Kontsevich's integral of a KNOT is a far-reaching generalization of the GAUSS INTEGRAL FORMULA for the LINKING NUMBER. It provides a tool to construct the universal VASSILIEV INVARIANT. Any Vassiliev knot invariant can be derived from it.

Represent the three-dimensional space $\mathbb{R}^3$ as a direct product of a complex line $\mathbb{C}$ with coordinate z and a real line $\mathbb{R}$ with coordinate t. The integral is defined for Morse knots, i.e. knots K embedded in $\mathbb{R}^3 = \mathbb{C}_z \times \mathbb{R}_t$ in such a way that the coordinate t is a MORSE FUNCTION on K (all CRITICAL POINTS are nondegenerate and all CRITICAL LEVELS are different). Its values belong to the GRADED COMPLETION $\overline{\mathcal{A}}$ of the ALGEBRA OF CHORD DIAGRAMS $\mathcal{A}$.

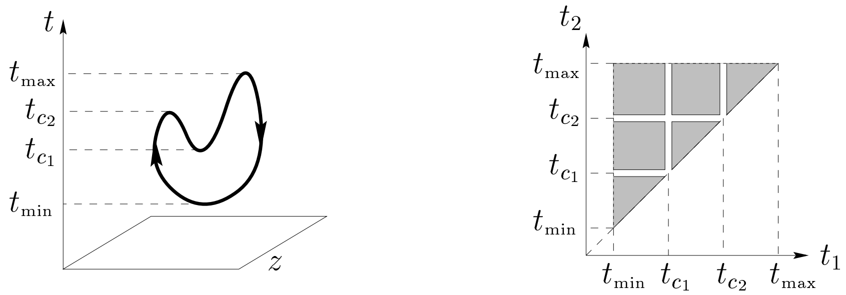
The *Kontsevich integral* $Z(K)$ of the knot K is defined as follows:

$$Z(K) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int\limits_{\substack{t_{\min}<t_1<\ldots<t_m<t_{\max} \\ t_j \text{ are noncritical}}} \sum_{P=\{(z_j,z'_j)\}} (-1)^{\downarrow} D_P \bigwedge_{j=1}^{m} \frac{dz_j - dz'_j}{z_j - z'_j} .$$

The ingredients of this formula have the following meaning.

The real numbers $t_{\min}$ and $t_{\max}$ are the minimum and the maximum of the function t on K.

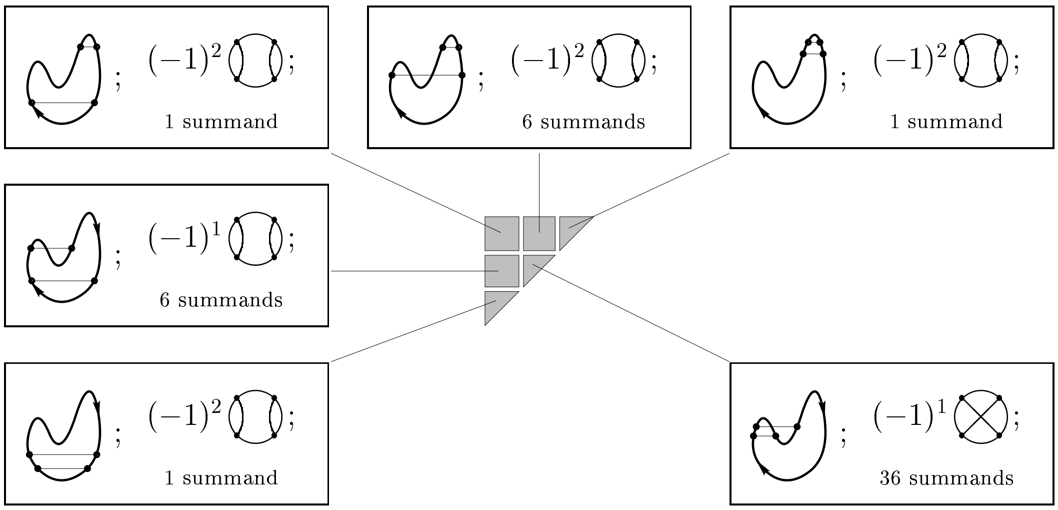
The integration domain is the m-dimensional simplex $t_{\min} < t_1 < \ldots < t_m < t_{\max}$ divided by the critical values into a certain number of *connected components*. For example, for the following embedding of the unknot and $m = 2$ the corresponding integration domain has six connected components and looks like



The number of summands in the integrand is constant in each connected component of the integration domain, but can be different for different components. In each plane $\{t = t_j\} \subset \mathbb{R}^3$ choose an unordered pair of distinct points (z_j, t_j) and (z'_j, t_j) on K, so that $z_j(t_j)$ and $z'_j(t_j)$ are continuous functions. We denote by $P = \{(z_j, z'_j)\}$ the set of such pairs for $j = 1, \ldots, m$. The integrand is the sum over all choices of P. In the example above for the component $\{t_{\min} < t_1 < t_{c_1},\ t_{c_2} < t_2 < t_{\max}\}$ we have only one possible pair of points on the levels $\{t = t_1\}$ and $\{t = t_2\}$. Therefore, the sum over P for this component consists of only one summand. Unlike this, in the component $\{t_{\min} < t_1 < t_{c_1},\ t_{c_1} < t_2 < t_{c_2}\}$ we still have only one possibility for the level $\{t = t_1\}$, but the plane $\{t = t_2\}$ intersects our knot K in four points. So we have $\binom{4}{2} = 6$ possible pairs (z_2, z'_2) and the total number of summands is six (see the picture below).

For a pairing P the symbol '$\downarrow$' denotes the number of points (z_j, t_j) or (z'_j, t_j) in P where the coordinate t decreases along the orientation of K.

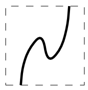
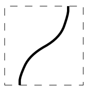
Fix a pairing P. Consider our knot K as an oriented circle and connect the points (z_j, t_j) and (z_j', t_j) by a chord. We obtain a chord diagram with m chords. The corresponding element of the algebra $\mathcal{A}$ is denoted by D_P. In the picture below, for each connected component in our example, we show one of the possible pairings, the corresponding chord diagram with the sign $(-1)^{\downarrow}$ and the number of summands of the integrand (some of which are equal to zero in $\mathcal{A}$ due to ONE-TERM RELATION).



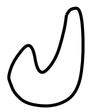
Over each connected component z_j and z_j' are smooth functions in t_j. By $\bigwedge\limits_{j=1}^{m} \dfrac{dz_j - dz_j'}{z_j - z_j'}$ we mean the PULLBACK of this form to the integration domain of variables $t_1, \ldots, t_m$. The integration domain is considered with the ORIENTATION of the space $\mathbb{R}^m$ defined by the natural order of the coordinates t_1, ..., t_m.

By convention, the term in the Kontsevich integral corresponding to $m = 0$ is the (only) chord diagram of order 0 with coefficient one. It represents the unit of the algebra $\mathcal{A}$.

The Kontsevich integral is convergent thanks to ONE-TERM RELATIONS. It is invariant under DEFORMATIONS of the knot in the class of Morse knots. Unfortunately, the Kontsevich integral is not invariant under deformations that change the number of critical points of the function t. However, the following formula shows how the integral changes under such deformations:

$Z\Big($  $\Big) = Z(H) \cdot Z\Big($  $\Big).$

Here the first and the third pictures depict two embeddings of an arbitrary knot, differing only in the shown fragment,

$H =$ 

is the *hump* (unknot embedded in $\mathbb{R}^3$ in the specified way), and the product is the product in the completed algebra $\overline{\mathcal{A}}$ of chord diagrams. The last equality allows to define the universal Vassiliev invariant by the formula

$$I(K) = \frac{Z(K)}{Z(H)^{c/2}},$$

where c denotes the number of critical points of K and quotient means division in the algebra $\overline{A}$ according to the rule $(1+a)^{-1} = 1 - a + a^2 - a^3 + \ldots$

The universal Vassiliev invariant $I(K)$ is invariant under an arbitrary deformation of K.

Consider a function w on the set of chord diagrams with m chords statisfying ONE- AND FOUR-TERM RELATIONS (a WEIGHT SYSTEM). Applying this function to the universal Vassiliev invariant $w(I(K))$, we get a numerical knot invariant. This invariant will be a Vassiliev invariant of order m and any Vassiliev invariant can be obtained in this way.

The Kontsevich integral behaves in a nice way with respect to the natural operations on knots, such as mirror reflection, changing the orientation of the knot, mutation of knots. In a proper normalization it is multiplicative under the connected sum of knots: $I'(K_1 \# K_2) = I'(K_1)I'(K_2)$, where $I'(K) = Z(H)I(K)$. For any knot K the coefficients in the expansion of $Z(K)$ over an arbitrary basis consisting of chord diagrams are rational [K, LM].

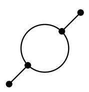
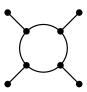
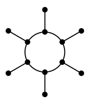
The task of computing the Kontsevich integral is very difficult. The explicit expression of the universal Vassiliev invariant $I(K)$ is currently known only for the unknot [BGRT]:

$$I(O) = \exp \sum_{n=0}^{\infty} b_{2n} w_{2n} = 1 + (\sum_{n=0}^{\infty} b_{2n} w_{2n}) + \frac{1}{2}(\sum_{n=0}^{\infty} b_{2n} w_{2n})^2 + \ldots$$

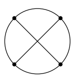
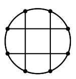
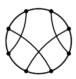
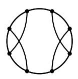
Here b_{2n} are MODIFIED BERNOULLI NUMBERS, i.e. the coefficients of the Taylor series

$$\sum_{n=0}^{\infty} b_{2n} x^{2n} = \frac{1}{2} \ln \frac{e^{x/2} - e^{-x/2}}{x/2},$$

($b_2 = 1/48$, $b_4 = -1/5760$, ...), and w_{2n} are the *wheels*, i.e. DIAGRAMS of the form

$w_2 =$  , $w_4 =$  , $w_6 =$  , ...

The linear combination is understood as an element of the ALGEBRA OF CHINESE CHARACTERS $\mathcal{B}$, which is isomorphic to the ALGEBRA OF CHORD DIAGRAMS $\mathcal{A}$. Expressed through chord diagrams, the beginning of this series looks as follows:

$I(O) = 1 - \frac{1}{24}$  $- \frac{1}{5760}$  $+ \frac{1}{1152}$  $+ \frac{1}{2880}$  $+ \ldots$

The Kontsevich integral was invented by M.Kontsevich [K]. Detailed expositions can be found in [Ar, BN, CD].

See also: VASSILIEV INVARIANTS, CHORD DIAGRAMS.

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