A NEW APPROACH TO REPRESENTATION THEORY OF SYMMETRIC GROUPS

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§0. Introduction

The aim of this paper is to give a new, simple and direct approach to the representation theory of $S(n)$.

Basically there are two ways to construct irreducible complex representations of $S(n)$. The first is based on the representation theory of $GL(N)$ and duality between $S(n)$ and $GL(N)$ in the tensors

$$\underbrace{\mathbb{C}^N \otimes \mathbb{C}^N \otimes \cdots \otimes \mathbb{C}^N}_{n \text{ times}},$$

which is called the Schur–Weyl duality (see [W]). The Schur functions, which are characters of $GL(N)$, play the key role in this approach. Using Schur functions, Frobenius described the characters of $S(n)$ as explained in the book [M].

The other way, usually attributed to Young with later contributions by Weyl and von Neumann, is based on the combinatorics of Young diagrams and tableaux. In this approach the irreducible representations (sometimes called the Specht's modules) arise as common components of two simple induced representations. This way is traditional and one can find it in almost all textbooks and monographs on the

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subject, for example, in one of the last books [JK]. It requires considerable efforts to obtain any explicit formulas for representations or characters.

Both ways are important as well as indirect; they rest upon deep and non-trivial auxiliary constructions. There is a natural question, that is, whether one can arrive at the main combinatorial objects of the theory (diagrams, tableaux etc.) in a more direct and natural fashion. We believe that representation theory of the symmetric groups must satisfy the three following conditions:

(1) Symmetric groups form a natural series and their representation theory should be recursive with respect to the series, which means that representation theory of $S(n)$ should rely on representation theory of $S(n-1)$ for all $n = 1, 2 \dots$.
(2) The combinatorics of the Young diagrams and Young tableaux, which reflects the branching rule for restriction

$$S(n) \downarrow S(n-1),$$

must not be introduced as an auxiliary tool of construction, not *a priori*; it should be deduced starting from the inside structure of the symmetric groups. In this case the branching rule (which is one of the main theorem of the theory) will appear naturally and not as a last corollary after developing the whole theory,
(3) Symmetric groups are Coxeter groups and the methods of their representation theory should apply to all classical series of Coxeter groups.

In this paper we suggest a new approach which satisfies the above principles and makes the whole theory more natural and simple. For our method the following notions are very important:

(1) Gelfand-Zetlin (GZ) algebra and basis,
(2) Jucys-Murphy (JM) elements,
(3) algebras with local system of the generators (ALSG) as a general context for the theory

One of the main advantages of our version of the representation theory of symmetric (and of other series of the Coxeter groups) is the following: *we obtain the branching rule simultaneously with the description of the representations and introduce Young diagrams and tableaux using only an analysis of the spectra of JM elements of the GZ algebra.*

The GZ basis was defined for the unitary and orthogonal groups by I. M. Gelfand and M. L. Zetlin in the fifties [GZ1-2]. The general notion of GZ algebra for inductive limits of algebras can be introduced in the same way for an arbitrary inductive limit of semisimple algebras (it was done, for example, in [KV]). The notion of ALSG generalizes the relation of Coxeter groups, braid groups, Hecke algebras and so on (see [V1]). This idea gives us the rule of induction process for the construction of the representations.

Very convenient special generators of the GZ algebra of the symmetric group $S(n)$ were found independently by A.-A.A.Jucys [Ju] and G.E.Murphy [Mu]. These JM generators are as follows:

$$X_i = (1\,i) + (2\,i) + \cdots + (i-1\,i), \quad i = 1, 2, \dots, n.$$

There exist an invariant way to define them (see below) which is good for a very general class of ALSG algebras, in particular for all Coxeter groups.

The complexity of the symmetric group (compared to $GL(N)$) lies in the fact that the Coxeter relations

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

for the generators of $S(k)$ are not commutation relations. Moreover, there is no large commutative subgroup of $S(k)$ that could play the role of a Cartan subgroup. However, our approach somehow resembles the Cartan highest weight theory with the role of Cartan subgroup played by the commutative GZ subalgebra in $\mathbb{C}[S(n)]$. The JM generators of this subalgebra diagonalize simultaneously in any representation of $S(n)$ and the whole representation theory of $S(n)$ is encoded in their spectra. The problem is, therefore, to describe those spectra, that is, to understand which eigenvalues of JM elements are possible and which occur in the same irreducible representation.

This problem is parallel to the description of the dominant weights of a reductive group. We solve it using induction on n and elementary analysis of the commutation relation

$$s_i X_i + 1 = X_{i+1} s_i\,, \quad i = 1, \dots, n-1 \tag{0.1}$$

between the JM elements the the Coxeter generators s_i. In a sense, the algebra $H(2)$ generated by s_i and two commuting elements X_i, X_{i+1} subject to the relation (0.1) plays in our paper the same role as $\mathfrak{gl}(2)$ plays in representation theory of reductive groups.

As an application of these results we derive the classical Young formulas for the action of generators s_i of $S(n)$ and a new proof of the Murnaghan-Nakayama rule for the characters of $S(n)$. The final step in the proof of the Young formulas is the same as in [Mu]; in fact, Young formulas were Murphy's motivation for introducing the elements X_i and calculating their eigenvalues. The novelty of our approach is that we do not assume any knowledge of the representation theory of $S(n)$ and build the theory starting from simple commutation relations.

The first attempt to develop a new version of the representation theory of the symmetric group was made in the papers [V1, V2] where the notion of ALSG was introduced. The branching rule and Young orthogonal form was deduced in [V1] from Coxeter relation for generators of $S(n)$ and the assumption that the branching graph (see below) of $S(n)$ is a Hasse diagram of a *distributive lattice*. The approach presented in this paper does not require any special assumptions.

Our scheme can be applied to some other ALSG and first of all to Coxeter groups of B-C-D series and to wreath products of the symmetric group with some finite groups. All these generalizations will be considered elsewhere.

We do not attempt to give here a complete bibliography on the subject. Proper analogs of Jucys-Murphy elements for the infinite symmetric group $S(\infty)$ proved themselves to be an extremely powerful tool in the infinite-dimensional representation theory [Ok1–2,Ol1–2]. About infinite symmetric group see also [KOV,V3,VK] and the forthcoming paper [KOOV]. In a series of papers [V1–3] the second author developed new approaches to representation theory of $S(n)$ in connection with $n \to \infty$ asymptotic problems. Applications of JM elements to classical representation theory are numerous (see, for example, [DG]; we learned about this important preprint after our paper was completed). The JM elements arise naturally in connection with higher Capelli identities [Ok3]. In the papers [C,D] the JM elements were considered in the context of the theory of the degenerate affine Hecke algebras. JM elements for Coxeter groups were defined in [N,R].

The reader is supposed to be familiar only with the elementary facts from abstract representation theory of finite groups. We will not use any facts from the representation theory of the symmetric groups.

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A short announcement of our results was made in [OV].

§1. Gelfand-Zetlin Basis for Inductive Families

Let

$$\{1\} = G(0) \subset G(1) \subset G(2) \subset \dots . \tag{1.1}$$

be a chain of finite groups. By $G(n)^\wedge$ denote the set of equivalence classes of irreducible complex representations of the group $G(n)$. The *branching graph* (or Bratteli diagram) of (1.1) is by definition the following oriented graph. The vertices of the branching graph are the elements of the set

$$\bigcup_{n\geq 0} G(n)^\wedge .$$

Two vertices $\mu \in G(n-1)^\wedge$ and $\lambda \in G(n)^\wedge$ are joined by k oriented edges (from μ to λ) if

$$k = \dim \operatorname{Hom}_{G(n-1)}(V^\mu, V^\lambda),$$

that is if k is the multiplicity of μ in the restriction of the representation λ to the group $G(n-1)$. We call the set $G(n)^\wedge$ the n-th *level* of the branching graph. Write

$$\mu \nearrow \lambda$$

if μ and λ are connected by an edge in the branching graph; write

$$\mu \subset \lambda,$$

where $\mu \in G(k)^\wedge$, $\lambda \in G(n)^\wedge$, and $k \leq n$, if the multiplicity of μ in λ is nonzero. In other words, $\mu \subset \lambda$ if there is a path from μ to λ in the branching graph. Denote by $\varnothing$ the unique element of $G(0)^\wedge$. The same definition of the branching graph is good for any chain

$$M(0) \subset M(1) \subset M(2) \subset \dots$$

of finite-dimensional semisimple associative algebras.

In the case

$$G(n) = S(n), \quad n = 1, 2, \dots$$

we always have

$$k \in \{0, 1\},$$

which means that the branching graph is multiplicity free. A proof of this well known fact (see, for example [JK]) will be given below. We assume $k \in \{0,1\}$ in the sequel. In this case the decomposition

$$V^\lambda = \bigoplus_{\mu \in G(n-1)^\wedge, \mu \nearrow \lambda} V^\mu$$

into the sum of irreducible $G(n-1)$-modules is canonical. By induction, we obtain a canonical decomposition of the module V^λ into irreducible $G(0)$-modules (that is simply 1-dimensional subspaces)

$$V^\lambda = \bigoplus_T V_T\,,$$

indexed by the all possible chains

$$T = \lambda_0 \nearrow \lambda_1 \nearrow \ldots \nearrow \lambda_n \tag{1.2}$$

where $\lambda_i \in G(i)^\wedge$ and $\lambda_n = \lambda$. Those are *paths* from $\varnothing$ to λ in the branching graph.

Choose a vector $v_T \in V_T$ such that

$$(v_T, v_T) = 1\,,$$

where $(\cdot\,,\,\cdot)$ is the $G(n)$-invariant inner product in V^λ. This basis $\{v_T\}$ is called the *Gelfand-Zetlin (GZ) basis.* In [GZ1,GZ2] such a basis was defined for representation of $SO(n)$ and $U(n)$; we use the same name in the general situation (see [VK]). We shall consider also non-normalized Gelfand-Zetlin basis vectors. By definition of v_T,

$$\mathbb{C}[G(i)] \cdot v_T, \quad i = 1, 2, \ldots, n, \tag{1.3}$$

is the irreducible $G(i)$-module V^{λ_i}. It is clear also, that v_T is the unique (up to a scalar factor) vector with this property.

By $Z(n)$ denote the center of $\mathbb{C}[G(n)]$. Let $A(n) \subset \mathbb{C}[G(n)]$ be the algebra generated by the subalgebras

$$Z(1), Z(2), \ldots, Z(n)$$

of $\mathbb{C}[G(n)]$. It is readily seen that $A(n)$ is commutative. The algebra $A(n)$ is called the *Gelfand-Zetlin subalgebra* (GZ-algebra). Recall the following fundamental isomorphism

$$\mathbb{C}[G(n)] = \bigoplus_{\lambda \in G(n)^\wedge} \operatorname{End}(V^\lambda)\,. \tag{1.4}$$

Proposition 1.1. *The GZ-algebra $A(n)$ is the algebra of of all operators diagonal in GZ-basis. In particular, it is a maximal commutative subalgebra of $\mathbb{C}[G(n)]$.*

Proof. Denote by $P_T \in A(n)$ the product of central idempotents

$$P_{\lambda_1} P_{\lambda_2} \ldots P_\lambda, \quad P_{\lambda_i} \in Z(i)\,,$$

corresponding to $\lambda_1, \lambda_2, \dots, \lambda$ respectively. Clearly, P_T is the projection onto V_T. Hence $A(n)$ contains the algebra of operators diagonal in the basis $\{v_T\}$, which is a maximal commutative subalgebra of $\mathbb{C}[G(n)]$. Since $A(n)$ is commutative the proposition is proved. $\square$

Remark 1.2. Note that by the above proposition any vector from the Gelfand-Zetlin basis in any irreducible representation of $G(n)$ is uniquely (up to a scalar) determined by the eigenvalues of the elements of $A(n)$ on it.

Remark 1.3. In general, the GZ-subalgebra is a maximal commutative subalgebra iff the branching graph has no multiple edges.

§2. Jucys-Murphy Elements

From now on we consider the case

$$G(n) = S(n)\,.$$

In section 4 we shall prove that any irreducible $S(n)$-module is indeed a multiplicity-free $S(n-1)$-module. The Gelfand-Zetlin basis for the symmetric group is known as the *Young basis.*

For $i = 1, 2, \dots, n$ consider the following elements $X_i \in \mathbb{C}[S(n)]$

$$X_i = (1\,i) + (2\,i) + \cdots + (i-1\ i)\,.$$

In particular, $X_1 = 0$. Following M. Nazarov we call them *Jucys-Murphy elements* or JM-elements; they were introduced independently in [Ju] and [Mu].

It is clear that

$$\begin{aligned} X_i = &\ \textit{sum of all transpositions in } S(i) - \\ &\ \textit{sum of all transpositions in } S(i-1)\,, \end{aligned} \tag{2.1}$$

that is a difference of an element of $Z(i)$ and an element of $Z(i-1)$. Therefore $X_i \in A(n)$ for all $i \le n$. In particular, the JM elements commute. The following proposition will be proved in Section 4. The fact that JM elements generate a maximal commutative subalgebra in $\mathbb{C}[S(n)]$ is well-known (see [DG]).

Proposition 2.1. *The elements $X_1, \dots, X_n$ generate the algebra $A(n)$.*

Another way to introduce the elements X_i is the following. There exists the unique map

$$S(n+1) \to S(n)$$

which commutes with the right and left multiplication by elements of $S(n)$; it removes the number $n+1$ from the cyclic notation of a permutation [KOV]. Extend this map to a map of group algebras by linearity

$$\mathbb{C}[S(n+1)] \to \mathbb{C}[S(n)]\,.$$

Proposition 2.2. *The affine space in the algebra $\mathbb{C}[S(n+1)]$ which is the intersection of the preimage (under that projection) of the identity element of $\mathbb{C}[S(n)]$ with the centralizer of the subalgebra $\mathbb{C}[S(n)]$ is spanned by identity element of $\mathbb{C}[S(n+1)]$ and the element $\frac{1}{n}X_{n+1}$.*

This proposition gives a general invariant definition of JM elements; it can be extended to many series of ALSG algebras and groups which have a similar projection. For example, the JM elements for general Coxeter groups can be defined in such a way. We will not use this proposition.

The Young basis is the common eigenbasis of JM elements. If v is a Young basis vector, then by

$$\alpha(v) = (a_1, \dots, a_n) \in \mathbb{C}^n$$

denote the eigenvalues of $X_1, \dots, X_n$ on v. Let us call the vector $\alpha(v)$ the *weight* of v. Denote by

$$\mathrm{Spec}(n) = \{\alpha(v),\ v \in \text{ Young basis}\}$$

the spectrum of JM elements. By Proposition 2.1 and Remark 1.2 a point $\alpha(v) \in \mathrm{Spec}(n)$ determines v up to a scalar factor (denote by v_α any Young basis vector corresponding to a point $\alpha \in \mathrm{Spec}(n)$). It follows that

$$|\,\mathrm{Spec}(n)| = \sum_{\lambda \in S(n)^\wedge} \dim \lambda\,.$$

By definition of the Young basis the set $\mathrm{Spec}(n)$ is in a natural bijection with the set of all paths (1.2) in the branching graph. Denote these correspondences by

$$T \mapsto \alpha(T), \quad \alpha \mapsto T_\alpha\,.$$

There is the following natural equivalence relation $\sim$ on $\mathrm{Spec}(n)$. Write

$$\alpha \sim \beta, \quad \alpha, \beta \in \mathrm{Spec}(n)\,,$$

if v_α and v_β belong to the same irreducible $S(n)$-module, or, equivalently, the paths T_α and T_β have the same end. Clearly,

$$|\,\mathrm{Spec}(n)/\sim| = |S(n)^\wedge|\,.$$

Our plan is to

(1) describe the set $\mathrm{Spec}(n)$,
(2) describe the equivalence relation $\sim$,
(3) calculate matrix elements in the Young basis,
(4) calculate characters of irreducible representations.

§3. Action of Coxeter generators and the algebra $H(2)$

The Coxeter generators of the group $S(n)$

$$s_i = (i\ i+1), \quad i = 1, \dots, n-1,$$

commute except for neighbors. Such generators were called *local* generators in [V1]. Here, as in physics, "local" means that remote generators do not affect each other (commute). The Young basis is also local in the following sense.

Proposition 3.1. *For any vector*

$$v_T, \quad T = \lambda_0 \nearrow \dots \nearrow \lambda_n, \quad \lambda_i \in S(i)^\wedge$$

and any $k = 1, \dots, n-1$ *the vector*

$$s_k \cdot v_T$$

is a linear combination of vectors

$$v_{T'}, \quad T' = \lambda'_0 \nearrow \dots \nearrow \lambda'_n, \quad \lambda'_i \in S(i)^\wedge$$

such that

$$\lambda'_i = \lambda_i, \quad i \neq k\,.$$

In other words, the action of s_k *affects only the* k*-th level of the branching graph.*

Proof. Suppose $i > k$. Since $s_k \in S(i)$ and the module

$$\mathbb{C}[S(i)] \cdot v_T$$

is irreducible we have

$$\mathbb{C}[S(i)]\, s_k \cdot v_T = \mathbb{C}[S(i)] \cdot v_T = V^{\lambda_i}\,, \tag{3.1}$$

where V^{λ_i} is the irreducible $S(i)$-module indexed by $\lambda_i \in S(i)^\wedge$.

Suppose $i < k$. Since s_k commutes with $S(i)$ we have (3.1) again. Now it follows from (1.3) that $s_k \cdot v_T$ is a linear combination of desired vectors. □

In the same way it is easy to show the the coefficients of this linear combination depend only on $\lambda_{k-1}, \lambda_k, \lambda'_k, \lambda_{k+1}$ and the choice of the scalar factors in Young basis vectors. That is,, the action of s_k affects only the k-th level and depends only on levels number $k-1, k, k+1$ of the branching graph.

Proposition 3.1 can be also easily deduced from the obvious relations

$$s_i X_j = X_j s_i, \quad j \neq i, i+1\,. \tag{3.2}$$

The elements s_i, X_i, and X_{i+1} satisfy a more interesting (and well-known relation)

$$s_i X_i + 1 = X_{i+1} s_i\,, \tag{3.2}$$

which is evident rewritten as

$$s_i X_i s_i + s_i = X_{i+1}\,.$$

The action of JM elements on the Young basis is also local. It readily follows from (2.1) that if

$$T = \lambda_0 \nearrow \dots \nearrow \lambda_n$$

and

$$\alpha(T) = (a_1, \dots, a_n)$$

Then a_k is the difference of a function of λ_k and a function of λ_{k-1} for all k.

Denote by $H(2)$ the algebra generated by elements s, Y_1, Y_2 subject to the following relations

$$s^2 = 1, \quad Y_1 Y_2 = Y_2 Y_1, \quad s Y_1 + 1 = Y_2 s\,.$$

This algebra will play the central role in the sequel. It is the simplest example of the degenerate affine Hecke algebra. Its importance is based on Theorem 4.1 proved in the next section.

Another important property of Coxeter generators and JM elements is that the relations between them are stable under a shift of indices. Such relations were called *stationary* relations in [V1].

§4. Centralizers

Suppose we have a group G, an irreducible G-module V, a subgroup $H \subset G$, and an irreducible H-module U. The multiplicity of U in V equals the dimension of the vector space

$$\operatorname{Hom}_H(U,V)\,.$$

This vector space, as it follows from (1.4), is an irreducible module over the centralizer

$$\mathbb{C}[G]^H$$

of the group H in $\mathbb{C}[G]$.

In this section we study the centralizers

$$Z(l,k)=\mathbb{C}[S(l+k)]^{S(l)}.$$

In the algebra $Z(l,k)$ there is a natural linear basis of delta-functions on the orbits of the $S(l)$-action on $S(l+k)$ by conjugation. Those orbits can be described in terms of the certain combinatorial objects called *marked permutations* in [Ol2]. The following combinatorics and Theorem 4.1 below will be used in full generality in the proof of the Murnaghan-Nakayama rule only; for the derivation of the branching rule it suffices to consider the cases $k=1,2$ which are quite elementary.

Take an arbitrary permutation of the set $\{l+1,\dots,l+k\}$, for example

$$(l+1\to l+2\to l+1)(l+3\to l+4\to\ldots\to l+3)\,,$$

and assign to each arrow an arbitrary nonnegative integer (called *tag*), for example

$$(l+1\xrightarrow{2} l+2\xrightarrow{0} l+1)(l+3\xrightarrow{1} l+4\xrightarrow{3}\ldots\xrightarrow{1} l+3)\,.$$

Add also some empty cycles with tags, for example, as follows:

$$(l+1\xrightarrow{2} l+2\xrightarrow{0} l+1)(l+3\xrightarrow{1} l+4\xrightarrow{3}\ldots\xrightarrow{1} l+3)(\xrightarrow{3})(\xrightarrow{1})\,. \tag{4.1}$$

Such combinatorial objects we shall call *marked permutations* of the set $\{l+1,\dots,l+k\}$.

It is easy to see that the orbits of the $S(l)$-action on $S(l+k)$ by conjugation are in natural bijection with marked permutations of $\{l+1,\dots,l+k\}$ such that the sum of all tags is equal to l. For example, the orbit, corresponding to the marked permutation (4.1) consists of all elements of $S(l+k)$ of the form

$$\begin{aligned}(l+1\to *\to *\to l+2\to l+1)(l+3\to *\to l+4\to *\to *\to *\to\ldots\\ \cdots\to *\to l+3)(*\to *\to *)(*)\,,\end{aligned}$$

where the l asterisks stand for some permutation of the numbers $1,2,\dots,l$. In general, an arrow $\xrightarrow{m}$ gets replaced by m asterisks, that is m different numbers from $\{1,2,\dots,l\}$.

Let us omit the trivial cycles of the form

$$(\xrightarrow{1})\quad\text{and}\quad(a\xrightarrow{0}a)\quad a\in\{l+1,\dots,l+k\}\,.$$

Then the JM element

$$X_{l+1}\in Z(l,k)$$

corresponds to the marked permutation

$$(l+1\xrightarrow{1} l+1)\,.$$

The following theorem was proved by G. Olshanski, see [Ol1], theorem 11.

Theorem 4.1 [Ol1]. *The algebra $Z(l,k)$ is generated by the elements*

$$X_{l+1}, \dots, X_{l+k}\,,$$

the group $S(k)$, which permutes the numbers $\{l+1,\dots,l+k\}$, and the center $Z(l)$ of $\mathbb{C}[S(l)]$.

Proof. Let us assume that $k=2$. The general case is similar. Consider the elements

$$C_m = (\xrightarrow{m}) \in Z(l) \subset Z(l,k)\,, \quad m = 2,3,\dots,l\,.$$

Consider the following filtration in $\mathbb{C}[S(l+k)]$: the i-th subspace of this filtration, where $i=0,\dots,l+k$, is spanned by permutations with $l+k-i$ fixed points. This filtration is invariant under conjugation; we consider the induced filtration of $Z(l,k)$. It suffices to prove that s_{l+1}, X_{l+1}, X_{l+2}, and C_m, $m=2,\dots,l$ generate the graded algebra corresponding to this filtration of $Z(l,k)$.

It is easy to see that

$$X_{l+1}^a (X_{l+2}-s_{l+1})^b C_{m_1} C_{m_2} \cdots = \\ (l+1 \xrightarrow{a} l+1)(l+2 \xrightarrow{b} l+2)(\xrightarrow{m_1})(\xrightarrow{m_2})\dots \quad + \quad \text{lower terms}$$

provided $a+b+m_1+m_2+\dots \le l$. Here "lower terms" stands for elements of the lower filtration space. Similarly,

$$s_{l+1} X_{l+1}^a (X_{l+2}-s_{l+1})^b C_{m_1} C_{m_2} \cdots = \\ (l+1 \xrightarrow{a} l+2 \xrightarrow{b} l+1)(\xrightarrow{m_1})(\xrightarrow{m_2})\dots \quad + \quad \text{lower terms}$$

provided $a+b+m_1+m_2+\dots \le l$. This concludes the proof. $\square$

In particular, the algebra

$$\mathbb{C}[S(n+2)]^{S(n)}$$

is generated by $Z(n)$ and homomorphic image of $H(2)$.

Now we can prove Proposition 2.1

Corollary 4.2. *The elements $X_1,\dots,X_n$ generate the algebra $A(n)$.*

Proof. By induction, it suffices to check that $Z(n)$ belongs to the algebra generated by $Z(n-1)$ and X_n. This follows from the above theorem and obvious inclusion

$$Z(n) \subset Z(n-1,1)\,. \quad \square$$

Corollary 4.3. *Let V^μ be an irreducible $S(l)$-module, and let V^λ be an irreducible $S(l+k)$-module. Then the multiplicity of V^μ in V^λ is not greater than $k!$. In particular, if $k=1$ then V^λ is a multiplicity-free $S(l)$-module.*

Proof. The multiplicity is the dimension of

$$\operatorname{Hom}_{S(l)}(V^\mu, V^\lambda)\,, \tag{4.2}$$

which is an irreducible $Z(l,k)$-module. The algebra $Z(l)$ acts by scalar operators in this module. The elements $X_{l+1},\dots,X_{l+k}$ commute and hence have a common eigenvector v in (4.2). It follows from the relations (3.2,3.3) that the vector space spanned by the vectors

$$s \cdot v, \quad s \in S(k),$$

is $Z(l,k)$-invariant and hence equals (4.2). It follows that

$$\dim \operatorname{Hom}_{S(l)}(V^\mu, V^\lambda) \leq k!\,. \quad \square$$

It is easy to see from the explicit construction of representation (see below) that this upper bound is sharp.

In terms of the branching graph Corollary 4.3 can be restated as follows. By the very definition of the Young basis, the space (4.2) has a natural basis indexed by paths from μ to λ in the branching graph. Suppose

$$T_0 = \mu \nearrow \dots \nearrow \lambda$$

is such a path. Then the corresponding homomorphism

$$V^\mu \to V^\lambda$$

takes a vector $v_T \in V^\mu$ to the vector

$$v_{T+T_0}\,,$$

where $T + T_0$ stands for junction of T and T_0.

Corollary 4.3 for $k = 2$ implies that there are only three possibilities:

(1) the multiplicity of μ in λ equals 0 and μ and λ are not connected in the branching graph;

(2) the multiplicity equals 1 and the interval between μ and λ is a chain

$$\mu \text{---} \nu \text{---} \lambda\,;$$

(3) the multiplicity equals 2 and the interval between μ and λ is a square

$$\begin{array}{ccccc} & \diagup & \nu & \diagdown & \\ \mu & & & & \lambda\,. \\ & \diagdown & \eta & \diagup & \end{array}$$

In the case of the chain the generator s_{l+1} multiplies all vectors

$$v_T, \quad T = \dots \nearrow \mu \nearrow \nu \nearrow \lambda \nearrow \dots$$

by a constant, which equals ± 1 by virtue of $s_{l+1}^2 = 1$.

We consider the case of the square in the next section.

Remark 4.4. The algebra generated by the group $S(n)$ and commuting elements $Y_1, \dots, Y_n$ subject to relations

$$s_i Y_i + 1 = Y_{i+1} s_i, \quad s_i Y_j = Y_j s_i,\ j \neq i, i+1\,,$$

is called the *degenerate affine Hecke algebra*, see [D,C]. We denote it by $H(n)$. We see that $Z(l,k)$ is generated by $Z(l)$ and homomorphic image of $H(k)$. In particular, (4.2) is an irreducible $H(k)$-module where Y_i acts as X_{l+i}.

§5. Irreducible representations of $H(2)$

We already know that all irreducible representations of $H(2)$ are at most 2-dimensional and have a vector v such that

$$Y_1 v = av, \quad Y_2 v = bv, \quad a, b \in \mathbb{C}.$$

If the vectors v and sv are linearly independent then the relation

$$sY_1 + 1 = Y_2 s \tag{5.1}$$

implies that Y_1 and Y_2 act in the basis v, sv as follows

$$Y_1 = \begin{pmatrix} a & -1 \\ 0 & b \end{pmatrix}, \quad Y_2 = \begin{pmatrix} b & 1 \\ 0 & a \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If $b \neq a \pm 1$ this representation is irreducible; denote it by $\pi_{a,b}$. If v and sv are proportional then

$$sv = \pm v$$

and it follows from (5.1) that

$$b = a \pm 1$$

in this case.

If $a = b$ then the operators $\pi_{a,b}(Y_i)$ are not semisimple and therefore such representations cannot appear in the action on the Young basis. If $a \neq b$ then the operators $\pi_{a,b}(Y_i)$ can be diagonalized, for example, as follows:

$$Y_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad Y_2 = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}, \quad s = \begin{pmatrix} \frac{1}{b-a} & 1 - \frac{1}{(b-a)^2} \\ 1 & \frac{1}{a-b} \end{pmatrix}. \tag{5.2}$$

Let us formulate our results as a proposition.

Proposition 5.1. *Suppose*

$$\alpha = (a_1, \dots, a_i, a_{i+1}, \dots, a_n) \in \mathrm{Spec}(n).$$

Then

(1) $a_i \neq a_{i+1}$ *for all* i,
(2) *if* $a_{i+1} = a_i \pm 1$ *then* $s_i \cdot v_\alpha = \pm v_\alpha$,
(3) *if* $a_{i+1} \neq a_i \pm 1$ *then*

$$\alpha' = s_i \cdot \alpha = (a_1, \dots, a_{i+1}, a_i, \dots, a_n) \in \mathrm{Spec}(n)$$

and $\alpha' \sim \alpha$. *Moreover,*

$$v_{\alpha'} = \left(s_i - \frac{1}{a_{i+1} - a_i} \right) v_\alpha$$

and the elements s_i, X_i, X_{i+1} *act in the basis* $v_\alpha, v_{\alpha'}$ *by formulas* (5.2) *with* Y_1 *replaced by* X_i *and* Y_2 *replaced by* X_{i+1}.

Let us call a transposition s_i as in part (3) of the proposition an *admissible* transposition. Admissible transpositions preserve the set $\mathrm{Spec}(n)$ and the set $\mathrm{Cont}(n)$ defined in the next section. Evidently, the two cases in this proposition are the cases of the chain and square from the previous section.

Note that if $a_{i+1} \neq a_i \pm 1$ then in the basis

$$\left\{ v_\alpha, \left(1 - \frac{1}{(a_{i+1} - a_i)^2}\right)^{-1/2} \left(s_i - \frac{1}{a_{i+1} - a_i}\right) v_\alpha \right\}$$

the matrix of the transposition s_i is orthogonal

$$s_i = \begin{pmatrix} 1/r & \sqrt{1 - 1/r^2} \\ \sqrt{1 - 1/r^2} & -1/r \end{pmatrix},$$

where $r = a_{i+1} - a_i$.

§6. Main theorems

In this section we shall describe the set $\mathrm{Spec}(n)$ and the equivalence relation $\sim$. Introduce the set $\mathrm{Cont}(n)$ of *content vectors* of length n. By definition

$$\alpha = (a_1, \dots, a_n) \in \mathrm{Cont}(n)$$

if α satisfies the following conditions

(1) $a_1 = 0$,
(2) $\{a_q - 1, a_q + 1\} \cap \{a_1, \dots, a_{q-1}\} \neq \varnothing$ for all $q > 1$,
(3) if $a_p = a_q = a$ for some $p < q$ then

$$\{a - 1, a + 1\} \subset \{a_{p+1}, \dots, a_{q-1}\}.$$

It is clear that

$$\mathrm{Cont}(n) \subset \mathbb{Z}^n.$$

Theorem 6.1.

$$\mathrm{Spec}(n) \subset \mathrm{Cont}(n). \tag{6.1}$$

We shall need the following lemma

Lemma 6.2. *Suppose*

$$\alpha = (a_1, \dots, a_n)$$

and $a_i = a_{i+2} = a_{i+1} - 1$ *for some* i. *Then*

$$\alpha \notin \mathrm{Spec}(n).$$

Proof of the Lemma. Suppose $\alpha \in \mathrm{Spec}(n)$. By proposition 5.1, part (2)

$$s_i v_\alpha = v_\alpha,\ s_{i+1} v_\alpha = -v_\alpha,$$

which contradicts the Coxeter relation

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} . \quad \square$$

Proof of the Theorem. Suppose $\alpha = (a_1, \dots, a_n) \in \operatorname{Spec}(n)$. Since $X_1 = 0$ we have $a_1 = 0$.

Let us verify the conditions (2) and (3) by induction on n. The case $n = 2$ is clear. Suppose $\{a_n - 1, a_n + 1\} \cap \{a_1, \dots, a_{n-1}\} = \varnothing$. Then the transposition of $n-1$ and n is admissible and

$$(a_1, \dots, a_{n-2}, a_n, a_{n-1}) \in \operatorname{Spec}(n) .$$

Hence $(a_1, \dots, a_{n-2}, a_n) \in \operatorname{Spec}(n-1)$ and clearly

$$\{a_n - 1, a_n + 1\} \cap \{a_1, \dots, a_{n-2}\} = \varnothing$$

in contradiction to the inductive assumption. This proves the necessity of (2).

Suppose $a_p = a_n = a$ for some $p < n$ and,

$$a - 1 \notin \{a_{p+1}, \dots, a_{n-1}\} .$$

We can assume, that p is chosen maximal, that is

$$a \notin \{a_{p+1}, \dots, a_{n-1}\} .$$

Then, by the inductive assumption, the number $a+1$ appears in $\{a_{p+1}, \dots, a_{n-1}\}$ at most once. That is we have two possibilities: either

$$(a_p, \dots, a_n) = (a, *, \dots, *, a) ,$$

or

$$(a_p, \dots, a_n) = (a, *, \dots, *, a+1, *, \dots, *, a) ,$$

where $*, \dots, *$ stands for a sequence of numbers different from $a-1$, a, $a+1$.

In the first case applying $n - p - 1$ admissible transpositions we get

$$\alpha \sim \alpha' = (\dots, a, a, \dots) ,$$

which contradicts proposition 5.1, part (1).

In the second case by the same argument

$$\alpha \sim \alpha' = (\dots, a, a+1, a, \dots) ,$$

which is impossible by the lemma. $\square$

We shall need one more equivalence relation. Write

$$\alpha \approx \beta, \quad \alpha, \beta \in \mathbb{C}^n ,$$

if β is a permutation of entries of α. The set $\mathrm{Cont}(n)$ with the relation $\approx$ has the following simple combinatorial interpretation.

Denote by $\mathbb{Y}$ the Young graph. By definition, the vertices of $\mathbb{Y}$ are Young diagrams and two vertices ν and η are joined by an oriented edge iff $\nu \subset \eta$ and η/ν is a single box. Write $\nu \nearrow \eta$ in this case. Recall that, given a box $\square \in \eta$, the number

$$c(\square) = x\text{-coordinate of } \square - y\text{-coordinate of } \square$$

is called the *content* of $\square$. By $\mathrm{Tab}(\nu)$ denote the set of paths in $\mathbb{Y}$ from $\varnothing$ to ν, such paths are called *standard* or Young tableaux. The convenient way to represent a path $T \in \mathrm{Tab}(\nu)$

$$\varnothing = \nu_0 \nearrow \ldots \nearrow \nu_n = \nu$$

is to write the numbers $1, \ldots, n$ in the boxes $\nu_1/\nu_0, \ldots, \nu_n/\nu_{n-1}$ of ν_n respectively. Put

$$\mathrm{Tab}(n) = \bigcup_{|\nu|=n} \mathrm{Tab}(\nu).$$

The following proposition can be easily checked

Proposition 6.3. *Suppose*

$$T = \nu_0 \nearrow \ldots \nearrow \nu_n \in \mathrm{Tab}(n).$$

The map

$$T \mapsto (c(\nu_1/\nu_0), \ldots, c(\nu_n/\nu_{n-1}))$$

is a bijection of $\mathrm{Tab}(n)$ *and* $\mathrm{Cont}(n)$. *We have* $\alpha \approx \beta$, $\alpha, \beta \in \mathrm{Cont}(n)$ *iff the corresponding paths have the same end, that is iff they are tableaux on the same diagram.*

In terms of Young tableaux, admissible transpositions are transpositions of numbers from different rows and columns.

Lemma 6.4. *Any two Young tableaux* $T_1, T_2 \in \mathrm{Tab}(\nu)$ *on a diagram* ν *can be obtained from each other by admissible transpositions. In other words, if* $\alpha, \beta \in \mathrm{Cont}(n)$ *and* $\alpha \approx \beta$ *then* β *can be obtained from* α *by admissible transpositions.*

Proof. Let us show that by admissible transpositions we can take any Young tableau $T \in \mathrm{Tab}(\nu)$ to the following tableau

$$T^\nu = \begin{array}{ccccc} 1 & 2 & \ldots & \ldots & \nu_1 \\ \nu_1 + 1 & \ldots & \nu_1 + \nu_2 & & \\ \ldots & & & & \end{array},$$

which corresponds to the following element of $\mathrm{Cont}(n)$

$$\alpha(T^\nu) = (0, 1, 2, \ldots, \nu_1 - 1, -1, 0, \ldots, \nu_2 - 2, -2, -1, \ldots).$$

To that end consider the last box of the last row of ν. Let i be the number written in this box of T. Transpose i and $i+1$, then $i+1$ and $i+2$,..., and finally $n-1$ and n. Clearly, all these transpositions are admissible, and we obtain a tableau with the number n written in the last box of the last column.

Now the number n is in the right position, so we can forget about it and repeat the same procedure for $n-1, n-2, \ldots$. $\square$

Corollary 6.5. *If* $\alpha \in \mathrm{Spec}(n)$ *and* $\alpha \approx \beta$, $\beta \in \mathrm{Cont}(n)$, *then* $\beta \in \mathrm{Spec}(n)$ *and* $\alpha \sim \beta$.

Remark 6.6. Our chain of transpositions linking T and T^ν is a minimal possible in the following sense. Denote by s the permutation, which maps T to T^ν. Let $\ell(s)$ be the number of inversions in s, that is

$$\ell(s) = \#\{(i,j) \in \{1,\dots,n\} \,|\, i < j, s(i) > s(j)\}\,.$$

It is well known that s can be written as a product of $\ell(s)$ transpositions s_i and cannot be written as a shorter product[3]. It is easy to see that our chain contains exactly $\ell(s)$ admissible transpositions. In other words, $\mathrm{Cont}(n)$ is a "totally geodesic" subset of $\mathbb{Z}^n$ for the action of $S(n)$.

There is an important transformation on the set of all tableaux on a given diagram which takes a tableau to the next tableau in the lexicographic order (see [VK] and references therein). This transformation was called the *Young automorphism* by the second author.

Theorem 6.7. $\mathbb{Y}$ *is the branching graph of the symmetric groups,* $\mathrm{Spec}(n) = \mathrm{Cont}(n)$ *and* $\sim = \approx$.

Proof. Consider the quotient $\mathrm{Cont}(n)/\approx$. We have

$$\#\{\,\mathrm{Cont}(n)/\approx\,\} = \#\{\text{partitions of } \lambda\}\,.$$

By Corollary 6.3 each equivalence class in $\mathrm{Cont}(n)/\approx$ either contains no elements of $\mathrm{Spec}(n)$ or is a subset of an equivalence class in $\mathrm{Spec}(n)/\sim$. Since

$$\#\{\,\mathrm{Spec}(n)/\sim\,\} = \#\{S(n)^\wedge\} = \#\{\text{partitions of } \lambda\}\,,$$

all equivalence classes in $\mathrm{Cont}(n)/\approx$ coincide with some equivalence classes in $\mathrm{Spec}(n)/\sim$. In other words,

$$\mathrm{Spec}(n) = \mathrm{Cont}(n) \quad \text{and} \quad \sim=\approx\,.$$

Clearly, this implies that $\mathbb{Y}$ is the branching graph of the chain of symmetric groups. □

§7. Young formulas

So far Young basis vectors v_T were considered up to scalar factors. In this section we shall specify the choice of these factors. The material in this section is standard.

Let us start with the tableau T^λ defined in the proof of Lemma 6.4. Choose any non-zero vector v_{T^λ} corresponding to this tableau.

Given a tableau $T \in \mathrm{Tab}(\lambda)$ put

$$\ell(T) = \ell(s)\,,$$

where s is the permutation, which maps T^λ to T. Recall that P_T denotes the orthonormal projection onto V_T (see §1). Put

$$v_T = P_T \cdot s \cdot v_{T^\lambda} \tag{7.1}$$

[3] simply because $\ell(s_i g) = \ell(g) \pm 1$ for all i and $g \in S(n)$.

By Lemma 6.4 the permutation s can be represented as a product of $\ell(T)$ admissible transpositions. Therefore, by definition (7.1) and the formulas (5.2)

$$s \cdot v_{T^\lambda} = v_T + \sum_{R \in \mathrm{Tab}(\lambda), \ell(R) < \ell(T)} \gamma_R \, v_R \tag{7.2}$$

where γ_R are some rational numbers. In particular, suppose $T' = s_i T$ and

$$\ell(T') > \ell(T) .$$

Let

$$\alpha(T) = (a_1, \ldots, a_n) \in \mathrm{Cont}(n)$$

be the sequence of contents of boxes in T. Then by (5.2), (7.1), and (7.2) we have

$$s_i \cdot v_T = v_{T'} + \frac{1}{a_{i+1} - a_i} v_T . \tag{7.3}$$

And again by (5.2)

$$s_i \cdot v_{T'} = \left(1 - \frac{1}{(a_{i+1} - a_i)^2} \right) v_T - \frac{1}{a_{i+1} - a_i} v_{T'} . \tag{7.4}$$

This proves the following

Proposition 7.1. *There exists a basis v_T of V^λ in which the generators s_i act by the formulas* (7.3,7.4). *All irreducible representations of $S(n)$ are defined over the field $\mathbb{Q}$.*

Another way to prove this proposition is to verify directly that these formulas define a representation of $S(n)$ (that is to verify the Coxeter relations).

This basis is called the Young *seminormal form* of V^λ. If we normalize all vectors v_T we obtain the Young *orthonormal form* of V^λ. This form is defined over $\mathbb{R}$. Denote the normalized vectors by the same letters v_T. Then s_i acts in the two-dimensional space spanned by v_T and $v_{T'}$ by an orthogonal matrix. Therefore

$$s_i = \begin{pmatrix} r^{-1} & \sqrt{1 - r^{-2}} \\ \sqrt{1 - r^{-2}} & -r^{-1} \end{pmatrix} , \tag{7.5}$$

where

$$r = a_{i+1} - a_i .$$

This number is usually called the *axial distance*, see [JK].

Proposition 7.2. *There exists an orthonormal basis v_T of V^λ in which the generators s_i act by the formulas* (7.5).

Remark 7.3. Since the weight $\alpha(T^\lambda)$ of v_{T^λ} is the maximal weight in V^λ with respect to the lexicographic order, we can call the weight $\alpha(T^\lambda)$ the *highest weight* of V^λ and call the vector v_{T^λ} the highest vector of V^λ.

The same formulas give the action of symmetric group in the representations associated to skew Young diagrams.

Suppose $|\mu| = l$, $|\lambda| = l + k$. Denote by $V^{\lambda/\mu}$ the $Z(l,k)$-module

$$V^{\lambda/\mu} = \mathrm{Hom}_{S(l)}(V^\mu, V^\lambda) ,$$

which was considered in §4. Clearly, this module has a similar Young orthonormal basis indexed by all Young tableaux on the skew diagram λ/μ. The generators

$$X_{l+i} , \quad i = 1, \ldots, k$$

of $Z(l,k)$ act in this basis by multiplication by the content of the i-th box and the Coxeter generators of the subgroup $S(k) \subset Z(l,k)$ act by formulas (7.5).

§8. Characters of symmetric groups

In this section we prove the Murnaghan-Nakayama rule for characters of the symmetric groups. The key fact we use is Proposition 8.3, which is based on Theorem 4.1

Recall that a Young diagram γ is called a *hook* if $\gamma = (a+1, 1^b)$ for some $a, b \in \mathbb{Z}_+$. The number b is called the height of the hook γ. Recall also that a skew diagram λ/μ is called a *skew hook*, if it is connected and does not have two boxes on the same diagonal. In other words, λ/μ is a skew hook if the contents of the boxes of λ/μ are some $|\lambda/\mu|$ consecutive integers. The number of rows occupied by λ/μ minus 1 is called the *height* of λ/μ and is denoted by $\langle\lambda/\mu\rangle$. Put $k = |\lambda/\mu|$. Let $V^{\lambda/\mu}$ be the $S(k)$-module corresponding to the skew diagram λ/μ, and let $\chi^{\lambda/\mu}$ be the corresponding character. Our aim is to prove the following well-known theorem

Theorem 8.1.

$$\chi^{\lambda/\mu}((12\dots k)) = \begin{cases} (-1)^{\langle\lambda/\mu\rangle}, & \textit{if } \lambda/\mu \textit{ is a skew hook,} \\ 0, & \textit{otherwise}. \end{cases} \tag{8.1}$$

Now suppose that ρ is a partition of k. Consider the following permutation from the conjugacy class corresponding to ρ

$$(12\dots\rho_1)(\rho_1+1\ \dots\ \rho_1+\rho_2)(\dots)\dots.$$

It is clear that applying Theorem 8.1 several times we get the following classical

Murnaghan-Nakayama rule. *Let ρ be a partition of k. The value $\chi^{\lambda/\mu}_\rho$ of the character $\chi^{\lambda/\mu}$ on a permutation with cycle type ρ equals*

$$\chi^{\lambda/\mu}_\rho = \sum_S (-1)^{\langle S\rangle},$$

where the sum is over all sequences S

$$\mu = \lambda_0 \subset \lambda_1 \subset \lambda_2 \dots = \lambda,$$

such that λ_i/λ_{i-1} is a skew hook with ρ_i boxes, and

$$\langle S\rangle = \sum_i \langle\lambda_i/\lambda_{i-1}\rangle$$

It is well-known and can be easily proved (see, for example [M], Ch. 1, Ex. 3.11) that this rule is equivalent to all other descriptions of the characters of symmetric groups, for example, to the following symmetric functions relation [M]

$$p_\rho = \sum_\lambda \chi^\lambda_\rho s_\lambda,$$

or to the determinantal formula [M,JK]. Note that the theorem we are going to prove is a special case of the Murnaghan-Nakayama rule.

The same proof of the following proposition was also given in [DG].

Proposition 8.2. *The formula* (8.1) *is true for* $\mu = \varnothing$.

Proof. It is easy to see (for example, by induction) that

$$X_2 X_3 \dots X_k = \text{sum of all } k\text{-cycles in } S(k)\ . \tag{8.2}$$

The eigenvalue of (8.2) on any Young basis vector in V^λ equals

$$(-1)^b b!\,(k-b-1)!$$

if λ is a hook of height b, and equals zero otherwise. Clearly, the number of k-cycles in $S(k)$ equals $(k-1)!$ and

$$\dim V^\lambda = \binom{k-1}{b},$$

if λ is a hook of height b. Taking the trace of (8.2) in V^λ proves the proposition. □

Proposition 8.3. *For any vector v from the Young basis of $V^{\lambda/\mu}$*

$$\mathbb{C}[S(k)] \cdot v = V^{\lambda/\mu}$$

Proof. The vector space $V^{\lambda/\mu}$ is an irreducible $H(k)$-module. The vector v is by assumption a common eigenvector for all X_i. By commutation relations in $H(k)$ the space

$$\mathbb{C}[S(k)] \cdot v$$

is $H(k)$-invariant and hence equals $V^{\lambda/\mu}$. □

Proposition 8.4. *If λ/μ is not connected then*

$$\chi^{\lambda/\mu}((12\dots k)) = 0\,.$$

Proof. Suppose $\lambda/\mu = \nu_1 \cup \nu_2$, where ν_1, ν_2 are two skew Young diagram which have no edge in common. Put $a = |\nu_1|$, $b = |\nu_2|$. Consider the Young tableaux on λ/μ that have the numbers $1, 2, \dots, a$ in ν_1 and the numbers $a+1, \dots, k$ in ν_2. Consider the subspace of $V^{\lambda/\mu}$ spanned by all Young basis vectors with such tableaux and consider the action of the subgroup $S(a) \times S(b)$ of $S(k)$ on this subspace. It is clear that as $S(a) \times S(b)$-module this subspace is isomorphic to

$$V^{\nu_1} \otimes V^{\nu_2}\,.$$

By Proposition 8.3 we have the following epimorphism

$$\operatorname{Ind}_{S(a)\times S(b)}^{S(k)} V^{\nu_1} \otimes V^{\nu_2} \longrightarrow V^{\lambda/\mu}\,. \tag{8.3}$$

It is easy to see that the dimension of the both sides of (8.3) equals

$$\binom{k}{a} \dim \nu_1 \, \dim \nu_2\,.$$

Hence (8.3) is an isomorphism.

In the natural basis of the induced representation, the permutation $(12\dots k)$ (as well as any permutation which is not conjugated to an element of $S(a) \times S(b)$) has only zeroes on the diagonal. This proves the proposition. □

Proposition 8.5. *If λ/μ has two boxes on the same diagonal then*

$$\chi^{\lambda/\mu}((12\dots k)) = 0\,.$$

Proof. Suppose there are two such boxes. Then there is a diagram η such that

$$\mu \subset \eta \subset \lambda$$

and η/μ is a 2×2 square

$$\eta/\mu = \boxplus\,.$$

That is $V^{\lambda/\mu}$ contains a $S(4)$-submodule $V^{\boxplus}$. By Proposition 8.3 we have an epimorphism

$$\operatorname{Ind}_{S(4)}^{S(k)} V^{\boxplus} \longrightarrow V^{\lambda/\mu}\,. \tag{8.4}$$

By the branching rule and Frobenius reciprocity, the left-hand side of (8.4) contains only such $S(k)$-modules V^{δ} that $\boxplus \subset \delta$. In particular, δ cannot be a hook and therefore

$$\chi^{\delta}((12\dots k)) = 0\,,$$

by proposition 8.2. This proves the proposition. $\square$

In fact, we have proved that under the assumptions of the proposition 8.5

$$\operatorname{Hom}_{S(k)}(V^{\gamma}, V^{\lambda/\mu}) = 0\,,$$

for all hook diagram γ.

Proposition 8.6. *Suppose λ/μ is a skew hook. Then for any hook $\gamma = (a+1, 1^b)$*

$$\operatorname{Hom}_{S(k)}(V^{\gamma}, V^{\lambda/\mu}) = \begin{cases} \mathbb{C}, & b = \langle \lambda/\mu \rangle\,, \\ 0, & \textit{otherwise}\,. \end{cases}$$

Proof. Since translation of λ/μ does not change the corresponding $S(k)$-module, we can assume that λ and μ are chosen minimal, that is

$$\lambda_1 > \mu_1, \quad \lambda_1' > \mu_1'\,.$$

Show that if $b < \langle \lambda/\mu \rangle$ then

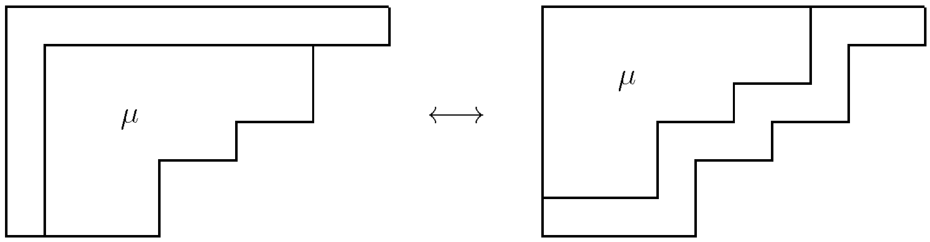
$$\operatorname{Hom}_{S(k)}(V^{\gamma}, V^{\lambda/\mu}) = 0\,.$$

Indeed, the module V^{γ} has a nonzero $S(k-b)$-invariant vector and there is no such vectors in $V^{\lambda/\mu}$ since there is no such vectors in V^{λ} (this follows from the branching rule). The case $b > \langle \lambda/\mu \rangle$ is similar.

Now suppose $b = \langle \lambda/\mu \rangle$. Consider the vector space

$$\operatorname{Hom}_{S(k)}(V^{\gamma}, V^{\lambda})\,.$$

It is easy to see from the following picture (and Young formulas) that



this space is the irreducible $S(|\mu|)$-module V^μ. Therefore

$$\mathrm{Hom}_{S(k)\times S(|\mu|)}(V^\gamma \otimes V^\mu, V^\lambda) = \mathbb{C},$$

and thus

$$\mathrm{Hom}_{S(k)}(V^\gamma, V^{\lambda/\mu}) = \mathbb{C}. \quad \square$$

The theorem follows evidently from the proved propositions.

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