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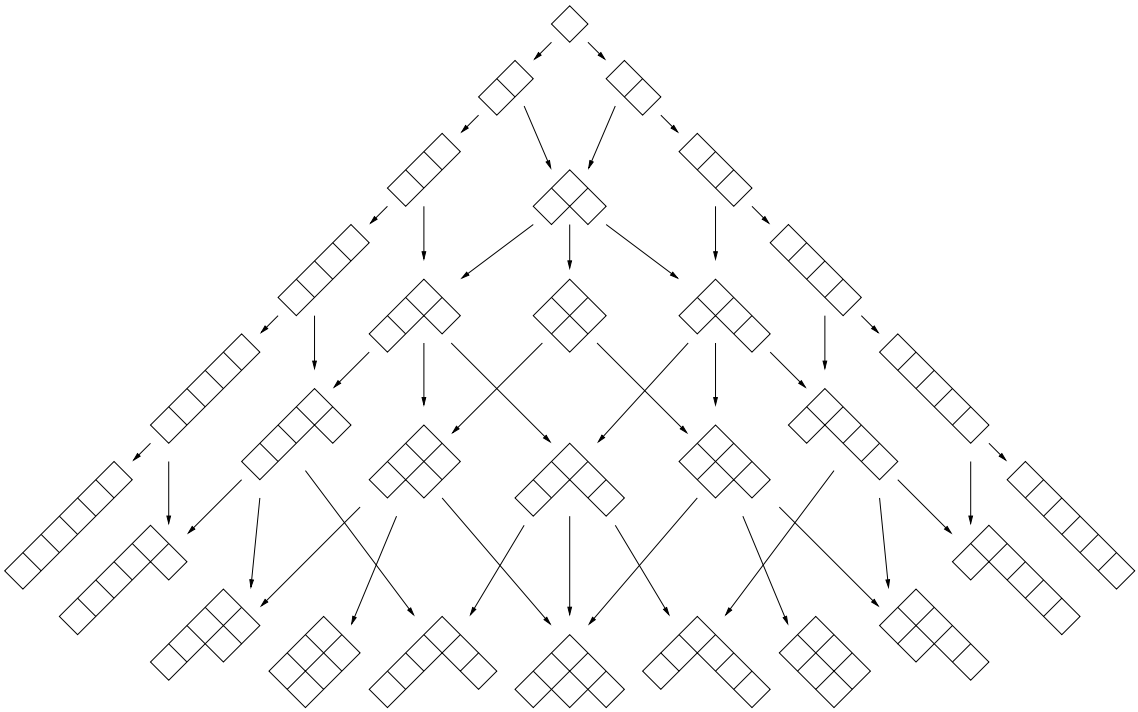
COMBINATORICS OF SYMMETRIC GROUPS

Lecture 4: Representations of the symmetric group

From the combinatorial point of view, the main theorem of the representation theory of symmetric groups claims the isomorphism of two infinite graphs: the *Young graph*, built out of Young diagrams, and the *Frobenius graph* that comes from irreducible representations of the symmetric groups.

Definition. The vertices of the *Young graph* $\mathcal{Y}$ are Young diagrams. Two diagrams, $[\lambda]$ and $[\mu]$, are connected by an edge going from $[\lambda]$ to $[\mu]$, if $|\mu| = |\lambda| + 1$ and $[\mu]$ can be obtained from $[\lambda]$ by the addition of one cell.



Definition. The vertices of the *Frobenius graph* $\mathcal{F}$ are equivalence classes of the irreducible representations of groups S_n for all $n \geq 1$. Two representations, U of S_{n-1} and V of S_n, are connected by an edge, if U appears in the decomposition of the representation V, restricted to the subgroup $S_{n-1} \subset S_n$. [We remind that (1) the multiplicity of U in the restriction can only be 0 or 1, and (2) this multiplicity is equal to the multiplicity of V in the representation of S_n, induced from U.]

Main theorem. $\mathcal{Y} \cong \mathcal{F}$.