History

The notion of finite type (Vassiliev) KNOT INVARIANTS was independently invented by V.Vassiliev and M.Goussarov around 1989.

Vassiliev's approach is based on the study of *discriminants* in the (infinite-dimensional) spaces of SMOOTH MAPS from one MANIFOLD into another. By definition, the discriminant consists of all maps with SINGULARITIES.

For example, consider the space of all smooth maps from the circle into 3-space $\mathcal{M} = \{f : S^1 \to \mathbb{R}^3\}$. If f is an EMBEDDING (has no singular points), it represents a knot. The complement of the set of all knots is the discriminant $\Sigma \subset \mathcal{M}$. It consists of all smooth maps from S^1 into $\mathbb{R}^3$ that have singularities, either *local*, where $f' = 0$, or *non-local*, where f is not injective. Two knots are equivalent, if and only if they can be joined by a path in the space $\mathcal{M}$ that does not intersect the discriminant. Therefore, knot types are in one-to-one correspondence with the connected components of the complement $\mathcal{M} \setminus \Sigma$, and knot invariants with values in an Abelian group G are nothing but COHOMOLOGY classes from $H^0(\mathcal{M} \setminus \Sigma, G)$. The filtration of Σ by subspaces corresponding to singular knots with a given number of simple double points gives rise to a SPECTRAL SEQUENCE, which contains, in particular, the spaces of finite type invariants.

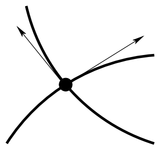
Birman and Lin (1993) have contributed a lot to the simplification of the Vassiliev's original techniques. They, in particular, explained the relation betwen Jones polynomial and finite type invariants and emphasized the role of the algebra of CHORD DIAGRAMS. Kontsevich (1993) proved the first difficult theorem about Vassiliev invariants with the help of KONTSEVICH'S INTEGRAL. Bar-Natan undertook a thorough study of Vassiliev invariants; in particular, he showed the importance of the algebra of Feynman diagrams and diagrams with uni- and tri-valent vertices; his paper (1995) is the most authoritative source on the subject.

Definitions

Expressed in simple terms, Vassiliev's fundamental idea is to study the prolongation of KNOT INVARIANTS to *singular knots* — immersions $f : S^1 \to \mathbb{R}^3$ having a finite number of simple double points.

A point $p \in \mathrm{im}(f) \subset \mathbb{R}^3$ is a *simple double point* if its preimage under f consists of two values t_1 and t_2 and the two tangent vectors $f'(t_1)$ and $f'(t_2)$

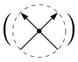
are non-collinear. Geometrically, this means that, in a neighborhood of p, the curve consists of 2 transversal branches:



Simple double point

Let X_n denote the set of equivalence classes of singular knots with n double points and no other singularities. The following definition is based on a recursion which allows to extend a knot invariant from X_0 to X_1, then to X_2, etc. and thus finally to the whole of $X = \bigcup_n X_n$.

Definition. Vassiliev's prolongation of knot invariants. Given a knot invariant $v : X_0 \to \mathbb{Q}$, we define its prolongation $\hat{v} : X \to \mathbb{Q}$ by the rules

1. $\hat{v}|_{X_0} \equiv v$.
2. $\hat{v}($$) = \hat{v}($$) - \hat{v}($$)$ (*Vassiliev's skein relation*).

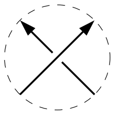
The right hand side of Vassiliev's skein relation refers to the two resolutions of the double point — positive and negative. A crucial observation is that each of them is well-defined (does not depend on the plane projection used to express this relation).

Definition. A knot invariant v is called a *Vassiliev invariant of order* $\leq n$ if its prolongation $\hat{v}$ vanishes on all knots with more than n double points.

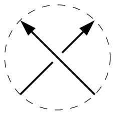
Example 1. The simplest non-trivial Vassiliev invariant v_2 has the following explicit description. Let D be an arbitrary KNOT DIAGRAM of the given knot K and $\star$ an arbitrary distinguished point on D, different from all crossings. Then

$$v_2(K) = \sum_{\substack{i\ j\ i\ j\\ U\,O\,O\,U}} \varepsilon_i \varepsilon_j \ ,$$

where the summation spreads over all pairs of crossing points i, j such that (1) during one complete turn of the diagram in the positive direction starting from point $\star$ the points i and j are encountered in the order i, j, i, j (2) the four corresponding passages through these crossing points are underpass, overpass, overpass and underpass respectively. The numbers ε_i, ε_j stand for the local WRITHE at points i and j defined according to the picture:



Local writhe $+1$



Local writhe -1

Example 2. The n-th coefficient of the CONWAY POLYNOMIAL is a Vassiliev invariant of order n. In particular, the second coefficient coincides with v_2.

Some basic facts

Vassiliev invariants are at least as strong as all known polynomial knot invariants: ALEXANDER, JONES, KAUFFMAN and HOMFLY polynomials. This means that if two knots, K_1 and K_2, can be distinguished by such a polynomial, then there is a Vassiliev invariant that takes different values on K_1 and K_2.

The set of all $\mathbb{Q}$-valued Vassiliev invariants $V = \bigcup_n V_n$ forms a vector space over the rationals, with the increasing FILTRATION $\mathbb{Q} = V_0 \subset V_1 \subset V_2 \subset \ldots$ The ASSOCIATED GRADED SPACE $\bigoplus_n V_n/V_{n-1}$ has a structure of a HOPF ALGEBRA and can be interpreted as the algebra of CHORD DIAGRAMS.

The number of independent Vassiliev invariants of a given degree n (i.e. the dimension of V_n) is known for $n \leq 12$ (J.Kneissler, 1997):

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim V_n$	1	1	2	3	6	10	19	33	60	104	184	316	548

The totality of all Vassiliev invariants is equivalent to one UNIVERSAL VASSILIEV INVARIANT defined through the KONTSEVICH INTEGRAL.

Two of the most important problems about Vassiliev invariants that were raised in 1990 and are still open:

1. Is it true that Vassiliev invariants distinguish knots? In other words, given two non-equivalent knots K_1 and K_2, is it always possible to indicate a finite type invariant v such that $v(K_1) \neq v(K_2)$?

2. Is it true that Vassiliev invariants can detect knot orientation? More specifically, is there a knot K and a finite type invariant v such that $v(K) \neq v(\bar{K})$, where $\bar{K}$ differs from K by a change of parameterization that reverses the orientation?

See also: Kontsevich's Integral, Chord diagrams.

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