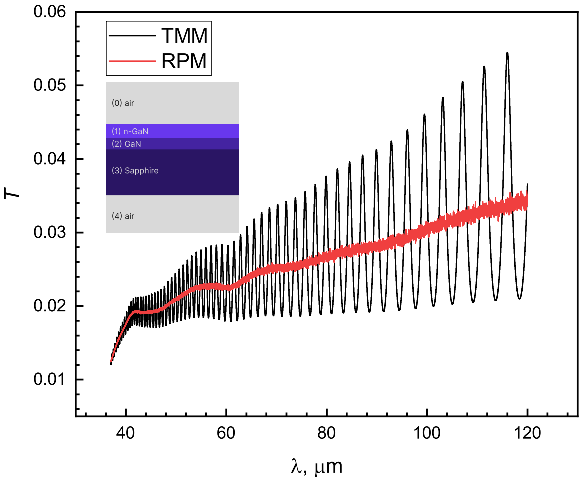

Fig. 1: Comparison of calculated transmission (T) spectra in the range from 30 to 130 µm by TMM (black line) and RPM (red line). The inset schematically depicts the design of the structure.

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A simple range characterization for spherical mean transform in odd and even dimensions

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We present a new and simple range characterization for the spherical mean transform of functions supported in the unit ball in odd and even dimensions. The range description in odd dimensions comprises of a set of symmetry relations between the values of certain differential operators acting on the coefficients of the spherical harmonics expansion of the function in the range of the transform whereas in even dimensions it consists of symmetry relations, using a special kind of elliptic integrals involving these coefficients. As part of the proof, we introduce a pair of identities involving normalized Bessel functions of the first and the second kind. The first result is an integral cross-product identity for Bessel functions for integer and half-integer orders. The second result is a new Nicholson-type identity. As part of the proof of one of the theorems, we derive an interesting equality involving elliptic integrals, which may be of independent interest. Finally, we present some applications of the range characterization in the odd dimensional case. This talk is based on joint works with Divyansh Agrawal, Gaik Ambartsoumian and Nisha Singhal.

Excitation of twisted waves by a circular phased array of loop antennas with the axes tangential to the array circumference in a magnetized plasma

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Over the past decade, a considerable effort has gone into the study of phased antenna arrays immersed in a magnetized plasma [1]. A significant portion of this effort has been devoted to the radiation characteristics of circular arrays that are capable of exciting twisted waves, which have helical phase fronts and carry orbital angular momentum (OAM). The ability of the OAM projection onto the main axis of wave propagation to take an infinite number of discrete values is of significant interest with regard to information transmission and wave–particle interactions in plasmas [1]. An important question to be answered in planning the future satellite experiments with such arrays is that of how to increase the number of selectively excited OAM-carrying waves with desired azimuthal indices and maximum excitation efficiency. The answer to this question consists in increasing the number of radiating elements in the array. Usually, relatively small magnetic loop antennas are used as such elements because of simplicity of their matching with a feeding source. If the axes of loop antennas are parallel to the symmetry axis of a circular array, an increase in the number of even small antennas eventually results in their overlapping [2, 3], which is undesirable in practice. To overcome this issue, it was proposed to orient the loop axes tangentially to the array circumference [2]. However, so far no theoretical analysis has been carried out for such orientation of the loop antennas in the arrays embedded in a magnetized plasma.

In this work, we study the radiation from a circular phased array with its symmetry axis aligned with an external static magnetic field superimposed on the plasma in the case where the radiating elements of the array are small loop antennas whose axes are tangential to the array circumference. We have found an exact solution for the field excited by such an array and determined its total radiated power and partial powers going to different azimuthal field harmonics. Conditions have been ascertained under which the array is capable of selectively exciting twisted waves with given azimuthal indices. Numerical calculations of the radiation characteristics of the described array have been performed in the whistler frequency range for plasma parameters typical of the Earth's ionosphere. The results obtained show that the considered phased array can be used as an efficient source of twisted whistler waves with the desired helicity of phase fronts in a magnetized plasma.

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Inverse scattering for the multipoint potentials of Bethe–Peierls–Thomas–Fermi type

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We consider the Schrödinger equation with a multipoint potential of the Bethe–Peierls–Thomas–Fermi type. We show that such a potential in dimension $d = 2$ or $d = 3$ is uniquely determined by its scattering amplitude at a fixed positive energy. Moreover, we show that there is no non-zero potential of this type with zero scattering amplitude at a fixed positive energy and a fixed incident direction.

Nevertheless, we also show that a multipoint potential of this type is not uniquely determined by its scattering amplitude at a positive energy E and a fixed incident direction. Our proofs also contribute to the theory of inverse source problem for the Helmholtz equation with multipoint source.

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Sloshing in vertical cylinders with circular walls and porous, radial baffles: examples of explicit solutions

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Several examples of explicit solutions are obtained for the sloshing problem in a circular/annular vertical cylinder containing one or more porous, radial baffles stretched throughout the constant (possibly infinite) depth of an inviscid, incompressible, heavy fluid. The fluid transmission across the baffle is modelled using Darcy's law (see, for example, [2]); namely, the velocity of the fluid at the baffle is directly proportional to the pressure difference across the baffle itself.

For the simplest geometry shown in fig. 1, the problem formulation is as follows:

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \ \text{ in } W, \quad (1)$$

$$\phi_z = \nu\phi \ \text{ on } F = \{(r,\theta,0) : r \in (0,1),\, \theta \in (0,2\pi)\}, \quad (2)$$

$$\phi_z = 0 \ \text{ on } B = \{(r,\theta,-h) : r \in (0,1),\, \theta \in (0,2\pi)\}, \quad (3)$$

$$\phi_r = 0 \ \text{ on } S = \{(1,\theta,z) : \theta \in (0,2\pi),\, z \in (-h,0)\}, \quad (4)$$

$$\phi_y(r,0,z) = \phi_y(r,2\pi,z) = \mathrm{i}\beta\,\omega[\phi(r,0,z) - \phi(r,2\pi,z)] \quad \text{for } r \in (0,1),\ z \in (-h,0). \quad (5)$$

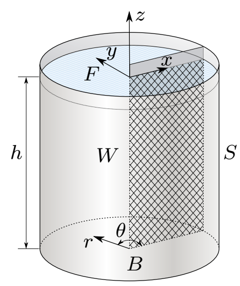

Fig. 1: A sketch of the circular cylinder with a baffle.

Here the velocity potential ϕ is assumed to be complex valued and belonging to the Sobolev space $H^1(W)$. Moreover, the frequency ω is also complex as well as the coefficient $\beta = \beta_r + \mathrm{i}\beta_i$ ($\beta_r, \beta_i > 0$), which characterizes the porosity of the baffle; g is the acceleration due to gravity.

A characteristic feature of the obtained explicit solutions is the fact that across each of the baffles the corresponding pressure difference vanishes. Therefore, these solutions coincide with those that describe sloshing in the case of rigid baffles. It means that porous baffles with any β are equivalent to the rigid ones having the same configuration. The damping efficiency of the latter baffles is due to changes of the velocity field in the fluid and this was investigated by the authors earlier; see, for example, [1, fig. 6]. Hence the damping efficiency of porous baffles is the same as that of rigid ones at the frequencies used in the considered examples. The obtained eigenfrequencies are real, and so the corresponding sloshing oscillations are non-decaying in time. At the same time, the problem's formulation is complex valued and this suggests that generally speaking oscillations should decay exponentially in time; see, for example, the model investigated in [2].

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Dispersion of optical vortices in twisted anisotropic optical fibers with torsional mechanic stress

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We have analytically studied topological, polarization and hybrid dispersions of optical vortices (OVs) in a twisted anisotropic fiber (TAF) with torsional mechanical stresses (TMS) in the presence of the spin-orbit interaction (SOI). We have shown that for particular choice of parameters, the modes of TAF with TMS are represented by either linearly polarized OVs or by circularly polarized OVs. We have demonstrated that in the case of prevailing anisotropy, the dispersion for OVs with topological charges (TCs) ± 1 and the same polarization can be significantly suppressed resulting in group delay times of the order of several femtoseconds. Also for such fibers the magnitudes of this topological dispersion for other values of TC comply with the standard requirements for conventional single-mode fibers. We have also found that, at dominating twisting, one can control by changing the twist pitch value the position of zero dispersion point. By selecting the fiber's parameters it is possible for a chosen group of OVs to make simultaneously zero all the three types of dispersion near the second transparency window.

About some specifications of the Landau problem in the Trubnikov–Godfrey Relativistic formalism

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The Landau problem is newly studied in the covariant formulation the Trubnikov–Godfrey formalism. The perturbation functions of the plasma electrons' velocities equilibrium Juettner–Singe distribution are newly found to be not arbitrary, but constrained after the Relativistic treatment with Langmuir dispersion relations. Applications to Relativistic winds from pulsars are newly provided with.

On the Trubnikov–Lamborn Relativistic plasma dispersion relations

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The Trubnikov dispersion relations are analyzed in the Lamborn notation. The linearized regime is newly studied. It is here newly explained that the non-Relativistic-temperatures limit is not enough

to set the non-relativistic limit of the dispersion relations, not even in the linearized regime. The request is here newly stipulated, that the non-relativistic limit of the heat function per unit plasma volume should be calculated; after these tasks, it is correct to take into consideration the rest mass only: i.e., the plasma is newly requested to undergo the regime of radial flow. Applications to the study of cosmic rays are recapitulated.

Asymptotics of the fundamental solution describing ballistic heat transport in a harmonic square lattice

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We investigate ballistic heat transport (a regime, which is observed in nanomaterials (see, e.g., [1, 2]) and at which quasiparticles (phonons) do not interact) within a harmonic square lattice with nearest-neighbor interactions. The initial conditions correspond to an instantaneous point heat pulse with no heat flux. This problem was previously solved in [3] concerning the kinetic temperature in the continuum approximation obtained by using the covariance approach. The present study aims to clarify this solution by deriving it in closed form.

An approach to derive the continuum fundamental solution for the kinetic temperature via asymptotic estimation of the exact discrete solution for the particle velocity field on a moving front, proposed by S. N. Gavrilov in [4] for the harmonic monoatomic chain, is extended to a harmonic square lattice. By employing the stationary phase approximation [5], a closed-form solution for the particle velocity is obtained. It is shown that wave propagation in the lattice generally exhibits two fronts. The particle velocity approximation is used to derive the kinetic temperature field in both discrete-continuum and continuum formulations. It is demonstrated that the continuum solution decays as $1/t^2$ (where t is time) everywhere except at the fronts, and as $1/t^{3/2}$ at the corners of the front closest to the source.

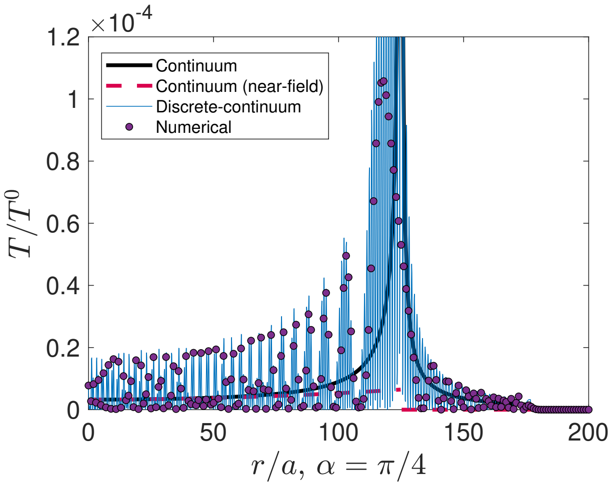

Fig. 1: Field of the kinetic temperature on the main diagonal of the square lattice.

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The history of superconductivity: from Nobel prizes to prison

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We present the history of the discovery and study of high-temperature superconductivity, which spans more than a century: from the creation of a gas liquefaction plant and the production of liquid helium (Tc = 4.2 K) by Heike Kamerlingh-Onnes (1853–1926) at the Leiden cryogenic laboratory (Netherlands) in 1908 to the demonstration of superconductivity at room temperature in 2020 [1]. It is a century-long history of the impressive development of technology, experiment and theory, and the history of six Nobel Prizes.

In 1911, Kamerlingh-Onnes first observed a sharp drop in the resistance of a mercury ring at temperatures below 4.2 K. This phenomenon was called superconductivity. However, at that time, the discovery of superconductivity itself was not appreciated and in 1913 Kamerlingh-Onnes was awarded the Nobel Prize in Physics with the following wording: “For his investigations on the properties of matter at low temperatures which led, inter alia, to the production of liquid helium”. Kamerlingh-Onnes earned the honorary nickname “Mr. Absolute Zero” from his colleagues.

Over the next 60 years, superconductivity was discovered in many metals such as Pb, Nb and binary compounds such as Nb_3Al, V_3Si. A huge amount of experimental material was accumulated, which made it possible to create a theory that explained superconductivity by the phonon mechanism. This theory was named “BCS theory” after the names of three American scientists J. Bardeen, L. N. Cooper and J. R. Schrieffer, who worked independently, but together they found an answer to the mystery of the mechanism of superconductivity (1957) [2]. In 1972, the Nobel Committee awarded them the prize with the wording “for the creation of the theory of superconductivity, usually called the BCS theory”. Among the numerous facts associated with this discovery, we note three. Firstly, John Bardeen (1908–1991) was the only person to be awarded the Nobel Prize in Physics twice: first in 1956 with William Shockley and Walter Houser Brattain for their invention of the transistor; and again for the BCS theory. Secondly, Leon Cooper (1930–2024) developed the concept of electron pairs, named Cooper pairs in his honor. The concept of Cooper pairs, i.e. the transition from fermions (electrons) to bosons (bound pairs of electrons) for which Bose condensation is possible, explained the transition to the superconducting state. Thirdly, the third Nobel laureate, John Schrieffer (1931–2019), was a malicious violator of traffic rules and his driver’s license was taken away. In 2004, a sports car driven by 74-year-old John Schrieffer crashed into a minibus carrying several people at 100 miles per hour (160 km/hour), killing one of the minibus passengers and injuring seven others. The court found Schrieffer guilty of negligent homicide and sentenced him to two years in prison, which he spent entirely in prison, continuing to write scientific articles.

Brian Josephson (born 4 January 1940), as a graduate student, at the age of twenty-two, theoretically predicted the phenomenon of electrons passing through a thin layer of dielectric, placed between two superconducting metals (later called the Josephson effect). The effect was discovered experimentally in 1963, and in 1973, at the age of 33, Brian Josephson became a Nobel Prize laureate in physics.

As is known, there can be three Nobel Prize winners each year. Therefore, in 1973, half of the monetary reward was received by Josephson, and the other half was equally divided between

Leo Esaki and Ivar Giaever "for their experimental discoveries regarding tunneling phenomena in semiconductors and superconductors, respectively". It is interesting to note that Leo Esaki was born on March 12, 1925 and celebrated his 100th birthday this year.

In 1978, Pyotr Leonidovich Kapitsa was awarded the Nobel Prize in Physics "for fundamental inventions and discoveries in the field of low-temperature physics". In fact, it was a one-page article reporting the discovery of superfluidity: "The important fact that liquid helium has a specific density ρ of about 0.15, not very different from that of an ordinary fluid, while its viscosity μ is very small comparable to that of a gas, makes its kinematic viscosity $\nu = \mu/\rho$ extraordinarily small" [3]. Contrary to tradition, Pyotr Kapitsa devoted his Nobel speech not to the works that were awarded the Nobel Prize, but to modern research. He referred to the fact that he had moved away from issues in the field of low-temperature physics about 30 years ago and was now fascinated by other ideas. Kapitsa's Nobel speech was called "Plasma and Controlled Thermonuclear Reaction".

A new stage in the history of superconductivity began in 1986, when Johannes Bednorz (born 16 May 1950) and Karl Alexander Müller (1927–2023) published a paper on superconductivity in the perovskite-like compound $La_{2-x}Ba_xCuO_4$ with $T_c \sim 30$ K. This work was rejected by Nature and published in Zeitschrift für Physik B [4]. Building on Bednorz–Müller's paper, less than a year later C. W. Chu's group published an article in Phys. Rev. Letters announcing superconductivity in a perovskite Y-Ba-Cu-O structure at 93 K [5]. The described structure was called the YBaCu and it has generated unprecedented interest among researchers in a wide range of fields of technology, experiment and theory. The fact is that superconductivity occurs at a temperature higher than the temperature of liquid nitrogen T = 77 K, which can be kept in Dewar vessels even at home. Making YBaCu ceramics is also very simple, by purchasing three oxides, mixing them in the right proportions and baking the ceramic samples in the kitchen on a gas stove. That's why hundreds of scientific laboratories around the world have switched their research to searching for new high-temperature superconductors. It is not surprising that the Nobel Prize was awarded to Bednorz and Müller in 1987 ("for their important breakthrough in the discovery of superconductivity in ceramic materials"), less than a year after the discovery of superconductivity in perovskites; this case has no precedent in the history of Nobel Prizes in Physics.

Superconductivity was later discovered in more complex compounds with a perovskite structure, namely in $Bi_2Sr_2Ca_{n-1}Cu_nO_{2n+4}$, ($T_c \sim 108$ K) [6], $Tl_2Ba_2CaCu_3O_{10}$, $T_c \sim 125$ K [7], $HgBa_2Ca_{n-1}Cu_nO_{2n+2+x}$, $T_c \sim 135$ K [8].

The last, sixth Nobel Prize in the field of superconductivity was awarded in 2003 to Alexey Abrikosov (1928–2017), Vitaly Ginzburg (1916–2009) and Anthony Leggett (born 26 March 1938) "for the creation of the theory of superconductivity of the second kind and the theory of superfluidity of liquid helium-3".

As noted at the very beginning, in 2020, a paper was published in the Nature [1] reporting the discovery of superconductivity at temperature of 287±1.2 K and pressure of 267±10 gigapascals. It seems that the experimental part of the work on superconductivity is nearing completion, while new discoveries can be expected in theory and practical application.

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Three identical quantum particles on a straight line with contact interaction in pairs as a solvable problem

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We study spectral properties of the Hamiltonian which is semi-bounded and corresponds to three 1D quantum particles with delta-interaction in pairs. The particles are assumed to be identical. We say that a problem at hand is solvable if one could construct the spectrum and (generalised) eigenfunctions of the corresponding Hamiltonian in an explicit form (i.e. by quadrature).

In this report we study the discrete part of the spectrum and construct the corresponding eigenfunctions. The continuous spectrum is also tractable and will be considered elsewhere. Our approach makes use of the symmetry of the problem, because the particles are identical, as well as appropriate integral representations for the solutions, reduction to functional equations and some explicit analysis.

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Automatically differentiable parabolic equation for solving a class of inverse tomography problems

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Most methods for solving inverse problems are based on minimizing a certain functional using one or another optimization algorithm [1]. This functional incorporates the forward problem operator (in our case, solving a parabolic equation in an inhomogeneous medium). The task is to determine the parameters of the wave propagation medium based on measurements of the wave field at certain points in space. Typically, optimization algorithms require repeated computation of the gradient of the considered functional. The gradient must be computed quickly and remain stable.

To numerically compute the gradient, we propose using the method of automatic differentiation [2], applied to a numerical scheme. The step-by-step numerical scheme for solving the parabolic equation is represented as a computational graph. The sought-after functions corresponding to the medium parameters are approximated using deep neural networks. For optimization, the L-BFGS-B and Adam algorithms were employed. It is shown that the use of neural networks significantly enhances the performance of local optimization methods in finding the global minimum. In fact, neural network methods manage to find the global optimum in a highly nonlinear, non-convex, large-dimensional problem. Unlike the Physics-Informed Neural Networks (PINN) approach, the proposed method is not only "physics-informed" but also "numerical method-informed".

The presentation will include simulation results for the inversion of the tropospheric refractive index [3] and the inversion of the underwater sound speed profile [4].

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On the existence of elastic waves in topographic waveguides

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The existence of elastic waves with displacements localized at the tip — for isotropic topographic waveguides with an aperture angle less than $\frac{\pi}{2}$ — was rigorously proven by V. M. Babich in [1]. We extend Babich's result by proving the existence of localized waves for interior angles exceeding $\frac{\pi}{2}$. Furthermore, our ansatz allows us to separately analyze the existence of symmetric and antisymmetric modes.

This study was supported by the Russian Science Foundation No. 24-21-00286.

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Dynamic inverse problem for Jacobi matrices and complex moment problem

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We establish the relationships between the dynamic inverse problem for a discrete-time dynamical system associated with the complex Jacobi matrix and the complex moment problem. In the finite-dimensional case we give sufficient conditions for a sequence of complex numbers to be a sequence of moments of some Borel measure on $\mathcal{C}$.

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Four sets of resonance frequencies of piezoelectric bodies and quality factors for vibrations at these frequencies

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Boundary value problems for describing steady-state oscillations of piezoelectric bodies with a circular frequency ω include a coupled system consisting of dynamic equations of motion and the equation of quasi-electrostatics of dielectrics; constitutive relations; equations for strain fields and the electric field, and boundary conditions. As a result, for a three-dimensional problem, a system of four equations can be obtained for three components of the complex amplitudes of the displacement vector $u_i(\mathbf{x})$ and the electric potential $\varphi(\mathbf{x})$.

Let us assume that when a piezoelectric body operates as a receiver (sensor), the vector of mechanical stresses $p_{\Gamma i}$ is applied to the part of its boundary Γ^m, or the displacement $u_{i\Gamma}$ is considered known (velocity $v_{\Gamma i} = j\omega u_{\Gamma i}$). Then we will assume that due to the piezoelectric effect, the electric potential φ_Γ and/or electric charge Q_Γ (current $I_\Gamma = j\omega Q_\Gamma$) are generated on the electrode boundary Γ_m.

On the contrary, when a piezoelectric body operates as a emitter (actuator), it can be assumed that the external influences are the electric potential φ_Γ or electric charge Q_Γ known on the electrode boundary Γ^e, and acoustic waves are generated into the external medium through the boundary Γ^m, i.e. $p_{\Gamma i}$ and/or $u_{\Gamma i}$.

In this case, an important role is played by the values of the acoustic (mechanical) impedance (resistance) Z^m_{il} or acoustic admittance (conductivity) Y^m_{il} ($\mathbf{Y}^m = (\mathbf{Z}^m)^{-1}$) and the values of the electrical impedance (resistance) Z^e or electrical admittance (conductivity) Y^e ($Y^e = (Z^e)^{-1}$), for which, with the accepted notations, the following formulas can be written: $p_{i\Gamma} = j\omega Z^m_{il} u_{\Gamma l}$, $\varphi_\Gamma = j\omega Z^e Q_\Gamma$.

Then, in the absence of damping, we can consider four sets of resonance frequencies, considering some external influence fixed:

1) $\omega \to \omega^m_{rk}$, $p_{i\Gamma} = \text{const}$, $||\mathbf{Y}^m|| \to \infty$, $||\mathbf{Z}^m|| \to 0$, $|\mathbf{u}|_\Gamma \to \infty$;

2) $\omega \to \omega^m_{ak}$, $u_{i\Gamma} = \text{const}$, $||\mathbf{Z}^m|| \to \infty$, $||\mathbf{Y}^m|| \to 0$, $|\mathbf{p}|_\Gamma \to \infty$;

3) $\omega \to \omega^e_{rk}$, $\varphi_\Gamma = \text{const}$, $|Y^e| \to \infty$, $|Z^e| \to 0$, $Q_\Gamma \to \infty$;

4) $\omega \to \omega^e_{ak}$, $Q_\Gamma = \text{const}$, $|Z^e| \to \infty$, $|Y^e| \to 0$, $\varphi_\Gamma \to \infty$.

For electrically active resonance modes, the frequencies ω^e_{rk} are called electrical resonance frequencies, and the frequencies ω^e_{ak} are called electrical antiresonance frequencies. However, similar terminology is not usually used for the acoustic frequencies ω^m_{rk} (acoustic resonance frequencies) and ω^m_{ak} (acoustic antiresonance frequencies).

The paper presents the basic mathematical properties of all the above-mentioned resonance frequencies and the corresponding eigenvectors, as well as the properties of the change in these frequencies when the sizes of the Γ^m and/or Γ^e boundaries change.

Further, for some canonical transducers operating on pronounced one-dimensional oscillation modes, the maxima of the output value amplitudes near the resonance frequencies for problems with damping are analyzed. It is shown that these maxima are expressed through the corresponding geometric and material parameters, which vary depending on the body geometry and types of boundary conditions. It is noted that the characteristic parameters and quality factors determining the efficiency of transducer operation can be optimized by selecting piezoceramic materials, including porous piezoceramics, as well as external effects.

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Localization of eigenfunctions in a thin-walled faceted Dirichlet glass

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Let ω_N be a regular N-polygon inscribed into the unit disk. For $N > 3$, we set

$$\omega_N^\varepsilon = \{y \in \omega_N : \mathrm{dist}(y, \partial\omega_N) < \varepsilon\},\ \Gamma_{NH}^\varepsilon = \omega_N^\varepsilon \times (0, H) \subset \mathbb{R}^3 \ni x = (y, z),$$

$$\Omega_{Nh}^\varepsilon = \{x : \, y := (x_1, x_2) \in \omega_N, z := x_3 \in (0, h\varepsilon)\} \, .$$

where H, h and ε are variable, variable and small positive parameters, respectively. In the domain $\sqcup_{NHh}^\varepsilon := \sqcup^\varepsilon = \Omega^\varepsilon \cup \Gamma^\varepsilon$ (a faceted glass), we consider the Dirichlet problem for the Laplace operator

$$-\Delta_x u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x),\ x \in \sqcup^\varepsilon, \quad u^\varepsilon(x) = 0,\ x \in \partial\sqcup^\varepsilon, \tag{1}$$

The main result to be presented in the talk reads: there exists $h_* \in \big(1, \sqrt{3}\big)$ such that, for $h \in \big(1/h_a st, h_a st)$ and a small $\varepsilon \in (0, \varepsilon(h_*)]$ the first N eigenfunctions of problem (1) are localized near the vertices of the bottom $\omega_N \times (0, h)$ and decay at exponential rate at a distance from them.

The key tool for the examination of the spectrum of problem (1) is to study of the essential and discrete spectra of the Dirichlet problem in the "thick" trihedral angle which generalizes methods of papers [1–3].

Several open questions will be formulated, in particular, one related to the case $N = 3$.

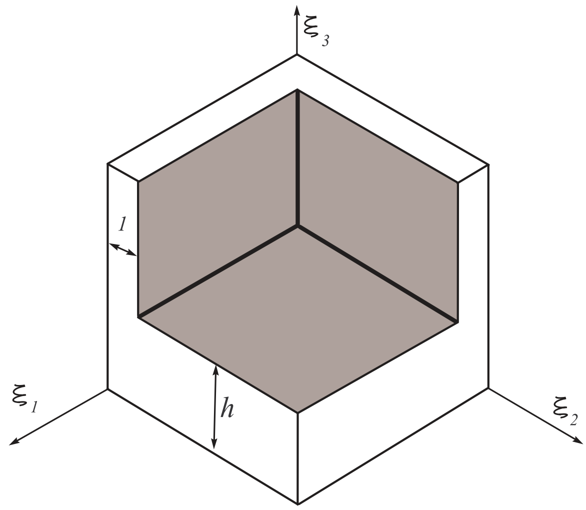

Fig. 1: "Thick" trihedral angle.

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Matrix representations of the group of formal diffeomorphisms and their applications

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Ray transfer matrices have been known for a long time in geometrical light and charged particle optics as a convenient computation tool for analizing the influence of the essential characteristic parameters such as lens focal distances, on the first order properties of optical systems, see [1]. They are often used in X-ray optical system, for example in EUV lithography, to give first preliminary estimations of projection systems [2]. Using matrix optics we can obtain a description of an optical system based on the properties of its constituting components by means of simple matrix multiplications. As optical designs in X-ray mirror optics become more and more sophisticated, with extensive use of aspherics, there arises a demand for tools which would allow one to advance beyond the first order properties and to estimate the influence of higher order parameters on higher order properties of imaging systems.

In this communication we present an approach based on the following premise: the analyzed system is divided into subsystems, each containing, if possible, one optical element. The optical mapping of the entire system is obtained by means of composition of mappings of the separate subsystems. For this end we require (i) a method for the composition of the entire system from its constituting subsystems and (ii) a method for the calculation of optical mappings of single subsystems. To achieve the first goal we propose a generalization of matrix optics to arbitrarily high nonlinear orders by means of matrix representation of formal diffeomorphism groups (in case of light and particle optics they are actually symplectomorphisms), based on a remark made in [3], Section 3. The calculation of optical mappings of single subsystems depends heavily on the physical law describing the system behaviour. In the case of light optics we have to analyze geometrical quantities like intersection conditions of rays with a mirror surface, whereas in the case of particle optics we calculate higher order variations for some suitable first order differential equations. For the latter case we use the Duhammel's principle formulated in a very abstract setting of Banach spaces to show some unexpected connections with a work of Li and Sinai in mathematical fluid mechanics [4].

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Mode transformation near degeneracy points of the crossing type for the 2D Dirac equation

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We study solutions of the two-dimensional Dirac equation describing the wave function of an electron in graphene immersed in external static electric and magnetic fields. We assume that the external fields are smooth and gradually varying functions of one variable x and that the semiclassical (also called adiabatic, WKB) approximation can be applied. This approximation implies a solution in the form

$$\mathbf{\Psi}_k(x,\hbar) = e^{\frac{i}{\hbar}\int^x \beta_k(\tilde{x})d\tilde{x}}(\boldsymbol{\varphi}_k(x) + \varepsilon\Phi_k^{(1)}(x) + \ldots), \qquad k = 1,2. \tag{1}$$

Here β_k and φ_k are the eigenvalue and the two-component eigenfunction, respectively, of the spectral problem obtained in the leading order in ϵ after substituting (1) into the Dirac equation.

We have found all the terms of the expansion (1) and shown that the higher order terms of this expansion have singularities at the degeneracy point $x = x_*$, that is, at the point, where the two eigenvalues coincide $\beta_1(x_*) = \beta_2(x_*) = 0$. We determine the direction of propagation of charge carriers by the sign of

$$J_k = (\boldsymbol{\varphi}_k, \sigma_1\boldsymbol{\varphi}_k), \tag{2}$$

$k = 1, 2$, σ_1 is the Pauli matrix. Note that β_k has the meaning of the momentum of a charge carrier, and the two solutions describe the wave function of carriers moving in opposite directions.

Our goal is to find the solution near the degeneracy point and calculate the scattering matrix. We use the method of matched asymptotic expansions.

We study two cases assuming that

$$\beta_2(x) - \beta_1(x) = 2Q(x - x_*)(1 + o(1)) \tag{3}$$

for $x \to x_*$, where Q is some positive constant. These cases differ in the behavior of the eigenfunctions when $x \to x_*$. The eigenfunctions $\boldsymbol{\varphi}_k(x_*)$, $k = 1, 2$ can become linearly dependent at $x \to x_*$ or remain linearly independent. As a result, we have two different scattering processes. Although the boundary layer width near x_* is of the same order $\varepsilon^{1/2}$ in these cases, the special function describing the internal expansion is different. The reflection and transfer coefficients are found in both cases.

The asymptotic expansions of the similar nature for another problems were presented in [1, 2], and [3].

The Dirac operator, which is a perturbation of the operator with degeneracy points of the crossing type, is also studied and the corresponding scattering matrix is found. Some applications related to electron-hole transmutations in graphene are discussed.

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Modified “complex source” solutions which are regular in whole space

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Relatively nondistorted waves (RNW) [1], play an important role in the theory of localized solutions of the wave equation

$$u_{tt} - c^2\Delta u = 0\,, \tag{1}$$

where $c > 0$ is the of wave speed, and $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ is 3D Laplacian. They are solutions of (1) having a form

$$u = gf(\theta)\,, \tag{2}$$

where an *amplitude* $g(t,\mathbf{R})$ and a *phase* $\theta(t,\mathbf{R})$ are fixed functions of time t and coordinates x, y, z, aggregated into a vector $\mathbf{R}$, and a *waveform* f is an arbitrary function of θ. In case of complex-valued function θ, f is an analytical function, which is regular in the area of values of θ [2].

An important example of RNW is the "complex source" family [3], for which $g = 1/R_*$, $\theta = R_* - ct$, where

$$R_* = \sqrt{r^2 + (z - ia)^2}\,, \tag{3}$$

$r = \sqrt{x^2 + y^2}$, and a is a real constant. In this case,

$$u(t,\mathbf{R}) = \frac{f(R_* - ct)}{R_*}\,. \tag{4}$$

For $a = 0$, (4) reduces to outgoing spherical wave generated by point source at the origin. For $a \neq 0$, the expression (3) branches on a circle $C = \{z = 0, r = |a|\}$, and has jumps on a 2D manifold (depending on definition of the branch cut) with the edge C. This surface is said to be antenna with sources, whose distribution has a rather complicated character [3].

In this paper, a simple modification of the function (4), is given, allowing to obtain a family of solutions of (1), which are regular in whole space-time. This family includes both earlier known and new solutions of (1). This modification has a form

$$u(t,\mathbf{R}) = \frac{f(R_* - ct) - f(-R_* - ct)}{R_*}\,. \tag{5}$$

For $a = 0$, (5) is a difference of two spherical waves, it satisfies homogeneous equation (1).

If $a \neq 0$, for any waveform f which is regular in a stripe $|\operatorname{Im}\theta| \le |a|$, the expression (5) represents a single-valued function which does not depend on choice of the branch of the square root in (3). On the circle C, the numerator and denominator vanish simultaneously, and (5) has a limit equal to $2f'(-ct)$. Thus, (5) continuous and satisfies (1) for all t and $\mathbf{R}$. The character of space-time localization of the solution (5) is determined by localization of the waveform in the stripe $|\operatorname{Im}\theta| \le |a|$. For localized solutions, some integral representations are obtained. If the derivative f' is square-integrable in a stripe mentioned above, the solution (5) has finite energy equal to

$$\mathcal{E} = \frac{2\pi}{a}\int_{-\infty}^{\infty} d\xi \int_{-|a|}^{|a|} |f'(\xi + i\eta)|^2\, d\eta\,. \tag{6}$$

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A finite-energy unidirectional solution of the wave equation with unexpected behavior at infinity

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In Mathematical Physics, frequently arise solutions of the wave equation

$$u_{xx} + u_{yy} + u_{zz} - \frac{1}{c^2}u_{tt} = 0, \quad c = const > 0, \tag{1}$$

that behave at infinity as a diverging spherical wave. This means that with a consistent removal to infinity in the direction of the unit vector $\mathbf{n}$ of a point (x, y, z), the asymptotics is

$$u \approx \frac{F(R - ct, \mathbf{n})}{R}, \quad R \to \infty, \quad t \to \infty. \tag{2}$$

Here, $R = \sqrt{x^2 + y^2 + z^2}$ is the length of the vector $\mathbf{R} = (x, y, z) \in \mathbb{R}^3$, and $\mathbf{n} = \mathbf{R}/R$ is the unit vector of the direction in which the point (x, y, z) moves to infinity.

In [1], a simple example of a function with finite energy is constructed, for which the right-hand side of (2) for all $\mathbf{n}$ is $O(\ln R/R)$. Here, we give an example of a function that has a finite energy, and in some directions has the asymptotics (2) but in others behaves as $O(\ln R/R)$. The example is based on a particular unidirectional solution of (1) found in [2] and having the form

$$V = \frac{1}{(z_* - S)S}, \tag{3}$$

$S(\mathbf{R}, t) = \sqrt{c^2 t_*^2 - \rho^2}$, where $\rho = \sqrt{x^2 + y^2}$, $z_* = z + i\zeta$, $ct_* = ct + i\tau$, ζ and τ are free real parameters obeying the condition $\zeta < c\tau$. The branch of the square root is such that $S|_{\rho=0} = ct_*$. This solution is unidirectional, i.e., its plane-wave constituents have only non-negative projections on the z-axis, which was established in [3]. It is useful in modelling low-cycle optical pulses [4].

Our example of a function with an exotic behavior is given by

$$u(\mathbf{R}, t) = c \int_{-\infty}^{t} V(\mathbf{R}, t')dt' = \frac{1}{\sqrt{z_*^2 + \rho^2}} \ln \frac{ct_* + S - z_* + \sqrt{z_*^2 + \rho^2}}{ct_* + S - z_* - \sqrt{z_*^2 + \rho^2}}. \tag{4}$$

We prove that it has a finite energy, behaves as $\ln R/R$ in every direction $\mathbf{n}$ having a positive projection on the z-axis, as $\ln R/2R$ in every direction orthogonal to it, and has a common asymptotics of the form (2) otherwise.

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G. D. Malyuzhinets' contribution to development of the parabolic equation method

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Being a devoted successor of A. Sommerfeld, M. A. Leontovich and V. A. Fock, Professor Malyuzhinets enthusiastically accepted and developed the physical ideas of the modern diffraction theory. Leaving aside the famous "first after Sommerfeld" exact solution of the diffraction problem by an impedance wedge, it is impossible to overestimate his contribution to the development of the parabolic equation method [1–3]. Although the works of G. D. Malyuzhinets are reflected in a review

[4] and the well-known monographs by F. Tappert (1977) and M. Levy (2000), I dare add some direct impressions from my communication with the outstanding scientist and teacher. A fundamental GDM's contribution was the understanding of the short-wave diffraction as an effect of transversal diffusion of the wave amplitude along the fronts of generalized geometrical optics [3]. Along with a qualitative picture of the phenomenon, this made it possible to give a uniform description of the wave field in the shadow region behind a convex cylinder [5, 6]. The use of the Malyuzhinets parabolic equation in ray coordinates made it possible also to better understand the mechanism of the edge wave occurrence on a sharp edge or a line of curvature discontinuity [7, 8] (this direction is now being actively developed by A. P. Kiselev and E. A. Zlobina [9]). As a graduate student of the Moscow Institute of Physics and Technology (MIPT), I was invited by GDM in 1964 to participate in the development of computational algorithms of wave field calculation using the parabolic equation method. Our work [10, 11] in collaboration with the of Computational Mathematics Department of Moscow State University laid a foundation for using PWE in a wide range of problems in acoustics, optics, and radiophysics (I cannot forget GDM solemnly demonstrating a plot of the field amplitude in an underwater sound duct at the Academic Council of the RAS Acoustics Institute). The transition from a rapidly oscillating wave amplitude to a smoothly changing envelope ensured the numerical efficiency and further development of the method in a wide range of problems. Without intending to review hundreds of articles, we will mention several works initiated by the author under the influence of Malyuzhinets' ideas:

- Low Order Modes Transformation in Smooth Waveguide Couplings [12].
- Generalization of the PWE for EM Waves in a dielectric layer of nonuniform thickness [13].
- Application of the parabolic wave equation to X-ray diffraction optics [14].
- EM wave propagation in curved waveguides (asymptotic analysis and parabolic equation [15].
- Modeling radio wave propagation in tunnels with a vectorial parabolic equation [16].
- Numerical Simulation of EM pulse propagation over nonuniform Earth surface [17].

The PWE ansatz can be considered as the first term of the Helmholtz equation expansion in the power series of the smoothness parameter. This idea was put forward by Malyuzhinets in the 1960s in his MIPT lectures. It received an early analytical implementation in the works of his students [18, 19], and has recently been actively developed by Petrov and Erhardt [20].

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Scattering and radiation of acoustic waves in discrete waveguides with several cylindrical outlets to infinity

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A discrete waveguide is a graph G that consists of several discrete semi-cylinders connected by a finite number of edges and nodes. By a discrete cylinder we mean a graph that is periodic when shifted by a given vector and has a finite periodicity cell. An equation of the form

$$-\operatorname{div} a\nabla u - \mu u = f \tag{1}$$

is considered on the graph G, where the given function f and the unknown function u are functions on the set V of nodes of the graph, and div and ∇ are difference analogs of the corresponding differential operators. The spectral parameter μ is assumed to be real and fixed. The weight function a is defined on the set of edges, is assumed to be positive and stabilizing at infinity with an exponential rate.

The *continuous spectrum eigenfunction* (CSE) is by definition a solution to the homogeneous problem (1) that is bounded and does not belong to $\ell_2(V)$. We construct a basis of CSEs subject to the asymptotics at infinity:

$$Y_j^+ = u_j^+ + \sum_{k=1}^{\Upsilon} S_{j,k} u_k^- + o(1).$$

Here $u_1^+, \dots, u_\Upsilon^+$ stand for *incoming waves*, while $u_1^-, \dots, u_\Upsilon^-$ stand for *outgoing waves*. The matrix $S = \|S_{j,k}\|$ is called the *scattering matrix*.

We establish a well-posted statement of problem (1) with the *intrinsic radiation conditions*:

$$u = c_1 u_1^- + \dots + c_\Upsilon u_\Upsilon^- + o(1).$$

The coefficients c_j are computed by the formulas $c_j = i(f, Y_j^-)_V$, where $(\cdot,\cdot)_V$ is the expansion of the inner product in $\ell_2(V)$ and $Y_1^-, \dots, Y_\Upsilon^-$ is another basis of CSEs given by

$$Y_j^- = \sum_{k=1}^{\Upsilon} (S^{-1})_{j,k} Y_k^+.$$

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The fixed angle inverse scattering problem for Riemannian metrics

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Wave propagation in an inhomogeneous acoustic medium may be modeled, for example, by the wave operators $\square + q(x)$, $\rho(x)\partial_t^2 - \Delta$ or $\partial_t^2 - \Delta_g$, for a function $q(x)$, a positive function $\rho(x)$ or a Riemannian metric $g(x)$, which are homogeneous outside a ball. The medium is probed by plane waves coming from a finite number (dimension dependent) of different directions, and the resultant time dependent waves are measured on the boundary of the ball. We describe our partial results about the recovery of q, ρ, g from these boundary measurements. These are long standing formally determined open problems. These results were obtained in collaboration with Lauri Oksanen and Mikko Salo.

Electric area of a pulse reflected by a thin layer of a medium with finite transverse dimensions

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The electric area of a pulse $\mathbf{S}_E = \int_{-\infty}^{+\infty} \mathbf{E}\, dt$, i.e. the integral of the electric field strength $\mathbf{E}$ over time t, determines the efficiency of the action of extremely short pulses on micro-objects [1]. In this regard, it is important to consider the properties of a vector field and its transformation on optical elements.

The problem of reflection a plane-wave pulse from a mirror in one-dimensional geometry has a simple solution [2, 3], see also [4] and the commentary to it [5]. The results are qualitatively different in cases where the mirror medium is a dielectric or contains free charges. Here, we consider this problem taking into account the finite transverse dimensions of the mirror medium. A quantitative analysis is simpler for a dielectric medium described by the Lorentz model. With a number of restrictions, finding the distribution of the electric area is reduced to the electrostatic problem of the field of two charged wires, which allows for a simple solution. Along with the asymptotics and illustrations of such a distribution, we note that a qualitatively similar distribution is obtained for a medium with free charges.

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On the semi-classical magnetic Schrödinger operator

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Consider a semi-classical quantum particle with an external potential $V(x)$, in a magnetic field $\mu A(x)$. Assume μ a large coupling constant, V has two non degenerate potential wells $x_\pm$ symmetric with respect to an hyperplane, and $A(x)$ vanishes at $x_\pm$. Its Hamiltonian is $P_A(x, hD_x) = (hD_x - \mu A(x))^2 + V(x)$ on $L^2(\mathbb{R}^d)$. The splitting between the low-lying eigenvalues of $P_A(x, hD_x)$ is related with the local L^2 decay of its first eigenfunction in the classically forbidden region (Agmon estimates). By a well-known localization procedure, it suffices to prove such a decay for the "one well problem" where we implement $P_A(x, hD_x)$ by Dirichlet boundary condition on some sufficiently large open set M. Let $\lambda_A(h)$ be its first eigenvalue (that we assume non degenerate) and $u_A(h)$ its corresponding eigenfunction.

For Schrödinger operator $P_0(x, hD_x) = -h^2\Delta + V(x)$ without a magnetic field, the rate of exponential decay of the first eigenfunction $u_0(h)$ associated with $\lambda_0(h)$ is related with a conformal (degenerate) Riemannian metric, so-called Agmon metric $\Phi_0(x) = d_0(x, U_0(h))$ where $U_0(h) = \{x \in M : V(x) - \lambda_0(h) \leq 0\}$.

When introducing a vector potential $A(x)$, the eigenfunction $u_A(h)$ still decays, but the geometric interpretation in term of a metric is lost. However Agmon estimates with Φ_0 provide an upper bound for $u_A(h)$, and when μ is large, we can still expect a faster decay.

Consider namely another Agmon metric $\Phi_1(x) = d_1(x, U_1(h))$ where

$$U_1(h) = \{x \in M : \mu^2|A(x)|^2(x) - \lambda_A(h) + \lambda_0(x) \leq 0\},$$

and

$$\Phi_A(x) = \max\{d_0(x, U_0(h)), d_1(x, U_1(h))\},$$

which defines a (degenerate) distance on M.

We define the complex valued function $v(h)$ by $u_A(h) = u_0(h)v(h)$ and its current $J_v = i(v\nabla\overline{v} - \overline{v}\nabla v)$. In particular, when $u_A(h) \neq 0$ in M, then $J_v(h) = |u_A(h)|^2\nabla(\arg u_A(h)^2)$. We prove

Theorem: *Let $u_A(h)$ be the first (possibly complex) eigenfunction of P_A on $L^2(M)$ with non degenerate eigenvalue $\lambda_A(h)$. Assume we have solved for $\Phi_I(x)$ the gauge equation*

$$2\langle J_v(x), \nabla\Phi_I(x)\rangle = \langle J_v(x), A(x)\rangle, \ x \in M$$

Then the following "Agmon estimate"

$$\int_M |\nabla e^{\Phi_A(x)/h} u_A(x; h)|^2\, dx + \int_M e^{2\Phi_A(x)/h}|u_A(x; h)|^2\, dx \leq C_\delta e^{2a(\delta)/h}$$

holds with $a(\delta) \to 0$ as $\delta \to 0$.

Our second result (work in progress) is related to phase-space tunneling. Theory of minimal coupling relates the canonical symplectic 2-form $\sigma = \sum_{j=1}^d d\xi_j \wedge dx_j$ on $T^*\mathbf{R}^d$ by lifting $B(x)$ to $T^*\mathbf{R}^d$ as the symplectic 2-form $\sigma_B = \sigma + \pi^*B(x)$.

The integral curves of $p_0 = \xi^2 + V(x)$ relative to $(T^*\mathbf{R}^d, \omega_B)$ and those of of $p_A(x, \xi) = (\xi - A(x))^2 + V(x)$ relative to $(T^*\mathbf{R}^d, \omega)$ are the same.

Mixing these two points of vue, we can define a new Dynamical System when $V = 0$, but not directly related with the natural Hamiltonian $p_A(x,\xi)$. Namely assume $d = 2$, and let $f_j(x,\xi) = \frac{1}{2}(\xi_j - A_j(x))^2$, $j = 1,2$. Then $\{f_1, f_2\}_B = 0$ everywhere iff the differential of $A(x)$ has determinant 1, and thus the system is integrable with respect to σ_B. This gives a foliation of the energy surface by invariant tori. In case $A(x) = (-x_2, x_1)$ the angle variables on these tori are again (x_1, x_2). If $A(x) = (-x_2, x_1) + A_p(x)$, where A_p a (periodic) vector field on $\mathbf{T}^2$, they form instead a smooth family of Lagrangian manifolds $L(I) = \{f_1 = I_1, f_2 = I_2\}$ supporting quasi-periodic motion with vector frequencies $\omega(I)$. In case $A(x) = -A(-x)$, some of these tori are disjoint in phase-space. This situation is the magnetic analogue of the "tunneling over the potential barrier" in the case of Schrödinger operator on $L^2(\mathbf{T}^2)$, see S. Dobrokhotov and A. Shafarevich (1999).

Analytical representation of heat waves on a finite interval in the framework of the hyperbolic heat equation

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The classic heat equation $\partial_t T - \alpha\Delta T = 0$, where T is temperature, and α is thermal diffusivity, is commonly used for description of heat conduction. The heat equation is obtained from the energy balance equation

$$C\partial_t T + \nabla\cdot\boldsymbol{q} = 0 \tag{1}$$

and Fourier's law $\boldsymbol{q} = -\kappa\nabla T$ (the constitutive equation), where C is volumetric heat capacity, $\boldsymbol{q}$ is the vector of heat flux, and κ is thermal conductivity. The energy balance equation (1) exactly expresses the first law of thermodynamics as applied to heat transfer: the conservation of thermal energy. Fourier's law, in turn, is an approximate relation between the heat flux and temperature. The heat equation violates the principle of causality due to the infinite speed of heat propagation in this model.

The simplest and most "famous" modification of Fourier's law is given by the constitutive equation [1–5]

$$\tau\partial_t\boldsymbol{q} + \boldsymbol{q} = -\kappa\nabla T, \tag{2}$$

where τ is the relaxation time. From the energy balance equation (1) and the modified Fourier's law (2) it follows that the temperature satisfies the so called hyperbolic heat equation

$$\tau\partial_t^2 T + \partial_t T - \alpha\Delta T = 0.$$

Since this equation is hyperbolic, it provides a finite speed of heat propagation, i. e., satisfies the principle of causality, and can describe heat waves. If boundary conditions set the temperature or heat flux behavior, then initial-boundary value problems (IBVPs) for the hyperbolic heat equation can be solved by the Fourier method or Laplace transform.

However, boundary conditions can describe Newton's law, which states that the heat flux at the boundary is directly proportional to the difference in the temperature of the physical system and ambient temperature. In this case IBVPs cannot be formulated for the hyperbolic heat equation, and must be formulated for the system of the equations (1) and (2). And in this case, classic integral transforms are not applicable.

In this paper we solve the IBVPs on a finite interval for the system of the equations (1) and (2) (in one-dimensional space) by the Fokas unified transform method (UTM) [6–8].

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On isospectral potentials on periodic discrete graphs

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We consider discrete Schrödinger operators $H = \Delta + Q$ with periodic (possibly complex valued) potentials Q on periodic graphs $\mathcal{G}$, where Δ is the discrete Laplacian. The spectrum of the operator H is a union of spectra of *Floquet* operators (matrices) $H(k)$ acting on the finite quotient graph $\mathcal{G}_*$ and depending on the parameter $k \in \mathbb{T}^d := \mathbb{R}^d/(2\pi\mathbb{Z})^d$ called *quasimomentum*, where d is the dimension of the periodic graph (i.e., the rank of its period lattice). The family of spectra $\sigma(H(k))$ of $H(k)$, $k \in \mathbb{T}^d$, is called the *Floquet spectrum* of the Schrödinger operator H. The spectrum of $H(0)$ is the *periodic spectrum* of H. Two potentials Q_1 and Q_2 are *Floquet isospectral* (respectively, just *isospectral*), if $\Delta + Q_1$ and $\Delta + Q_2$ have the same Floquet (respectively, periodic) spectrum.

We are interested in the following problem. To what extent is the potential Q determined by the periodic spectrum or by the Floquet spectrum of the Schrödinger operator H on an arbitrary (but fixed) periodic graph? The following partial answers are obtained.

• For a given potential Q there are generically $n!$ complex-valued potentials isospectral to Q, where n is the number of the vertices of the quotient graph. Moreover, for sufficiently large real potentials Q with pairwise distinct values at the vertices of the quotient graph, there are $n!$ *real* potentials isospectral to Q.

• If a real potential Q is isospectral to the zero potential, then $Q = 0$. If a real potential Q is isospectral to the *"degree"* potential æ, i.e., $æ(v) = æ_v$, where $æ_v$ is the degree of the vertex v, and the quotient graph has no loops, then $Q = æ$.

• We provide examples of periodic graphs for which the Floquet spectrum of the Schrödinger operator $H = \Delta + Q$ generically determines the potential Q uniquely up to symmetries in the quotient graph.

The proof of the obtained results is based on the spectral invariants of the Schrödinger operator on periodic graphs from [1]. Our results extend the results of Kappeler obtained for the lattice $\mathbb{Z}^d$ [2, 3] to the case of arbitrary periodic graphs.

This work is supported by the Russian Science Foundation (project No. 25-21-00157).

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On adiabatic evolution generated by a one-dimensional Schrödinger operator. Solutions corresponding to continuous spectrum

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As $\varepsilon \to 0$, we study a solution Ψ to the Schrödinger equation

$$i\frac{\partial \Psi}{\partial t} = -\frac{\partial^2 \Psi}{\partial x^2} + v(x, \varepsilon t)\Psi, \qquad x > 0, \qquad -\infty < t < +\infty, \qquad \Psi\big|_{x=0} = 0, \tag{1}$$

where the potential v is a finite square well that shrinks linearly with time:

$$v(x,\tau) = \begin{cases} -1 & \text{if} \quad 0 \le x \le 1-\tau, \\ 0 & \text{otherwise.} \end{cases}$$

One says that eq. (1) describes adiabatic evolution generated by the stationary operator $H(\varepsilon t) = -\frac{\partial^2}{\partial x^2} + v(x, \varepsilon t)$ with the Dirichlet boundary condition at zero. The spectrum of this operator consists of the (absolutely) continuous spectrum $[0, +\infty)$ and a finite number of negative eigenvalues. With time, the eigenvalues one by one approach the edge of the continuous spectrum and, having reached it, disappear.

For $\varepsilon t \le 1$, a set of solutions to (1), each close at some moment to an eigenfunction of the stationary operator, were constructed and asymptotically studied in [1–3]. One can say that these solutions correspond to the eigenvalues of the stationary operator. Now we construct and asymptotically describe an analogous solution Ψ corresponding to the continuous spectrum of the stationary operator.

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Far-field asymptotics of the Green's function near the Dirac point for a triangular phononic crystal

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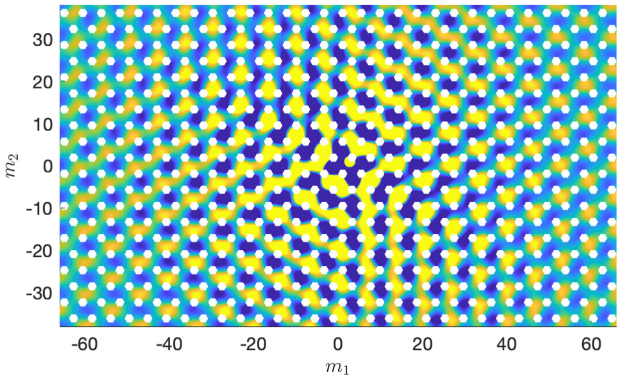
Phononic crystals are structured materials that can be used to control wave propagation. By introducing a repeating pattern at the wavelength scale, these materials can create band gaps– frequency ranges where wave propagation is forbidden — allowing for precise manipulation of wave behavior. This enables applications such as vibration isolation, sound filtering, waveguiding, and energy localization, which are not possible with conventional, homogeneous materials.

In this talk, we examine a triangular phononic crystal formed by a two-dimensional hexagonal lattice of identical inclusions embedded within an otherwise homogeneous medium. Among two-dimensional periodic structures, the triangular (or honeycomb) geometry stands out for its conical Dirac degeneracies. Near the Dirac point, the dispersion relation exhibits linear crossings that lead to interesting phenomena such as a pseudo-diffusion transportation [1]. We develop a rigorous Floquet–Bloch-integral framework for obtaining Green's function far-field approximation for a triangular phononic crystal near Dirac point [2]. Next, we construct a discrete analogue of the problem and solve for its Green's function numerically using finite-element modelling. Finally, we compare the direct modelling with the approximation to make sure that the approximation is valid.

The study was conducted under the state assignment of Lomonosov Moscow State University.



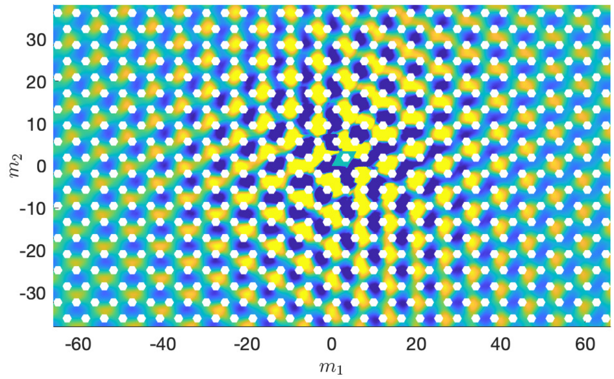


Fig. 1: Direct modelling of the Green's function (left) and asymptotic estimation (right).

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Asymptotic evaluation of three-dimensional integrals with singularities in application to transient acoustic radiation

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The following three-dimensional Fourier integral is considered:

$$u(z;\Lambda) = \int_\Gamma F(\xi)\, \exp\{i\Lambda G(\xi; z)\}\, d\xi, \tag{1}$$

where $\xi = (\xi_1, \xi_2, \xi_3)$ is the triplet of complex integration variables, $z = (z_1, z_2, z_3)$ are spatial or temporal real coordinates playing the role of parameters in the integral, Λ is a large real positive parameter, Γ is a sufficiently regular oriented domain of real dimension 3 (the integration domain), the function G is called the *phase function* of the integral,

$$d\xi = d\xi_1 \wedge d\xi_2 \wedge d\xi_3. \tag{2}$$

Our task is to evaluate the integral (1) as $\Lambda \to \infty$, retaining only terms that do not exponentially decrease with Λ. The asymptotic consideration will be carried out with fixed parameters z, so for brevity, we will replace $G(\xi; z)$ by $G(\xi)$.

It can be shown that the following points can contribute to the integral:

- Stationary phase points of function $G(\xi)$.
- Stationary phase points of the restriction of $G(\xi)$ on a single singularity.
- Stationary phase points of the restriction of $G(\xi)$ on a crossing of two singularities.
- Points of triple crossing of singularities.
- Conical points of singularities.
- Stationary phase points on the single integration domain boundary element.
- Stationary phase points on a crossing of two boundary elements.
- Points of triple crossing of boundary elements.
- Conical boundary points.

In the case of infinite integration domain, topological conditions for the existence of nonzero asymptotics are constructed, and the asymptotics themselves are derived. The proposed technique is tested on the example of the problem of transient acoustic radiation emitted by the point source traveling on the surface of the sine-shaped screen.

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Isotropic surfaces and complex vector bundles corresponding to the Schrödinger equation with a delta potential

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We consider the non-stationary Schrödinger equation with a potential given by the sum of a smooth function and a delta-shaped function. The initial data are chosen as rapidly oscillating wave packets. The Schrödinger operators with delta potentials are defined using the theory of extensions and boundary conditions on the surface supporting the delta singularity.

The semiclassical theory for equations with smooth coefficients was thoroughly developed by V. P. Maslov. However, the existing semiclassical approach is, in general, not applicable to equations with singular coefficients and requires modification.

To describe the solutions, we construct special geometric structures — Lagrangian manifolds or, in the case of complex phases, isotropic surfaces and complex vector bundles over them. These structures, as well as the corresponding asymptotic solutions, must satisfy both the initial conditions and the boundary conditions imposed on the singular support.

The work is supported with a scholarship by the Theoretical Physics and Mathematics Advancement Foundation “BASIS”.

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Derivation of equations of the Cosserat continuum of a special type and their analysis in the context of the Schrödinger equation and the Klein–Gordon equation

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We consider the Cosserat continuum of a special type: it is isotropic, linear elastic and on the linear elastic foundation that is modeled as the body force which is proportional to the displacement vector in the linear momentum balance equation and the body moment which is proportional to the rotation vector in the angular momentum balance equation. We make assumptions that the stress tensor and the moment stress tensor are antisymmetric and the divergence of the displacement vector and the rotation vector is equal to zero.

From the mathematical description of this mechanical model we obtain the equation that generalizes the Schrödinger equation and the Klein–Gordon equation. In contrast to the classical approach, we get the Schrödinger function as a complex vector. Through the introduction of the analogy between the mechanical variables and the quantities which characterize electrodynamic processes we suggest an electrodynamic interpretation of the Schrödinger function. Through the comparison of the obtained equations to the Schrödinger equation and the Klein–Gordon equation we represent parameters of the mechanical model through the physical parameters.

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A slope tomography algorithm based on the high-frequency asymptotics of the Double Square Root equation

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Reflection seismology is the leading exploration method in oil and gas prospecting. Accurate reconstruction of the subsurface velocity distribution is a crucial step in the seismic processing stack since an incorrect velocity model may distort seismic images and mislead the interpreters. We propose a novel slope tomography algorithm based on the high-frequency asymptotics of the Double Square Root (DSR) equation [1]. Our approach resembles the Controlled Directional Reception (CDR) method [2] and differs from it in the ray tracing engine and regularization technique.

Slope tomography is a distinct group of ray-based tomographic methods. Its input data comprises source and receiver coordinates, reflection traveltimes, and their horizontal derivatives [3]. In the CDR method one traces incident and reflected rays downward, stopping as soon as the total traveltime along the rays equals the observed one. When traced in the correct velocity model, the incident and the reflected rays meet at the reflection points. This principle forms the basis for the velocity update algorithm [2].

The DSR equation is a pseudodifferential equation describing the observed wavefield as a function of source and receiver coordinates [4]. In our work, we apply the high-frequency asymptotics of the

DSR equation [1] to describe the rays' propagation and find a suboptimal regularization parameter's estimate. We test our method on a synthetic dataset from [3]. A comparison of the true velocity model and the reconstructed one is shown in the Fig. 1. More examples will be included in the presentation.

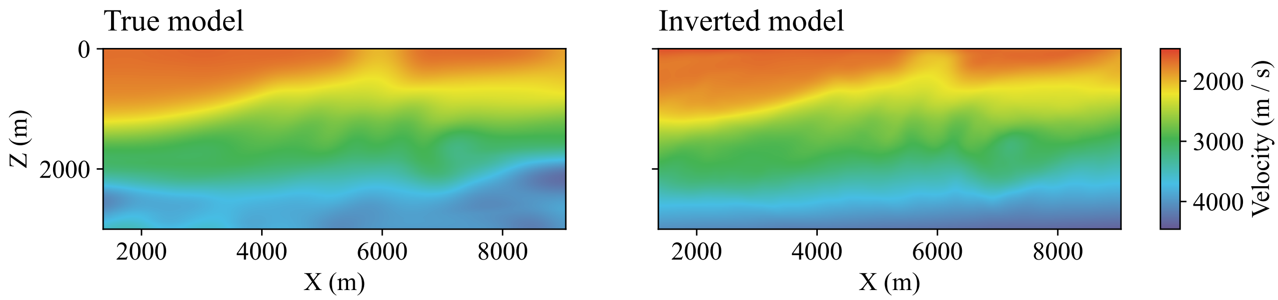

Fig. 1: Inversion result for a synthetic dataset.

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Evolution of travelling wave-fronts in Allen–Cahn model applied to microbial population dynamics

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Mathematical models are the key tools for analyzing the spatio-temporal behavior of bacteria. The classical Kolmogorov–Petrovsky–Piskunov equation, by coupling the diffusion term with the logistic growth term, successfully characterizes the smooth expansion process of bacterial colonies at the periphery in a homogeneous environment through its traveling wave properties [1]. However, the growth of bacteria in real environments may be regulated by more complex factors, resulting in certain non-smooth expansion characteristics. The assumptions of the traditional Fisher model are difficult to fully cover these situations. Therefore, the Allee–Cahn equation, as a generalized form of the Fisher equation, can more accurately describe the spatio-temporal dynamics of bacteria in non-homogeneous environments by introducing a more flexible reaction term. The wave speed of its traveling wave solution becomes the core indicator for analyzing the model behavior. The specific equation form can be expressed as follows:

$$\begin{aligned}\tau_B \frac{\partial B}{\partial t} &= \nabla \cdot (D_B \cdot R \nabla B) + aB(b-B)(B - c_{\max} + \theta \cdot \operatorname{atan}(eN)) \\ &\cdot \frac{1}{2}\left(1 + \tanh\left(\omega(B - c_{\max} + \theta \cdot \operatorname{atan}(eN))\right)\right),\end{aligned} \tag{1}$$

$$\tau_N \frac{\partial N}{\partial t} = D_N \Delta N - h \frac{\partial B}{\partial t}, \tag{2}$$

where $-l/2 < x < l/2$, $-l/2 < y < l/2$, $t > 0$; l is the linear size of the domain in m; $B(x, y, t)$ is the bacterial density in a.u./m^3; $N(x, y, t)$ is the nutrient concentration in a.u./m^3; R is a spatially correlated random function; D_B, D_N are the diffusion coefficients; $\tau_B, a, b, c_{\max}, \theta, e, \omega, \tau_N, h$ are the control positive parameters.

The governing equations (1)–(2) are supplemented by the initial as well as boundary conditions:

$$B(x, y, 0) = B_0 \exp\left(-\frac{x^2 + y^2}{q^2}\right), \tag{3}$$

$$N(x, y, 0) = N_0, \quad -l/2 < x < l/2, \quad -l/2 < y < l/2, \tag{4}$$

$$\left.\frac{\partial B}{\partial \mathbf{n}}\right|_\Gamma = 0, \quad \left.\frac{\partial N}{\partial \mathbf{n}}\right|_\Gamma = 0, \quad t > 0, \tag{5}$$

where B_0, N_0 correspond to the initial values; Γ is the boundary of the computational domain.

A numerical analysis platform was constructed using Matlab, and the sparse matrix storage technology was used to handle the two-dimensional partial differential equation system, achieving efficient solution for the complex reaction-diffusion system.The computational experiments focus on the study of the wave velocity characteristics of the Allee–Cahn model and couples the changes of nutrients in the bacterial environment.

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Semi-analytical solutions of (2+1)-dimensional biological population model using Homotopy Analysis Method

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In this paper, we use the Homotopy Analysis Method (HAM) to find approximate solutions to the (2+1)-dimensional biological Population model governed by Verhulst Law, which describes nonlinear population dynamics with density-dependent growth. This model is represented by parabolic partial differential equation (PDE), captures essential ecological phenomena such as spatial diffusion and carrying capacity limitations. Using HAM, we construct convergent series solutions and validate their accuracy through residual error analysis. The method provides physical interpretable solutions that reveals the relation between population dispersal and logistic growth. These results enhance our understanding of species distribution patterns and offer practical insights for ecological modelling and conservation strategies. The effectiveness of HAM is demonstrated by comparison with known exact solution.

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Asymptotic solution of Maxwell's equation with localized right-hand side

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We consider Maxwell's equation in an inhomogeneous medium with localized right-hand side

$$\hat{p} \times \hat{p} \times E(x,h) + q(x)E(x,h) = F\Big(\frac{x-\xi}{h}\Big), \tag{1}$$

here $E(x,h)$ is the electric field, $q(x)$ describes the inhomogeneity of the medium, h is small parameter, ξ is fixed point, F is fast decaying function. The leading part of the matrix symbol on the left-hand side of the equation has two eigenvalues: $H^0(x,p) = q(x) > 0$ (multiplicity 1, elliptic), $H^1(x,p) = q(x) - |p|^2$ (multiplicity 2, hyperbolic). Far from the source ξ the field is described by a canonical operator on a Lagrangian surface composed of trajectories of the Hamiltonian system emitted from the intersection of the vertical Lagrangian plane with the zero level surface of the Hamiltonian H^1. The far field is defined by the projection of the 3-vector $\widetilde{F}$ onto the two-dimensional eigenspace corresponding to the eigenvalue H^1.

The work was supported by the Russian Science Foundation grant No. 24-11-00213.

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Semiclassical approximation for Jacobi polynomials, defined by a difference equation, and the Bessel function

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We are developing a method for constructing global asymptotics for solutions of difference equations based on the semiclassical approximation [1]. The idea is to reduce the difference equation to a pseudodifferential one and apply the Maslov canonical operator associated with a geometric object in the phase space — the Lagrangian manifold.

While manifolds with a turning point in whose neighborhood the asymptotics is expressed in terms of the Airy function are well studied, the methods for the case when the asymptotics is determined by the Bessel function are not so well developed.

In the talk we will illustrate the discussed method using the example of Jacobi polynomials, for which asymptotics in terms of a Bessel function J_0 arises. In particular, we will describe the type of corresponding Lagrangian manifold and present an algorithm which allows one to obtain global formulas [2].

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Non-coherent coupling of one-dimensional vector laser solitons

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We investigate vector dissipative solitons and their interactions in the one-dimensional laser scheme with a nonlinear medium inside a resonator. The quasioptical nonlinear equation for the slow varying complex amplitudes of two circular polarisation components (generalized Ginzburg–Landau equation) was solved by numerical integration using split-step and Crank–Nicolson methods. Saturating absorption is modeled by a two-level scheme ([1]) and a four-level spin-flop scheme is adopted for saturable amplification introduced by [2].

In the present work we are taking into account polarisation state and forming stable solitons with non-uniform elliptical polarisation placing different scalar solitons in circular polarisation components. Also we investigate interactions between vector solitons and find stable soliton complexes with coherent and non-coherent coupling, example of such complex is shown at Fig. 1. There are two coupled solitons in E_+ component — symmetrical (left) and double, antisymmetrical (right). They have different frequencies, but remain coupled, just oscillating over time.

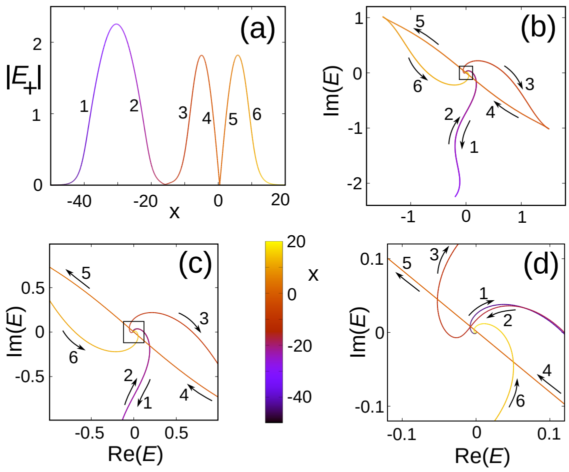

Fig. 1: Non-coherent coupling of vector solitons. Absolute value of E_+ circular component (a) and “phase diagram” of the complex field (b-d). Numbers 1–6 represent corresponding parts of trajectory, arrows show the x direction.

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Inversion of circular polarization in anisotropic optical fibers with torsional acoustic wave

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In this paper, we have studied the acousto-optic transformation of a circularly polarized fundamental mode in circular single-mode anisotropic optical fiber with traveling lowest-order torsional acoustic wave. The analytical expressions for resonance optical modes and the spectrum of propagation constants are obtained. We consistently describe the experimentally demonstrated mutual transformation of linearly polarized LP_0 modes and the corresponding shift of the optical frequency. The new effect of acoustically controlled sign inversion of spin angular momentum of fundamental mode is predicted. The possibility of creating optical beams with local entanglement of their polarization and frequency in the regime of linear optics is shown.

Theoretical investigation on possible indices of electromagnetic multipoles' singularities

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Electromagnetic multipoles characterized by vector spherical harmonics are the complete basis of solutions of the vector Helmholtz equation and form a foundational framework for analyzing electromagnetic fields. They are actively studied in both theoretical and applied contexts, particularly in nanophotonics, where they play a key role in understanding bound states in the continuum (BIC) which are non-radiating states of a system whose energy lies in the region of a continuous spectrum. Recent works have established the multipolar nature of BICs, showing that these states result from the destructive interference of multipoles in the direction of the open diffraction channel [1]. Notably, this direction of vanishing radiation corresponds to a singular point in the multipolar field distribution. Around such points in reciprocal space, polarization vortices emerge, each characterized by a topological charge, defined as the number of field rotations around the singularity [2–4].

Although numerous numerical and experimental studies have measured topological charges associated with BICs, an analytical understanding of how these polarization vortices form remains limited. Such a theoretical framework could significantly advance the field, especially in efforts to generate large topological charges — one of the current open issues in BIC research.

Previous work [5] has investigated the singularities of individual multipoles, concluding that for a dominant single multipole, the topological charge at the Γ-point corresponds to its azimuthal number m, while away from the Γ-point, the charge is restricted to 1 modulo at most. In our work, we generalize this analysis to arbitrary combinations of multipoles, aiming to develop a complete theory that describes the emergence of field rotation (vortices) around BIC points and to identify the conditions for generating higher-order topological charges.

We demonstrate that the structure of polarization vortices near a BIC point can be interpreted as a phase portrait of a two-dimensional dynamical system defined on the sphere. Each contributing multipole forms an individual portrait, and their sum determines the total field behavior near the singularity. If the Jacobi matrix of the field at the singular point is non-degenerate, then the maximum topological charge remains limited ($|q| \leq 1$). However, our analysis reveals that degeneracy of the Jacobian matrix can enable the formation of higher topological charges even away from the Γ-point. We estimate that the modulus of such a charge can, in principle, reach up to the Petrovskii number $|q| < \Pi_2(n+1)$, where n is a number equal to the degree of a polynomial of minimal degree when decomposing a field composed of vector spherical harmonics into a Taylor series.

For the types of multipoles typically excited in realistic photonic structures, we find that no natural degeneracy of the individual Jacobian matrices occurs in reciprocal space exept some special cases. Therefore, achieving degeneracy — and thus a high topological charge — requires careful tuning of the coefficients in the multipolar expansion. These coefficients must satisfy a system of linear equations that ensure the degeneracy of all partial derivatives of the entire field to a certain degree at a given point. The number of equations imposed by different combinations of multipoles and points in reciprocal space sets a lower bound on the number of multipoles needed to realize large topological charges.

At the Γ-point, our analysis shows that the topological charge is determined by the minimal non-zero degree in the asymptotic expansion of the multipole fields.

Overall, our work uncovers a rich and intricate relationship between the local structure of electromagnetic field's phase portraits and the resulting topological characteristics of BICs. The presented analytical framework provides a deeper theoretical insight into the formation mechanisms of topological charges in photonic systems.

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Asymptotics for long nonlinear coastal waves propagating along a sloping beach

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Coastal waves are understood as gravitational waves on water in a basin with depth $D(x, y)$, localized in the vicinity of the coastline. This report presents asymptotic solutions of a nonlinear shallow water equations system over a uniformly sloping bottom $D(x, y) = \gamma x$, describing waves traveling along the shoreline. The asymptotic solutions of the nonlinear shallow water equations system are written in the form of parametrically defined functions determined through exact solutions of the linearized system (see [1, 2]). In the linear problem, the variables are separable, and normal to the coastline the solution is expanded in terms of eigenfunctions of the operator $\hat{L} = -\partial^2/(\partial x^2) - x\partial/\partial x + (k^2 x - \omega^2/\gamma)$ with boundedness conditions at the shoreline $|\xi|_{x=0} < \infty$ and decay at infinity. The connection of the constructed solution with the classical (integrable) "billiard with a semi-rigid wall" (see [3]) is also discussed.

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Excitation of localized beams by electric dipole sources embedded into metamaterial dielectic layer

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The problem of radiation of electro-magnetic waves excited by electric vertical and horizontal dipoles located near to the interface separating two dielectric homogeneous media has been vital since the beginning of the twentieth century (see first of all [1]) in various areas of radio-science theoretical activity and engineering. In this paper we study a similar but of more complicated geometry problem in the frequency domain. This is a problem of excitation of localized beam into the upper dielectric half-space with dielectric permittivity ϵ_2 located above another horizontal dielectric layer with ϵ_1 with embedded electric dipole source. Both permittivities are positive. For the sake of simplicity it is assumed that magnetic permeabilities for the upper half-space and the horizontal layer are identical to unity, that is the media are not magnetic. The horizontal dielectric layer lays on a perfectly conducting screen. Firstly, in order to illustrate the problem, we study the case of a point vertical electric dipole embedded into a horizontal dielectric layer. Using the Sommerfeld solution ([1]) combined with the multiply reflected images summation method the exact solution to the problem is derived in terms of Hertz vector integral representations. It is shown that in the upper dielectric half-space in the far field zone the generated wave field represents a localized beam (a spherical wave with exponentially localized radiation pattern) if the following condition holds true $\epsilon_1 < \epsilon_2$. Thus, if the upper dielectric half-space is vacuum with $\epsilon_2 = 1$, and the horizontal layer is a metamaterial with $\epsilon_1 < 1$, the localization of the beam is observed. This localization is stronger the smaller the ration ϵ_1/ϵ_2.

Secondly, we study the case of a point horizontal electric dipole embedded into a horizontal dielectric layer that is more vital from the point of view of applications in radio-science of antenna theory (vertical dipole does not radiate much in the vertical direction). The corresponding exact solution that was obtained similarly to the previous case enables us to construct approximate solution for active horizontal thin wire of finite length (elementary radio vibrator antenna). On this way we apply analysis based on Pocklington integral equation (see for example [2]) to compute resonance frequencies and construct excited localised beam irradiated into the upper half-space. This approach is based on the long-wave approximation of thin wires. It is worth remarking that recently the approach based on the Pocklington type integral equation was successfully applied to studying a surface waves generation by a thin wire in the presence of impedance surface of metallic grid (see [3]). As far as the resonance electric currents of the thin wire (active vibrator) have been computed numerically we evaluate the exited near field in the form of Hertz vector integral representations of Sommerfeld type solution. In the far field zone, applying the steepest descend method of the short-wave approximation to these integral representations, for the wave field we obtain asymptotic expansions of the expanding bulky spherical waves in the upper half-space with the radiation pattern of a localized beam if the following condition holds true $\epsilon_1 < \epsilon_2$. In the numerical analysis we pay a particular attention to the shape of the radiation pattern of the localized beam irradiated into the upper half-space for the case when the upper dielectric half-space is vacuum with $\epsilon_2 = 1$, and the horizontal layer is a metamaterial with $\epsilon_1 \ll 1$.

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Paraxial diffraction by a delta potential

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Diffraction of high-frequency waves by delta singularities of refractive index, localized on codimension 1 surfaces, has recently attracted attention of various researchers. Mathematicians were interested in the development of Maslov's theory for non-tangential incidence in an arbitrary dimension [1, 2]. Physicists addressed a 2D problem to simulate the effect of a thin absorbing layer [3], basing on the Leontovich–Fock paraxial approach that reduces the high-frequency Helmholtz equation to the 'parabolic' Schrödinger equation.

Continuing our research [4], we investigate the following problem

$$\begin{cases} 2iku_x + u_{yy} + 2i\nu\delta(y)u = 0, \quad x > 0, \; -\infty < y < \infty; \\ u\big|_{x=0} = e^{-ik\alpha y} \end{cases} \tag{1}$$

Here, constants k and α are positive, and k is large, while the parameter ν is complex that enables to model absorbing, or active, or conservative thin layer. The problem (1) is inspired by the diffraction of plane wave $e^{ik(x\cos\alpha - y\sin\alpha)}$ by the singularity. A closed-form solution of (1) is

$$u = e^{-ik\left(y\alpha + x\frac{\alpha^2}{2}\right)} - \frac{i\eta}{i\eta - \alpha}F^{\alpha} - \frac{i\eta}{i\eta + \alpha}F^{-\alpha} + \frac{2\eta^2}{\eta^2 + \alpha^2}F^{i\eta}, \tag{2}$$

where

$$F^t = F^t(x, y) = e^{ik\left(|y|t - x\frac{t^2}{2}\right)}\Phi\left(-\sqrt{\frac{kx}{2}}|y| + \sqrt{\frac{kx}{2}}t\right) \tag{3}$$

and

$$\Phi(Z) = \frac{e^{-i\pi/4}}{\sqrt{\pi}}\int_{-\infty}^{Z} e^{it^2}dt \tag{4}$$

is the classical Fresnel integral.

We will report on the asymptotic study of the solution (2).

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Asymptotic solutions for water waves in a channel over a slow-varying bottom considering the reflection from a cross section wall

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Surface gravity water waves above a varying bottom are described by a pseudo-differential operator that takes into account the dispersion effect [1]. In this research we consider the case with one horizontal coordinate and solve the Cauchy problem with a localized initial condition and the Neumann boundary condition on a vertical wall. The reflection of the wave from the wall and the effect of dispersion on the initial disturbance are investigated.

The asymptotics of the problem are constructed in the form of the Maslov canonical operator with a boundary index. In the vicinity of the front the asymptotic behavior is expressed in terms of the Airy function and its derivative. It is possible as well to get the uniform asymptotics in terms of the Airy function [2], which is much more convenient than traditional matching of solutions in regular and singular maps when being applied to modern software packages.

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