Vassiliev invariants and combinatorial structures

Exercises and open problems

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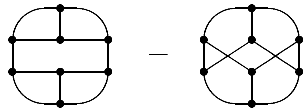
1 Exercises

1.
   - Find some bases of the spaces $\mathcal{A}_n$, $\mathcal{F}_n$, $\mathcal{B}_n$ for $n \leqslant 4$.
   - Indicate an explicit form of the isomorphisms $\mathcal{A} \cong \mathcal{F} \cong \mathcal{B}$ in these bases.
   - Compile the multiplication table for $\mathcal{B}_m \times \mathcal{B}_n \to \mathcal{B}_{m+n}$, $m + n \leq 4$, for the second product in $\mathcal{B}$ (the one pulled back from $\mathcal{F}$ along the isomorphism $\mathcal{F} \cong \mathcal{B}$).

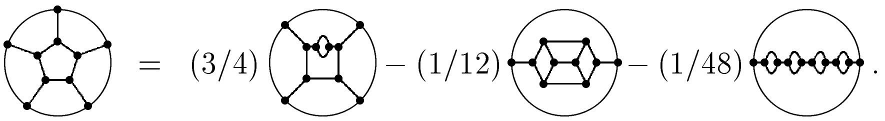
2. (Lecture 6). Check that the following linear combination is equal to a connected Feynman diagram:



3. (Lecture 8). If a tangle is closed into a circle, then the tangled 4 term relation becomes the usual 4 term relation for circular chord diagrams.

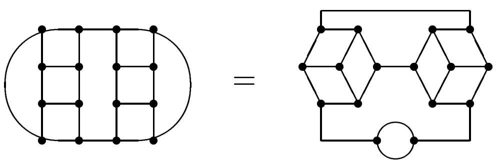
4. Prove that the mapping $\psi : \mathcal{B} \to S(\mathfrak{g})$ is well-defined, i.e. that the IHX relation follows from the Jacobi identity.

5. Prove the following identity in the space $\mathcal{F}$ of Feynman diagrams:

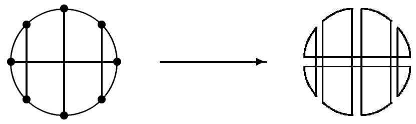


6. Prove the linear independence of the three elements in the right hand side of the last equality, using Lie algebra invariants.

7. Find explicitly a chain of IHX and AS relations that proves the following equality in the algebra Γ of 3-graphs:



8. (*S. Lando*). Denote by $s(D)$ the number of connected components of the curve obtained by doubling all chords of a chord diagram D. Example:

for $D =$ 

we have $s(D) = 3$.

Let N be a formal variable. Prove that:

- $s(D) - 1$ is equal to the corank of the adjacency matrix of the intersection graph $I(D)$,
- $N^{s(D)-1}$ is a multiplicative weight system, i.e. defines an algebra homomorphism $\mathcal{A} \to \mathbb{Z}[N]$.
- Find the relation of this weight system with the Lie algebra weight system for the algebra gl_N.

9. (*S. Lando*). Let N be a formal variable. Prove that $N^{\mathrm{corank} A(G)}$ defines an algebra homomorphism $\mathcal{L} \to \mathbb{Z}[N]$, where $\mathcal{L}$ is the graph algebra of Lando and $A(G)$ stands for the adjacency matrix of the graph G.

10. (Lectures 2 and 13). Let A be the arrow diagram with two intersecting chords. Denote by $G(K)$ the Gauss diagram corresponding to any plane diagram of the knot K. Prove that $K \mapsto \langle A, G(K) \rangle$ is a Vassiliev invariant of order 2.

11. Let A be a graded commutative Hopf algebra and P its primitive subspace. Set $f(t) = \sum(\dim P_n)t^n$ and $g(t) = \sum(\dim A_n)t^n$. Express $g(t)$ through $f(t)$.

12. Let $\mathcal{A}^*$ be the algebra of linear functions on chord diagrams satisfying 4-term relations, while $\tilde{\mathcal{A}}^*$ be its subalgebra consisting of all functions that also satisfy 1-term relations. Find an explicit formula for the projection $\pi : \mathcal{A}^* \to \tilde{\mathcal{A}}^*$ which splits the natural inclusion $\iota : \tilde{\mathcal{A}}^* \to \mathcal{A}^*$, i.e. such that $\pi \circ \iota = \mathrm{id}$.

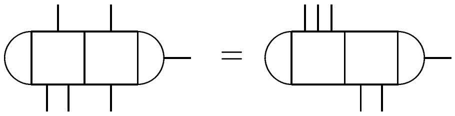
13. (Lecture 11). Prove that isomorphisms $\chi : \mathcal{B} \to \mathcal{A}$ (Bar-Natan) and PBW : $S(\mathfrak{g}) \to U(\mathfrak{g})$ (Poincaré-Birkhoff-Witt) carry the mapping $\psi_{\mathfrak{g}} : \mathcal{B} \to S(\mathfrak{g})$ into the mapping $\phi_{\mathfrak{g}} : \mathcal{A} \to U(\mathfrak{g})$.

14. (Lecture 11). Find a graphical rule to compute the invariant $\psi_{\mathfrak{g}}(b)$ for the Lie algebra $\mathfrak{g} = so_N$.

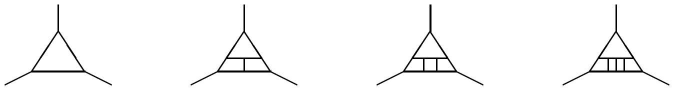
15. (Lecture 13). Call a Feynman diagram $f \in \mathcal{F}$ even (odd), if $\bar{f} = f$ (resp. $\bar{f} = -f$). Prove that under the isomorphism $\chi : \mathcal{B} \to \mathcal{F}$:

- the image of an even Feynman diagram in $\mathcal{B}$ is a linear combination of Chinese characters with an even number of legs,
- the image of an odd Feynman diagram in $\mathcal{B}$ is a linear combination of Chinese characters with an odd number of legs.

16. (*J. Kneissler*). Let $\mathcal{B}_{n,u}$ be the space of Chinese characters of degree n with u univalent vertices (modulo the AS and IHX relations). Denote by $\omega_{i_1 i_2 \ldots i_k}$ the element of $B_{i_1+\ldots+i_k+k-1, i_1+\ldots+i_k}$ represented by a "caterpillar" diagram consisting of k body segments with $i_1, \ldots, i_k$ "legs", respectively. Using AS and IHX relations, prove that $\omega_{i_1 i_2 \ldots i_k}$ is well-defined, i.e. for inner segments it makes no difference on which side of the body they are drawn). For example,

$\omega_{0321} =$ 

17. Let t, x_3, x_4, x_5, ... be the following elements of the space $X^{(3)}$:



- Prove that x_i's belong to Vogel's algebra Λ, i.e. that they are antisymmetric with respect to permutations of legs.
- Prove that x_k with even k can be expressed through t, x_3, x_5, ...

18. Let $\mathcal{A}$ be the algebra of chord diagrams, while $\vec{\mathcal{A}}$ be the algebra of oriented (arrow) chord diagrams. Prove that the natural mapping $\mathcal{A} \to \vec{\mathcal{A}}$ is well-defined, i.e. that the 6T relation implies the 4T relation.

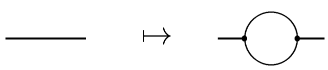
19. Let $\vec{\mathcal{A}}$ be the algebra of arrow diagrams, while $\vec{\mathcal{F}}$ be the algebra of acyclic arrow graphs (oriented Feynman diagrams). Prove that the natural mapping $\vec{\mathcal{A}} \to \vec{\mathcal{F}}$ is well-defined, i.e. that the $\vec{STU}$ relation implies the 6T relation.

2 Open problems

1. Is is true that any chord diagram of order 13 is equivalent to its mirror image modulo 4-term relations?

2. Is it true that a Chinese character with 3 univalent vertices is always equal to 0 as an element of the algebra $\mathcal{B}$?

3. Is it true that the operator of multiplication by the "bubble"

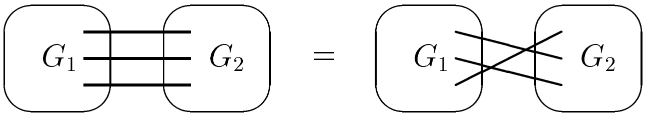


(in the algebras $\mathcal{F}$, Γ and Λ) has no kernel?

4. Let Γ be the algebra of 3-graphs.

- Is it true that Γ has no divisors of 0?
- In particular, is it true that multiplication by Θ has no kernel?
- Is it true that Γ is generated by plane graphs?

- Find generators and relations of the algebra Γ.
- Suppose that a graph $G \in \Gamma$ consists of two parts G_1 and G_2 connected by three edges. Is the following equality



an identity in Γ?

5. Let $X^{(k)}$ be the space of 1- and 3-valent graphs with k numbered legs. Is it true that under any transposition of two legs every element of $X^{(k)}$ remains invariant (changes sign), if k is even (odd)? (The first nontrivial case is when $k = 3$).

6. Let Λ be Vogel's algebra, i.e. the subspace of $X^{(3)}$ consisting of all antisymmetric elements.

   - Is it true that $\Lambda = X^{(3)}$?
   - Is it true that Λ is generated by the elements t and x_k (draw the picture!!)?
   - Is it true that the operator of multiplication by t has no kernel?

7. Let $X^{(3)}$ be the space of 1- and 3-valent graphs with 3 numbered vertices. The group S_3 acts in $X^{(3)}$, splitting it into 3 subspaces:

   - symmetric, which is isomorphic to $B^{(3)}$ (Chinese characters with 3 legs),
   - totally antisymmetric, which is Vogel's Λ by definition, and
   - some subspace Y, corresponding to the third irreducible representation of S_3.

   Question: is it true that $Y = 0$?

8. Define an analogue of the algebra of oriented Feynman diagrams $\vec{\mathcal{F}}$ generated by all graphs, not only acyclic. Denote it by $\vec{\mathcal{F}}_{\text{all}}$. Is it true that the natural mapping $\vec{\mathcal{F}} \to \vec{\mathcal{F}}_{\text{all}}$ is an isomorphism?

9. Construct the analogue of the algebra of Chinese characters $\mathcal{B}$ consisting of graphs with oriented edges.

10. Let $\lambda : A \to \mathcal{L}$ be the natural homomorphism from the algebra of chord diagrams into the graph algebra of Lando.

    - Find $\ker \lambda$ (unknown in degrees greater than 7).
    - Find $\operatorname{im} \lambda$ (unknown in degrees greater than 7).
    - Describe the primitive space $P(\mathcal{L})$.
    - $\mathcal{L}$ is the analogue of the algebra of chord diagrams in the case of intersection graphs. Are there any counterparts of algebras $\mathcal{F}$ and $\mathcal{B}$?

11. (*J. Kneissler*) (See Exercise 16 for the definition of caterpillar diagrams). Is it true that any caterpillar diagram in the algebra $\mathcal{B}$ can be expressed through caterpillar diagrams with even indices $i_1, \ldots, i_k$? Is it true that the primitive space $\mathcal{P}(\mathcal{B})$ (i.e. the space spanned by connected Chinese characters) is generated by caterpillar diagrams?