Lectures on Vassiliev knot invariants

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Contents

Preface

This text provides an introduction to the theory of finite type (Vassiliev) knot invariants, with a stress on its combinatorial aspects. It is intended for readers with no or little background in this area, and we care more about a clear explanation of the basic notions and constructions than about widening the exposition to more recent and more advanced material.

Historical remarks

The notion of finite type knot invariants was independently invented by V. Vassiliev [Va1] and M. Goussarov [G1] around 1989.

Goussarov did not publish anything until 1994, because, making first-class works, he simply did not like to write. Vassiliev was not very much interested in knot invariants, either: his main concern was the general theory of discriminants in the spaces of smooth maps.

It was V. I. Arnold [Ar] who understood the importance of the particular case of maps from the circle to 3-space in knot theory. He coined the name "Vassiliev invariants" and told about them Joan Birman, one of the world's leading experts on knots. Since that time, the term "Vassiliev invariants" became standard, although nowadays some people use the expression "Vassiliev-Goussarov invariants", which is more close to the historical truth.

Vassiliev's approach is based on the study of discriminants in the (infinite-dimensional) spaces of smooth maps from one manifold into another. By definition, the discriminant consists of all maps with singularities.

For example, consider the space of all smooth maps from the circle into 3-space $\mathcal{M} = \{f : S^1 \to \mathbb{R}^3\}$. If f is an embedding (has no singular points), it represents a knot. The complement of the set of all knots is the discriminant $\Sigma \subset \mathcal{M}$. It consists of all smooth maps from S^1 into $\mathbb{R}^3$ that have singularities, either local, where $f' = 0$, or non-local, where f is not injective. Two knots are equivalent, if and only if they can be joined by a path in the space $\mathcal{M}$ that does not intersect the discriminant. Therefore, knot types are in one-to-one correspondence with the connected components of the complement $\mathcal{M} \setminus \Sigma$, and knot invariants with values in an Abelian group G are nothing but cohomology classes from $H^0(\mathcal{M} \setminus \Sigma, G)$. The filtration of Σ by subspaces corresponding to singular knots with a given number of simple double points gives rise to a spectral sequence, which contains, in particular, the spaces of finite type invariants.

J. Birman and X.-S. Lin [BL] have contributed a lot to the simplification of Vassiliev's original techniques. They explained the relation betwen Jones polynomial and finite type invariants and emphasized the role of the algebra of chord diagrams. M. Kontsevich [Ko1] proved the first difficult theorem about Vassiliev invariants with the help of Kontsevich's integral. D. Bar-Natan undertook a thorough study of Vassiliev invariants; in particular, he showed the importance of the algebra of Feynman diagrams and diagrams with uni- and tri-valent vertices; his paper [BN1] is still the most authoritative source on the subject.

Among more recent important developments in the area of finite type knot invariants let me mention:

- T. Le and J. Murakami's proof that the coefficients of the Kontsevich integral are rational [LM2].
- Existence of non-Lie-algebraic weight systems (P. Vogel [Vo], J. Lieberum [Lieb]).
- J. Kneissler's fine analysis [Kn1, Kn2, Kn3] of the structure of the algebra Λ introduced by P. Vogel [Vo].
- Gauss diagram formulas invented by M. Polyak and O. Viro [PV1] and the proof by M. Goussarov [G2] that all finite type invariants can be obtained in this way.
- D. Bar-Natan's proof that Vassiliev invariants for *braids* separate braids [BN5][1] (this is the only situation where Vassiliev invariants are proved to be a complete systems of invariants).
- Computation of the Kontsevich's integral for the unknot [BGRT].
- Habiro's theory of claspers [Hab] (see also [G3]).

In these lectures, we confine ourselves to Vassiliev invariants of *knots* and do not mention other closely related areas, e. g. finite type invariants of 3-manifolds. The interested reader is referred to D. Bar-Natan's online bibliography regularly updated at

`http://www.ma.huji.ac.il/~drorbn/VasBib/`

On Jan. 24, 2002 it contained 465 entries, out of which 41 were added in 2001 and 5 in January 2002. This shows that the study of finite type invariants is going on at a steady pace. Nevertheless, we must underline that notwithstanding all efforts two of the most important problems risen in 1990 are still open:

1. Is it true that Vassiliev invariants distinguish knots?
2. Is it true that Vassiliev invariants can detect knot orientation?

Contents

The main object of this course is the graded Hopf algebra naturally associated with the filtred space of Vassiliev invariants, which appears in three different disguises: as the algebra of chord diagrams $\mathcal{A}$ (Lecture 3), as the algebra of Feynman diagrams $\mathcal{F}$ (Lecture 5), and as the algebra of Chinese characters $\mathcal{B}$ (Lecture 6).

[1] This fact is also an immediate consequence of T. Kohno's theorem [Koh], proved before the notion of finite type invariants was introduced.

After giving the basic definitions, we pass to the discussion of these algebras, then introduce the Kontsevich's integral crucial in the proof of the main (Vassiliev–Kontsevich) theorem (sec. 2.2). Then we explain the construction of Lie algebra weight systems and introduce other relevant combinatorial algebras. The last lecture is devoted to the known results about the dimensions of the space of Vassiliev invariants.

We conclude with the list of examination problems.

Acknowledgements

The text is based on my course "Vassiliev invariants and combinatorial structures" (spring 1999, Graduate School of Mathematics, University of Tokyo) and, partially, on my course "Combinatorial structures in topology" (Autumn 1998, Independent University of Moscow). I am obliged to my listeners M. Nagura. H. Yashima, S. Fuchizawa (Tokyo) and V. Kirichenko (Moscow) for the nice lecture notes that they provided. Ii is my pleasure to thank D. Bar-Natan, O. Dasbach, S. Lando and T. Kohno for many useful comments concerning the subjects touched upon below. My special thanks go to Sergei Chmutov, both for our innumerable conversations and calculations and because I have used fragments of our joint papers in the preparation of the present TeX and Xfig files.

The work on this text was made possible due to financial support, hospitality and creative atmosphere of the Graduate School of Mathematical Sciences, University of Tokyo, and Research Institute for Mathematical Sciences, Kyoto University, where my stays in 1999 and 2002 respectively were organized by T. Kohno and H. Murakami.

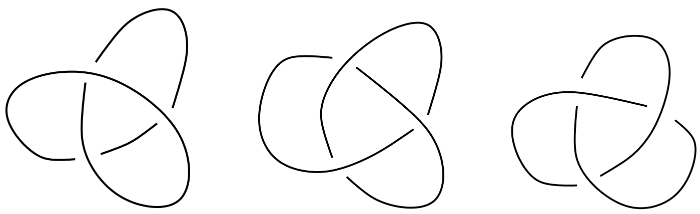
Lecture 1

Finite type invariants

1.1 Knots, Links and Invariants

Definition 1.1 *A knot is a smooth embedding* $f : S^1 \to \mathbb{R}^3$*, where* S^1 *is an oriented circle and* $\mathbb{R}^3$ *is oriented space.*

Examples of knots.



Trivial knot   Left trefoil   Right trefoil

Knots are a special case of links.

Definition 1.2 *A link is a smooth embedding* $S^1 \cup \cdots \cup S^1 \to \mathbb{R}^3$*, where* $S^1 \cup \cdots \cup S^1$ *is the disjoint union of several circles.*

Trivial 2-component link   Nontrivial 2-component link

To vizualise knots and links on paper, we use *knot diagrams* which are formally defined as plane projections with only transversal double points (crossings) as singularities, and at each crossing the overpass and the underpass are distinguished.

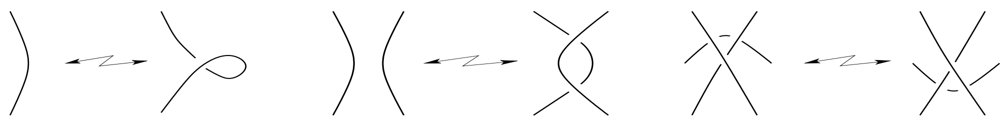
Definition 1.3 *Two knots,* f *and* g*, are equivalent, if there is a commutative diagram*

$$\begin{tikzcd} S^1 \arrow[r, "f"] \arrow[d, "\varphi"'] & \mathbb{R}^3 \arrow[d, "\psi"] \\ S^1 \arrow[r, "g"] & \mathbb{R}^3 \end{tikzcd}$$

where φ and ψ are orientation preserving diffeomorphisms of the circle S^1 and space $\mathbb{R}^3$, respectively. A similar definition is used for links, too.

In terms of knot diagrams, two non-oriented knots, K_1 and K_2, are equivalent if and only if a diagram of K_1 can be transformed into a diagram of K_2 by a sequence of plane diffeomorphisms and local moves of the following three types:

Ω_1 Ω_2 Ω_3



Reidemeister moves

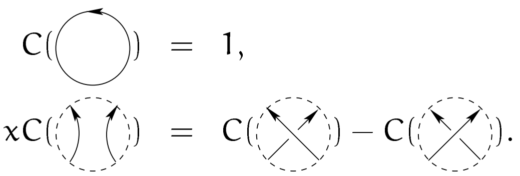
(To adjust this assertion to the oriented case, each of the Reidemeister moves has to be equipped with orientations in all possible ways.)

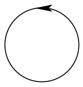
Let $\mathcal{K}$ be the set of all equivalence classes of knots.

Definition 1.4 *A knot invariant is a function of a knot which takes equal values on equivalent knots. In other words, it is a mapping $\nu : \mathcal{K} \to M$ from $\mathcal{K}$ to a certain set M.*

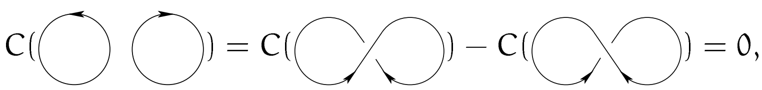
In general, the domain of a knot invariant can be an arbitrary set (e. g. the set of isomorphism classes of groups). In these lectures, however, we are interested in algebraic operations on the sets of invariants, therefore we will restrict ourselves to the case when M is an algebra over a field $\mathbb{F}$, e. g. $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

Example of a knot invariant. The Conway polynomial is an invariant of links (therefore, in particular, an invariant of knots) defined by the two properties:



Here  stands for the unknot (trivial knot) while the three pictures in the second line mean three knots that are identical everywhere except for the fragments shown.

Example of computation of the Conway polynomial:



because the two knots on the right are equivalent (both are trivial).

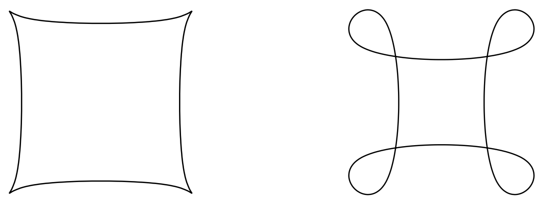
For every n, the coefficient of x^n in C is a numeric invariant of the knot.

1.2 Definition of Vassiliev invariants

The notion of *finite type* knot invariants appeared around 1989 in V.Vassiliev's investigations of discriminants in the (infinite-dimensional) spaces of smooth maps from one manifold into another[1]. By definition, the discriminant consists of all maps with singularities.

For example, consider the space of all smooth maps $\{f : S^1 \to \mathbb{R}^3\}$ If f is an embedding (has no singular points), then it represents a knot. The complement of the set of all knots is the discriminant Σ. It consists of all smooth maps from S^1 into $\mathbb{R}^3$ that have singularities, either *local*, where $f' = 0$, or *non-local*, where the mapping f is not injective.

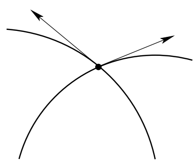
Examples of smooth maps with singularities:



Local singularities   Non-local singularities

Question. What singularity is the most often occuring one?
Answer. A simple double point.

A point $p \in \mathrm{im}(f) \subset \mathbb{R}^3$ is a *simple double point* of f if its preimage under f consists of two values t_1 and t_2 and the two tangent vectors $f'(t_1)$ and $f'(t_2)$ are non-collinear. Geometrically, this means that, in a neighborhood of the point p, the curve consists of two transversal branches. Smooth singular maps having one simple double points form a submanifold of codimension 1 in the space of all smooth maps, while for other types of singularities the codimension is greater. They form the principal stratum of the discriminant Σ.



Simple double point

Vassiliev's initial idea was to study the prolongation of knot invariants to singular knots.

Let $\mathcal{X}_n$ denote the set of (equivalence classes of) singular knots with n double points and no other singularities. By definition, $\mathcal{X}_0 = \mathcal{K}$ is the set of all non-singular knots. Put $\mathcal{X} = \cup_{n \geq 0} \mathcal{X}_n$.

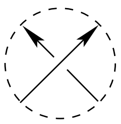
[1]A definition of finite type invariants equivalent to Vassiliev's one, but based on a different approach, was invented by M. Goussarov approximately at the same time. See page 14 for a short introduction to Goussarov's theory.

Definition 1.5 *(Vassiliev's prolongation of knot invariants.) Given a knot invariant* $v : \mathcal{K} \to M$*, we define its prolongation* $\widetilde{v} : \mathcal{X} \to M$ *by the rules*

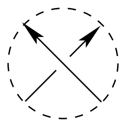
$$\begin{aligned} \widetilde{v}\big|_{\mathcal{K}} &\equiv v, \\ \widetilde{v}(\;) &= \widetilde{v}(\;) - \widetilde{v}(\;). \end{aligned} \tag{1.1}$$

(The last equality is known as Vassiliev's skein relation).

The right hand side of Vassiliev's skein relation refers to the two resolutions of the double point — positive and negative. In terms of the plane knot diagram, we say that the *local writhe* of the knot at the positive crossing is $+1$ and at the negative crossing is -1.



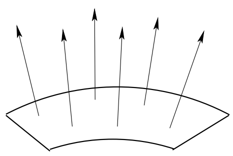
Positive crossing
(local writhe $= 1$)



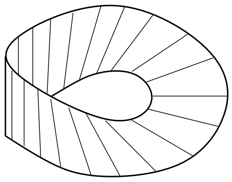
Negative crossing
(local writhe $= -1$)

A crucial observation is that each of these resolutions is well-defined (does not depend on the plane projection used to express this relation). To make sure of this fact, the reader is invited to look at Vassiliev's skein relation from the other side of the piece of paper it is written on.

A more formal way to state this fact is to say that the discriminant Σ has a coorientation, i. e. one can orient the normal bundle of Σ in a consistent way. The figure below on the left illustrates this notion. The figure on the right shows the simplest example of a hypersurface that has no coorientation — the Möbius band in $\mathbb{R}^3$.



Coorientation



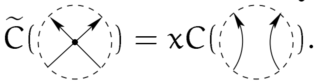
Möbius band

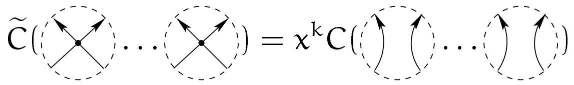
Definition 1.6 *A function* $v : \mathcal{K} \to M$ *is a Vassiliev invariant of order* $\leq n$ *if its prolongation vanishes on all singular knots with more than* n *double points:*

$$\widetilde{v}\big|_{\mathcal{X}_{n+1}} = 0$$

(and hence $\widetilde{v}\big|_{\mathcal{X}_m} = 0$ *for all* $m > n$*).*

Example. The n-th coefficient of the Conway polynomial is a Vassiliev invariant of order $\leq n$.

Indeed, the definition of Conway polynomial, together with Vassiliev's skein relation implies that . Applying this relation several times, we get

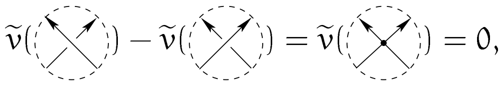


for a singular knot with k double points. If $k \geq n+1$, then the coefficient of x^n in this polynomial is 0.

Notation: $\mathcal{V}_n$ will stand for the set of $\mathbb{F}$-valued Vassiliev invariants of order $\leq n$ (it is a linear space over $\mathbb{F}$).

Proposition 1.1 $\mathcal{V}_0 = \{\text{const}\}$, $\dim \mathcal{V}_0 = 1$.

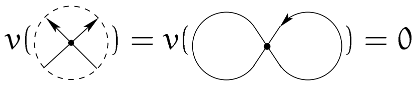
Proof. By definition, the assumption $v \in \mathcal{V}_0$ means that the value of v on any knot with at least one double point is 0. Due to Vassiliev's skein relation



which means that the value of $v(K)$ does not change when any undercrossing in the knot K becomes an overcrossing (or vice versa). Under such changes, all knots are equivalent between themselves. Therefore, v is a constant.

Proposition 1.2 $\mathcal{V}_1 = \{\text{const}\}$, $\dim \mathcal{V}_1 = 1$.

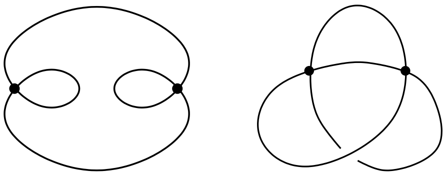
Proof. A similar argument shows that the value of v on any knot with 1 double point is equal to its value on the special figure eight singular knot and hence to 0:



Therefore, $\mathcal{V}_1 = \mathcal{V}_0$.

These two examples seem to lead to the following *conjecture*: $\dim \mathcal{V}_n = 1$ for any n. Of course, this cannot be true — otherwise there would be no theory of finite type invariants. Actually, in the case $n = 2$ we get the first nontrivial Vassiliev invariant.

Let me explain the reason why the argument like above fails in this case. Take a knot with two double points and try to transform it to a fixed knot with two double points using a smooth homotopy where additional double points are allowed. It is easy to understand that any knot with two double points can be reduced to one of the following two canonical knots:



First knot Second knot

— but these two knots are *not equivalent* to each other! The essential difference between these two knots is in the order of the double points on the knot.

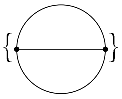
When travelling along the first knot, the two double points are encountered in the order 1122 (or 1221, 2211, 2112, if you start from a different initial point). For the second knot the sequence is 1212 (equivalent to 2121). The two sequences 1122 and 1212 are different even if cyclic permutations are allowed. This shows that $\dim \mathcal{V}_2 \leq 2$. Later on (see page 21) I will give an explicit construction of the invariant $v_2 \in \mathcal{V}_2$ which is different from a constant.

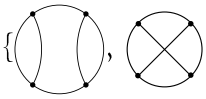
Once again, any singular knot with $n = 0, 1$ double points can be deformed into a canonical form, if knots with $n + 1$ double points are allowed in the process of deformation; for $n = 2$ this is not true: there are, in fact, *two* canonical forms for such knots.

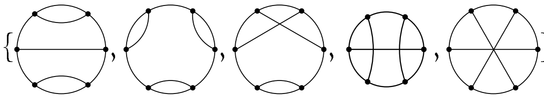
Now let us give a formal definition of the combinatorial structure which underlies the above analysis.

Definition 1.7 *A chord diagram of order n (or degree n) is an oriented circle with n pairs of distinct points, considered up to an orientation preserving diffeomorphism of the circle. The set of all chord diagrams of order n is denoted by CD_n.*

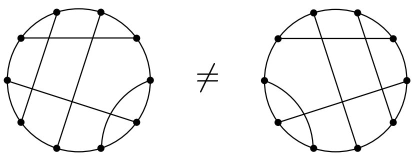
Examples.

$CD_1 = \{$  $\}$,

$CD_2 = \{$  $\}$,

$CD_3 = \{$  $\}$.

Remark. Chord diagrams that differ by a reflection with respect to a diameter are, in general, different:



This observation reflects the fact that we are studying *oriented* knots.

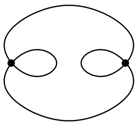
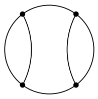
Another (equivalent) definition of chord diagrams can be given as follows.

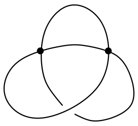
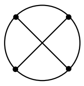
Definition 1.8 $CD_n = W_n/(S_n \times \mathbb{Z}_{2n})$, *where*

W_n: *words of length* $2n$ *in* n *letters each of which appears twice,*
S_n: *permutations (relabelling) of these* n *letters,*
$\mathbb{Z}_{2n}$: *cyclic shifts of the word.*

Definition 1.9 *The chord diagram* $\sigma(K) \in CD_n$ *of a singular knot* $K \in \mathcal{X}_n$ *is the parameterizing circle* S^1 *equipped with* n *pairs of points which are the inverse images of* n *double points of the knot.*

Examples.

$\sigma($  $)$ $=$  ,

$\sigma($  $)$ $=$  .

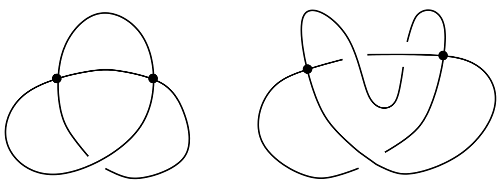
Proposition 1.3 *The value of* $v \in \mathcal{V}_n$ *(Vassiliev invariant of order* $\leq n$*) on a knot* $K \in \mathcal{X}_n$ *(knot with* n *simple double points) depends only on* $\sigma(K)$ *(the chord diagram of* K*):*

$$\sigma(K_1) = \sigma(K_2) \Rightarrow v(K_1) = v(K_2).$$

In other words, a mapping $\mathcal{X}_n \to \mathbb{F}$ *defined by a Vassiliev invariant of order* $\leq n$*, factors through the canonical mapping of passing to chord diagrams* $\sigma : \mathcal{X}_n \to CD_n$*:*

$$\begin{tikzcd} \mathcal{X}_n \arrow[d, "\sigma"'] \arrow[dr, "\forall"] & \\ CD_n \arrow[r, "\exists!"'] & \mathbb{F} \end{tikzcd}$$

Proof. Suppose that $\sigma(K_1) = \sigma(K_2)$. Then there is a one-to-one correspondence between the double points of K_1 and K_2 which is consistent with the two chord diagrams. Put K_1, K_2 in $\mathbb{R}^3$ so that the corresponding double points coincide together with both branches of the knot in the vicinity of each double point.



Knot K_1 Knot K_2

Now we can find a smooth isotopy from K_1 to K_2, identical in the neighbourhoods of their double points and entirely consisting of knots with n or $n+1$ double points. By Vasiliev's skein relation, this implies that $\nu(K_1) = \nu(K_2)$. □

Proposition 1.3 shows that there is a well-defined mapping $\alpha_n : \mathcal{V}_n \to \mathrm{CD}_n^*$ (linear space of $\mathbb{F}$-valued functions on the set CD_n). By definition, $[\alpha_n(\nu)](D) = \nu(K)$, where K is an arbitrary knot with $\sigma(K) = D$.

To understand the size and the structure of the space $\mathcal{V}_n$, it is very useful to have a description of the kernel and the image of this mapping α_n.

The first question is quite easy.

Observation: $\ker \alpha_n = \mathcal{V}_{n-1}$. This follows immediately from the definitions. Therefore, we obtain an injective linear mapping

$$\overline{\alpha}_n : \mathcal{V}_n/\mathcal{V}_{n-1} \to \mathrm{CD}_n^*. \tag{1.2}$$

Since the mapping α_n discards the order $(n-1)$ part of a Vassiliev invariant ν, we can, by analogy with differential operators, call the function $\alpha(\nu)$ on chord diagrams *the symbol of the Vassiliev invariant* ν.

A complete description of the image, $\operatorname{im} \alpha_n$, which is more involved, will be given in theorem 2.1, p. 17.

1.3 Goussarov's Approach

Vassiliev's definition of finite type invariants relies on the prolongation of invariants to singular knots which is only possible when a coorientation of the discriminant Σ is chosen.

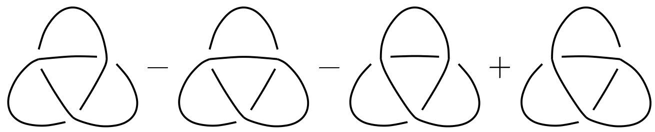
One can, however, give an equivalent definition which does not refer to the positive and negative resolutions of a knot self-intersection.

Definition 1.10 *(M.Goussarov). Let f be a knot invariant. We can think of f as a function of a knot diagram D, stable under Reidemeister moves. Denote by $D_{\varepsilon_1,\dots,\varepsilon_n}$ the diagram which is obtained from D by toggling the underpass and overpass at i-th crossing, if $\varepsilon_i = -1$, and doing nothing, if $\varepsilon_i = 1$. The invariant f is said to be a finite type invariant of degree (or order) at most $n-1$, if for any knot diagram with at least n crossings we have*

$$\sum_{\varepsilon} (-1)^{|\varepsilon|} f(D_{\varepsilon_1,\dots,\varepsilon_n}) = 0, \tag{1.3}$$

where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, $|\varepsilon|$ is the number of -1's in the sequence $\varepsilon_1, \dots, \varepsilon_n$, and the summation is spread over all the 2^n choices of signs that correspond to an arbitrarily chosen set of n crossings in D.

An example of such a sum is shown in the picture:



The fact that this definition is equivalent to Vassiliev's one can be seen as follows. Take a singular knot with n double points and, using Vassiliev's skein relation, expand it into an alternating sum of 2^n knots where all these points are resolved. This sum has the same appearance as equation 1.3.

Lecture 2

Four term relation

In section 1.2 (eq. 1.2) we constructed a linear inclusion

$$\overline{\alpha}_n : \mathcal{V}_n/\mathcal{V}_{n-1} \to \mathrm{CD}_n^*,$$

where $\mathcal{V}_n$ is the space of $\mathbb{F}$-valued Vassiliev invariants of order n and CD_n^*, the linear space of $\mathbb{F}$-valued functions on the set CD_n of chord diagrams of order n.

To describe the image of $\overline{\alpha}_n$, we will need the following definitions.

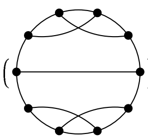
2.1 Definitions

Definition 2.1 *A weight system of order* n *is a function* $f \in \mathrm{CD}_n^*$ *that satisfies 4-term relations.*

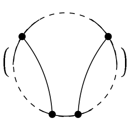
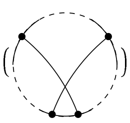
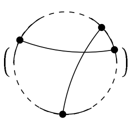
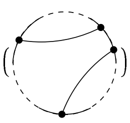
Definition 2.2 *A strict weight system of order* n *is a weight system* $f \in \mathrm{CD}_n^*$ *that also satisfies 1-term relations.*

Definition 2.3 *A 1-term relation says that* f *vanishes on every chord diagram with an isolated chord. An isolated chord is a chord that does not intersect any other chord of the diagram.*

Example.

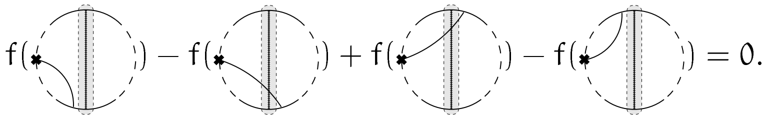
$f($  $) = 0,$

Definition 2.4 *A 4-term relation says that the alternating sum of the values of* f *on certain special quadruples of diagrams is zero:*

$f($  $) - f($  $) + f($  $) - f($  $) = 0.$ (2.1)

Here and below, dotted arcs suggest that there might be further chords attached to their points, while on the solid portions of the circle all the endpoints are explicitly shown.

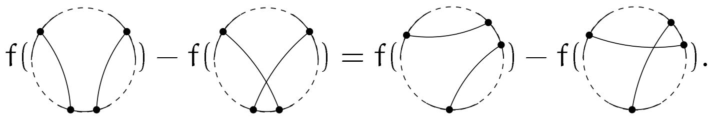
It is often useful to look at the 4T relation from the following viewpoint. Imagine that of the two chords that participate in equation 2.1 one ("second") is fixed, as well as one of the two endpoints of the first one, while another endpoint of that first chord makes a complete turn around the first chord, stopping in four places, adjacent to its endpoints; the resulting four diagrams are then summed up with alternating signs. Pictorially:



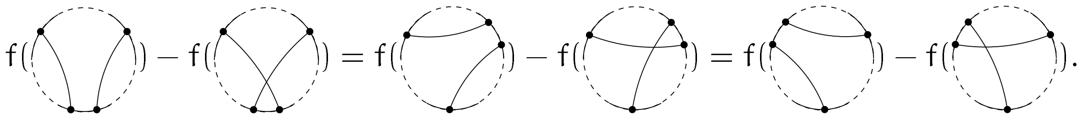
(2.2)

where the fixed endpoint of the first (moving) chord is marked by ✖.

Another way of writing the 4T relation, which will be useful in Lecture 5, is to split the four terms in two pairs:



Due to obvious symmetry, this can be completed as follows:



(2.3)

Note that for every order n there are as many 1-term relations as there are chord diagrams with an isolated chord and the choice of a specific 4-term relation depends on the following data:

- a diagram of order $n-1$,
- a distinguished chord of this diagram ("fixed chord"), and
- a distinguished arc on the circle of this diagram (to put the fixed endpoint).

Notation. Denote by W_n the subspace of CD_n^* consisting of all weight systems of order n and by W_n' the subspace of all strict weight systems of order n (those that satisfy not only 4T relations, but also 1T relations).

2.2 Main Theorem

Theorem 2.1 *(Vassiliev–Kontsevich) The mapping $\overline{\alpha}_n$ yields an isomorphism*

$$\overline{\alpha}_n : \mathcal{V}_n/\mathcal{V}_{n-1} \to W_n',$$

that is, the space of strict weight systems is a graded space associated with the filtered space of Vassiliev invariants.

The theorem consists of two assertions:

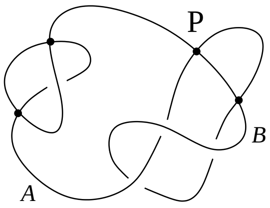
- (V. Vassiliev) The symbol of every Vassiliev invariant is a weight system.
- (M. Kontsevich) Every weight system is the symbol of a certain Vassiliev invariant.

We will now prove the first part of the theorem. The second part will be proved later (in Lecture 9) using the Kontsevich integral.

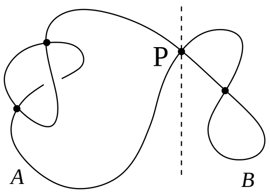
The first part of the theorem consists of two assertions, and we prove them one by one.

Proof that any $f \in CD_n^*$ *coming from a* $v \in \mathcal{V}_n$ *satisfies 1-term relations.*

Let $K \in \mathcal{X}_n$ be a singular knot whose chord diagram contains an isolated chord:



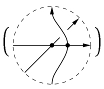
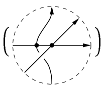
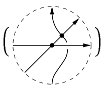
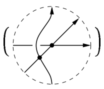
The double point P that corresponds to the isolated chord divides the knot into two parts: A and B. The fact that the chord is isolated means that these two parts do not have common double points. They may, however, be knotted between themselves in the sense that on any knot diagram there are crossing points between part A and part B. Due to Vassiliev's skein relation (eq. 1.1 on page 10), by changing undercrosses and overcrosses, we can untangle part A from part B obtaining a singular knot K' with the property that the two parts lie on either side of a plane in $\mathbb{R}^3$ passing through the double point P:



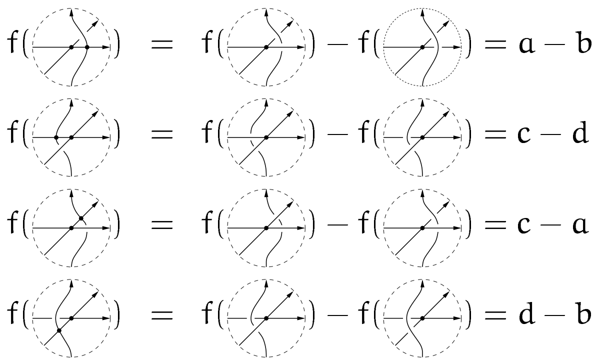
Here, the two resolutions of the double point P obviously give equivalent knots, therefore $v(K) = v(K') = v(K'_+) - v(K'_-) = 0$.

Proof that any $f \in CD_n^*$ *coming from a* $v \in \mathcal{V}_n$ *satisfies 4-term relations.*

Lemma (4-term relation for knots). Any Vassiliev invariant satisfies

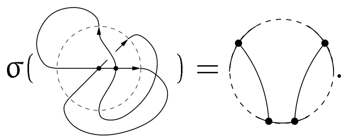
f() − f() + f() − f() = 0,

Proof. By Vassiliev's skein relation:



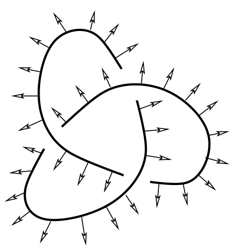
The alternating sum of these expressions is $(a-b)-(c-d)+(c-a)-(d-b)=0$, and the lemma is proved.

Now, to prove the 4-term relation for the symbols of Vassiliev invariants, let us choose, for the first diagram D_1 in the relation, an arbitrary singular knot K_1 such that $\sigma(K_1) = D_1$:



Then the three remaining knots K_2, K_3, K_4 that participate in the 4-term relation for knots, correspond to the three remaining chord diagrams of the 4-term relation for chord diagrams, and the claim follows from the lemma.

Remark. The 1-term relation is also called *framing independence relation*, because there exists a theory of Vassiliev invariants for framed knots, where the 4-term relation holds, while the 1-term relation does not. A *framed knot* is a knot equipped with a smooth section of its normal bundle, considered up to smooth deformations:



Like in the conventional case, we can introduce the prolongation of invariants to the singular framed knots, understood as knots with simples double points, and give the definition of finite type invariants. If we denote the space of such invariants by $\mathcal{FV}_n$, then we get a linear mapping $\mathcal{FV}_n/\mathcal{FV}_{n-1} \to \mathrm{CD}_n^*$ which satisfies the 4T relation for the same reasons, but does not satisfy the 1T relation, because the two knots, differing by a kink like in first Reidemeister move, are equivalent as usual knots, but not equivalent as framed knots. This proves that $\mathcal{FV}_n/\mathcal{FV}_{n-1}$ is a subspace of $\mathcal{A}_n^*$.

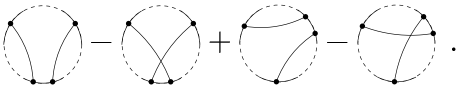
The equality $\mathcal{FV}_n/\mathcal{FV}_{n-1} = \mathcal{A}_n^*$ follows from the Kontsevich's integral for framed knots (see p. 71 below).

We see that 4T relations are more fundamental than 1T relations, therefore in the sequel we will study combinatorial structures where only 4T relations are used. Anyway, 1T relations can always be added, either by considering an appropriate subspace or an appropriate quotient space. This is especially easy to do in terms of primitive elements (see page 27), because the problems is reduced to simply leaving out one primitive generator.

2.3 Linear space of chord diagrams

A dual way to define the weight systems is to introduce the 1- and 4-term relations directly in the linear space generated by chord diagrams.

Definition 2.5 *The space $\mathcal{A}_n$ of chord diagrams of order n is the linear space generated by the set* CD_n *(all diagrams of order n) modulo the subspace spanned by all 4-term linear combinations*

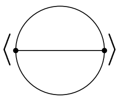


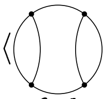
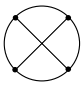
In these terms, the space of weight systems W_n is dual to the space of chord diagrams $\mathcal{A}_n$:

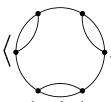
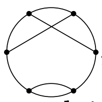
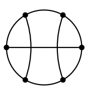
$$W_n = \mathrm{Hom}(\mathcal{A}_n, \mathbb{F}).$$

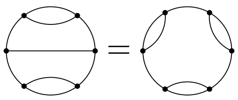
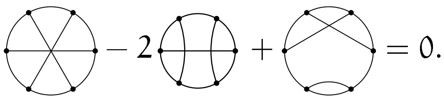
The quotient space of $\mathcal{A}_n$ modulo the subspace of one-term relations will be denoted by $\mathcal{A}'_n$ (with a prime).

Below, we list the dimensions and some bases of the spaces $\mathcal{A}_n$ for $n = 1$, 2 and 3:

$\mathcal{A}_1 = \langle$  $\rangle$, $\dim \mathcal{A}_1 = 1$.

$\mathcal{A}_2 = \langle$ ,  $\rangle$, $\dim \mathcal{A}_2 = 2$, because the only 4-term relation that can be set up out of chord diagrams of order 2 is trivial.

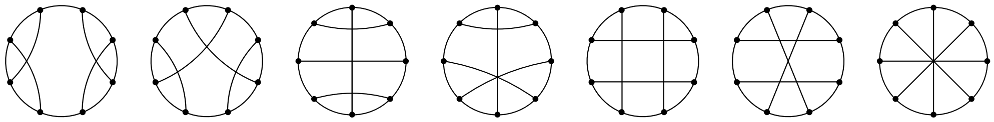
$\mathcal{A}_3 = \langle$ , ,  $\rangle$, $\dim \mathcal{A}_3 = 3$, because $\#(\mathrm{CD}_3) = 5$, and there are 2 independent 4-term relations:

 and 

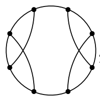
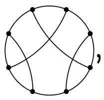
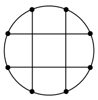
2.4 Exercises

Exercise 1. Find the space of weight systems W_4'.

Answer. The basic weight systems are:



1	0	-1	-1	0	-1	-2
0	1	1	2	0	1	3
0	0	0	0	1	1	1

The table shows that the three diagrams , ,  form a basis in the space $\mathcal{A}_4'$.

Exercise 2. Choose a plane diagram of a knot K with a base point and let

$$\nu_2(K) = \sum_{\substack{i\ j\ i\ j\\ u\,o\,o\,u}} \varepsilon_i \varepsilon_j \ ,$$

where i, j are numbers of two crossing points, ε_i, ε_j are local writhe numbers (see p. 10) at these points, and the sum is taken over all pairs of crossing points i, j such that, when moving along the knot in the positive direction starting from the base point, these points appear in the specified order (i, j, i, j) and the corresponding consecutive passages are undercrossing, overcrossing, overcrossing and undercrossing,

This is an example of the construction of Vassiliev invariants given by M. Polyak and O. Viro [PV1], see Lecture 13.

Lecture 3

Algebra of Chord Diagrams

Here, the main protagonist of this course appears in its first disguise — directly associated to Vassiliev invariants — as the algebra generated by chord diagrams modulo the four term relations.

3.1 Bialgebras

We begin with some general words about algebras, coalgebras and bialgebras.

Definition 3.1 *An algebra over a field $\mathbb{F}$ is an $\mathbb{F}$-linear space A equipped with a linear mapping $\mu : A \otimes A \to A$, called multiplication. We will only consider associative algebras with unity.*

Associativity means that the diagram

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu \otimes \mathrm{id}} & A \otimes A \\ {\scriptstyle \mathrm{id} \otimes \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow[\mu]{} & A \end{array}$$

is commutative.

Unity is a linear mapping $\iota : \mathbb{F} \to A$ (uniquely defined by the element $\iota(1) \in A$) that makes commutative the diagram

$$\begin{array}{ccc} \mathbb{F} \otimes A & \xrightarrow{\iota \otimes \mathrm{id}} & A \otimes A \\ \uparrow & & \downarrow{\scriptstyle \mu} \\ A & = & A \end{array}$$

where the left vertical arrow is the natural isomorphism.

We have on purpose formulated the notion of an algebra in this formal way: this allows us to derive the definition of a coalgebra by a mere reversion of arrows.

Definition 3.2 *A coassociative coalgebra with counity is a linear space A equipped with a linear mapping $\delta : A \to A \otimes A$, called multiplication, and a linear mapping $\varepsilon : A \to \mathbb{F}$, called counity, such that the following two diagrams commute:*

$$
\begin{tikzcd}
A \otimes A \otimes A & A \otimes A \arrow[l, "\delta \otimes \mathrm{id}"'] \\
A \otimes A \arrow[u, "\mathrm{id} \otimes \delta"] & A \arrow[l, "\delta"'] \arrow[u, "\delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{F} \otimes A \arrow[d] & A \otimes A \arrow[l, "\varepsilon \otimes \mathrm{id}"'] \\
A \arrow[r, equal] & A \arrow[u, "\delta"']
\end{tikzcd}
$$

The dual properties of commutativity of an algebra and cocommutativity of a coalgebra are defined through the following commutative diagrams:

$$
\begin{tikzcd}
A \otimes A \arrow[r, "\mu"] & A \arrow[d, equal] \\
A \otimes A \arrow[u, "\tau"] \arrow[r, "\mu"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \arrow[d, "\tau"'] & A \arrow[l, "\delta"'] \arrow[d, equal] \\
A \otimes A & A \arrow[l, "\delta"']
\end{tikzcd}
$$

where $\tau : A \otimes A \to A \otimes A$ is the permutation of tensor factors.

Definition 3.3 *A bialgebra is a linear space A with the structure of an algebra given by μ, ι and the of a coalgebra given by δ, ε which agree in the sense that the following identities hold:*

(1) $\varepsilon(1) = 1$
(2) $\delta(1) = 1 \otimes 1$
(3) $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$
(4) $\delta(ab) = \delta(a)\delta(b)$

Note that these conditions, taken in pairs, have the following meaning:
(1,3) $\Leftrightarrow$ ε is an algebra homomorphism.
(2,4) $\Leftrightarrow$ δ is an algebra homomorphism.
(1,2) $\Leftrightarrow$ i is a coalgebra homomorphism.
(3,4) $\Leftrightarrow$ μ is a coalgebra homomorphism.

The coherence of the two structures making the definition of a bialgebra can thus be stated in either of the two equivalent ways:

- ε and δ are algebra homomorphisms,
- μ and i are coalgebra homomorphisms.

Example 1. Algebra of functions on a finite group with pointwise multiplication. G is a finite group. A is the space of function $A = \{G \to \mathbb{F}\}$ with

$$(fg)(x) = f(x)g(x), \quad \delta(f)(xy) = f(xy).$$

Example 2. Algebra of functions on a finite group with convolution as multiplication. G and A have the same meaning as above. Operations are defined by

$$(fg)(x) = \sum_{y \in G} f(xy^{-1})g(y), \quad \delta(f) = f \otimes f.$$

Example 3. Graded polynomial algebra. Let $P = \oplus_{n\geq 0} P_n$ be a vector space, graded by finite-dimensional subspaces P_n. Let A be the symmetric algebra of P. Then A is a Hopf algebra with coproduct defined on the elements $p \in P \subset A$ by setting $\delta(p) = 1 \otimes p + p \otimes 1$ and then extended as an algebra homomorphism to the enture A.

Example 4. Universal enveloping algebra. Let $\mathfrak{g}$ be a Lie algebra, $A = U(\mathfrak{g})$ — its universal enveloping algebra with usual multiplication. Define $\delta(g) = 1 \otimes g + g \otimes 1$ for $g \in \mathfrak{g}$ and extend it to all of A by the axioms of bialgebra.

Exercise. Define the appropriate units and counits in each case.

Definition 3.4 *A Hopf algebra is a graded bialgebra. This means that the linear space A is graded by finite-dimensional subspaces*

$$A = \bigoplus_{k\geq 0} A_k$$

and the grading is compatible with the operations μ, ι, δ, ε in the following way:

$$\mu : A_m \otimes A_n \to A_{m+n}, \tag{3.1}$$

$$\delta : A_n \to \oplus_{k+l=n} A_k \otimes A_l. \tag{3.2}$$

A Hopf algebra A is said to be of finite type, if all its homogeneous components A_n are finite-dimensional. A Hopf algebra is said to be connected, if (1) $\iota : \mathbb{F} \to A$ is an isomorphism of $\mathbb{F}$ onto A_0 and (2) $\varepsilon |_{A_k} = 0$ for $k > 0$, while $\varepsilon |_{A_0}$ is an isomorphism between A_0 and $\mathbb{F}$.

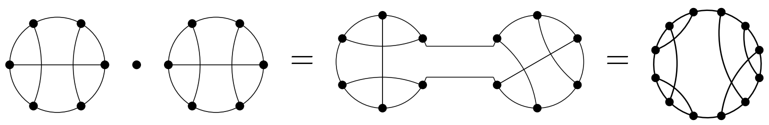
Remark. The above definition follows the classical paper [MiM]. In more recent times, it became customary to include one more operation, called *antipode*, in the definition of a Hopf algebra. The antipode is a linear mapping $S : A \to A$ such that $\mu \circ (S \otimes 1) \circ \delta = \mu \circ (1 \otimes S) \circ \delta = \iota \circ \varepsilon$. We remark that bialgebras we are interested in (those that satisfy the premises of Theorem 4.1 below) always have the antipode, defined by setting $S(1) = 1$ and $S(p) = -p$ for any primitive (see p. 27) element p.

3.2 Bialgebra of chord diagrams

Now we are ready to define the Hopf algebra of chord diagrams $\mathcal{A}$. As a linear space, we set

$$\mathcal{A} = \bigoplus_{k\geq 0} \mathcal{A}_k.$$

Definition 3.5 *The product of two chord diagrams D_1 and D_2 is defined by cutting and gluing the two circles as shown here:*

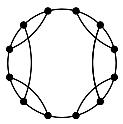


(mind the orientation of the circles!)

This mapping is then extended by linearity to

$$\mu : A_m \otimes A_n \to A_{m+n}.$$

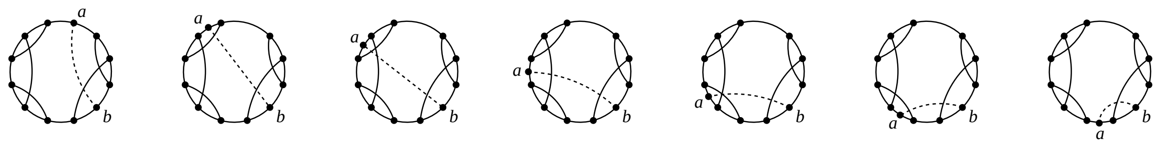
Remark. In the example above we could with an equal success cut the circle in another points and get a different result:



Lemma 3.1 *The result of multiplication is well-defined modulo 4T relations.*

Proof. **1.** In the case of the product of two diagrams, it is enough to prove that if one of the two diagrams, say D_2, is turned inside the product diagram by one "click" with respect to D_1, then the result is the same modulo 4T relations.

Note that this turn is equivalent to the following transformation. Pick a chord in D_2 with endpoints a and b such that a is adjacent to D_1. Then, fixing the endpoint b, move a all through the diagram D_1. In the process of this motion we obtain $2n+1$ diagrams P_0, P_1, ..., P_{2n}, where n is the order of D_1, and we must prove that $P_0 \equiv P_{2n}$ mod 4T. Now, if you think a little bit, you will see that the difference $P_0 - P_{2n}$ is in fact equal to the sum of all n four-term relations which are obtained by fixing the endpoint b and all chords of D_1, one by one. For example, if we consider the two products shown above and use the following notation:



P_0 P_1 P_2 P_3 P_4 P_5 P_6

then we must take the sum of the three linear combinations

$$\begin{gathered} P_0 - P_1 + P_2 - P_3, \\ P_1 - P_2 + P_4 - P_5, \\ P_3 - P_4 + P_5 - P_6, \end{gathered}$$

and the result is exactly $P_0 - P_6$.

2. Now we must prove that the product of any two linear combinations of chord diagrams is well-defined modulo 4T relations. This fact directly follows from **1** and standard linear algebra.

Definition 3.6 *The coproduct in the algebra* A

$$\delta : A_n \to \bigoplus_{k+l=n} A_k \otimes A_l$$

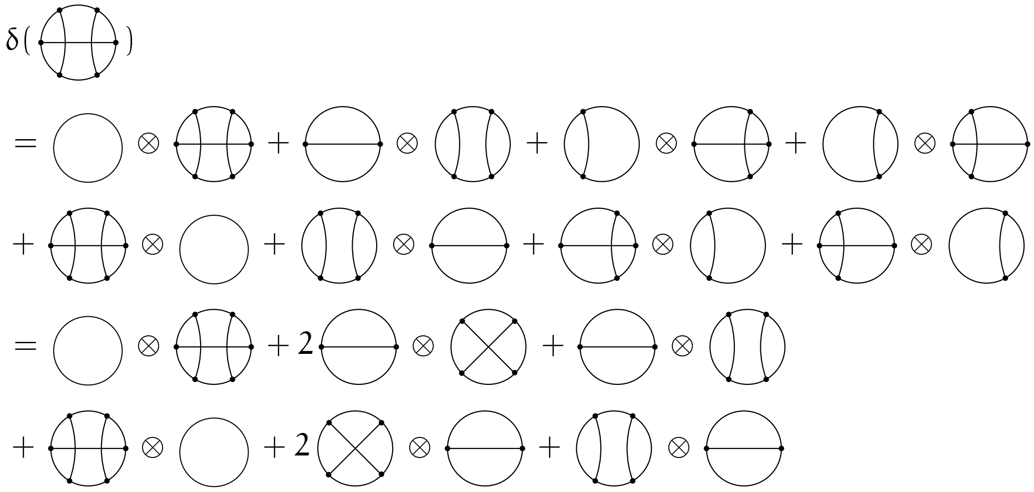
is defined as follows. For a diagram $D \in \mathcal{A}_n$ *we put*

$$\delta(D) = \sum_{J \subset [D]} D_J \otimes D'_J,$$

the summation taken over all subsets J *of the set of chords of* D. *Here* D_J *is the diagram consisting of the chords that belong to* J, D'_J *is the complementary diagram. To the entire space* $\mathcal{A}$ *the operator* δ *is extended by linearity.*

If D is a diagram of order n, then the total number of summands in the right hand side of the definition is 2^n.

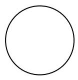
Example.

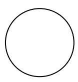


Lemma 3.2 *The coproduct* δ *is well-defined modulo 4T relations.*

Proof. Unlike the similar property of multiplication, this fact is fairly evident.

The unity and the counity in $\mathcal{A}$ are defined as follows:

$\iota : \mathbb{F} \to \mathcal{A}$, $\iota(x) = x$  ,

$\varepsilon : \mathcal{A} \to \mathbb{F}$, $\delta(x$  $+ \dots) = x$.

Exercise. Check the axioms of bialgebra.

Taking into account the 1-term relations leads to the quotient Hopf algebra $\mathcal{A}' = \mathcal{A}/(\Theta)$, where (Θ) is the ideal generated by the chord diagram with one chord.

Lecture 4

Primitive elements. Intersection graphs

The bialgebra of chord diagrams $\mathcal{A}$ has the following properties:

— it is commutative,

— it is cocommutative,

— it is connected,

— and of finite type.

(We do not list the properties of associativity and coassociativity which are presupposed for every bialgebra).

It turns out that these properties ensure that $\mathcal{A}$ has a straightforward description in terms of its *primitive subspace*.

4.1 Structure theorem for Hopf algebras

Definition 4.1 *An element* $p \in A$ *of a bialgebra* A *is said to be primitive, if*

$$\delta(p) = 1 \otimes p + p \otimes 1.$$

The axioms of a bialgebra imply that the set of all primitive elements forms a linear subspace $P = P(A) \subset A$ which is homogeneous in the sense that $P = \oplus_{n \geq 0} P \cap A_n$.

Theorem 4.1 *(Milnor–Moore [MiM]). Any commutative cocommutative connected Hopf algebra of finite type over a field of characteristic 0 is equal to the symmetric algebra of its primitive subspace:*

$$A = S(P(A)).$$

The word 'equal' (not just isomorphic) means that if the inclusion linear operator $P(A) \to A$ *is prolonged to a homomorphism* $S(P(A)) \to A$ *in the standard way, this gives a bialgebra isomorphism. In other words, if a linear basis is chosen in every homogeneous component* $P_n = P \cap A_n$*, then every element of* A *can be written uniquely as a polynomial in these variables.*

Proof. There are two assertions to prove:

(I) that every element of A is polynomially expressible (i. e. as a linear combination of products) through primitive elements,

(II) that the value of a nonzero polynomial on a set of linearly independent homogeneous primitive elements cannot vanish in A.

Let us start to prove assertion (I) for homogeneous elements of the algebra by induction on their degree.

First note that under our assumptions the coproduct of a homogeneous element $a \in A_n$ has the form

$$\delta(x) = 1 \otimes x + \cdots + x \otimes 1, \tag{4.1}$$

where the dots stand for an element of $A_1 \otimes A_{n-1} + \cdots + A_{n-1} \otimes A_1$. Indeed, we can always write $\delta(x) = 1 \otimes y + \cdots + z \otimes 1$. By cocommutativity $y = z$. Then, $x = (\varepsilon \otimes \mathrm{id})(\delta(x)) = y + 0 + \cdots + 0 = y$.

In particular, for any element $x \in A_1$ equation 4.1 ensures that $\delta(x) = 1 \otimes x + x \otimes 1$, so that $A_1 = P_1$.

Take an element $x \in A_2$. We have

$$\delta(x) = 1 \otimes x + \sum \lambda_{ij} p_i^1 \otimes p_j^1 + x \otimes 1,$$

where p_i^1 constitute a basis of $A_1 = P_1$ and λ_{ij} is a symmetric matrix over the ground field. Let

$$x' = \frac{1}{2} \sum \lambda_{ij} p_i^1 p_j^1.$$

Then

$$\delta(x') = 1 \otimes x' + \sum \lambda_{ij} p_i^1 \otimes p_j^1 + x' \otimes 1.$$

It follows that

$$\delta(x - x') = 1 \otimes x' + x' \otimes 1,$$

i. e. $x - x'$ is primitive, and x is expressed through primitive elements as $(x - x') + x'$, which is a polynomial, linear in P_2 and quadratic in P_1.

Proceeding in this way, the theorem can be proved in grading 3, 4, ... We omit the formal inductive argument.

The uniqueness of the polynomial representation in terms of basic primitive elements (assertion II) is a consequence of the following observation:

Denote by $R_n \subset A_n$ the subspace spanned by all nontrivial products of homogeneous elements of positive degree. Then there is a direct sum decomposition

$$A_n = P_n \oplus R_n.$$

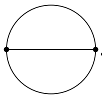
This completes the proof.

Corollary 1. The algebra A has no zero divisors.

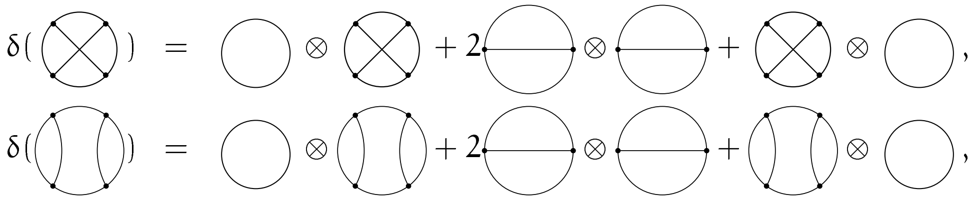
Corollary 2. $\mathcal{A}$ is a Hopf algebra with antipodal automorphism S defined on primitive elements by $S(p) = -p$ and identical on A_0.

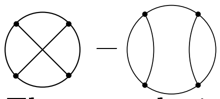
4.2 Primitive elements in $\mathcal{A}$

Since the algebra of chord diagrams $\mathcal{A}$ satisfies the premises of theorem 4.1, it also satisfies its implication, i. e. any chord diagram is a polynomial in primitive elements. Let us see what are the primitive elements of $\mathcal{A}$ in small dimensions.

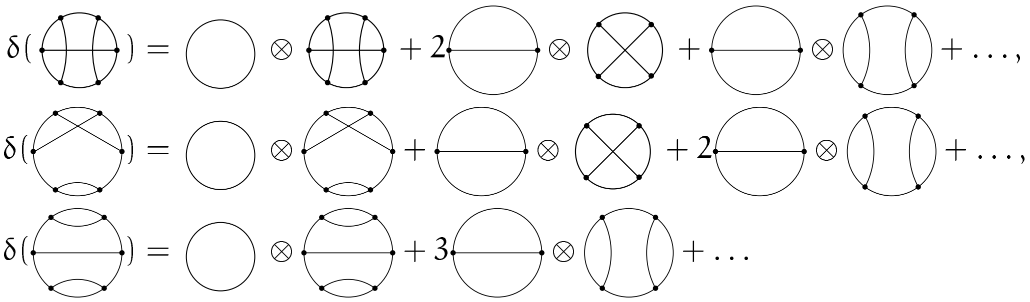
dim $= 1$. $\mathcal{P}_1 = \mathcal{A}_1$ is one-dimensional and generated by .

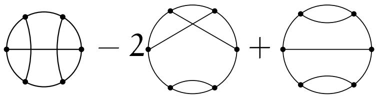
dim $= 2$. Since



the element  is primitive. It constitutes a basis of $\mathcal{P}_2$.

dim $= 3$. The coproducts of the 3 basic elements of $\mathcal{A}_3$ are

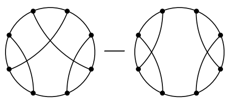
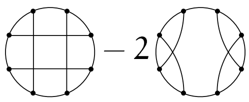


Therefore the element  is the only primitive homogeneous element of degree 3.

The dimensions of $\mathcal{P}_n$ for $n > 3$ are bigger than 1. They are now known up to $n = 12$ (the last three values, corresponding to $n = 10, 11, 12$ were found by J. Kneissler [Kn0]):

n	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{P}_n$	1	1	1	2	3	5	8	12	18	27	39	55

Exercise. Find a basis in the primitive space $\mathcal{P}_4$.

Answer. A possible basis consists of  and .

If the dimensions of $\mathcal{P}_n$ were known for all n, then the dimensions of $\mathcal{A}_n$ would also be known.

Example. Let us find the dimensions of $\mathcal{A}_n$, $n \le 5$, assuming that we know the values of $\dim \mathcal{P}_n$ for $n = 1, 2, 3, 4, 5$, which are equal to $1, 1, 1, 2, 3$. Let p_i be the basic element of $\mathcal{P}_i$, $i = 1, 2, 3$ and denote the bases of $\mathcal{P}_4$ and $\mathcal{P}_5$ as p_{41}, p_{42} and p_{51}, p_{52}, p_{53}, respectively. Nontrivial monomials up to degree 5 that can be made out of these basic elements are:

Degree 2 monomials: p_1^2 (number: 1).

Degree 3 monomials: p_1^3, p_1p_2 (number: 2).

Degree 4 monomials: p_1^4, $p_1^2p_2$, p_1p_3, p_2^2 (number: 4).

Degree 5 monomials: p_1^5, $p_1^3p_2$, $p_1^2p_3$, $p_1p_2^2$, p_1p_{41}, p_1p_{42}, p_2p_3 (number: 7).

A basis of each $\mathcal{A}_n$ can be made up of the primitive elements and their products of the corresponding degree. For $n = 0, 1, 2, 3, 4, 5$ we get: $\dim \mathcal{A}_0 = 1$, $\dim \mathcal{A}_1 = 1$, $\dim \mathcal{A}_2 = 1 + 1 = 2$, $\dim \mathcal{A}_3 = 2 + 1 = 3$, $\dim \mathcal{A}_4 = 4 + 2 = 6$, $\dim \mathcal{A}_5 = 7 + 3 = 10$.

The partial sums of this sequence give the dimensions of the spaces of framed Vassiliev invariants: $\dim \mathcal{FV}_0 = 1$, $\dim \mathcal{FV}_1 = 2$, $\dim \mathcal{FV}_2 = 4$, $\dim \mathcal{FV}_3 = 7$, $\dim \mathcal{FV}_4 = 13$, $\dim \mathcal{FV}_5 = 23$.

Analytical problem. Given the generating function for the sequence of dimensions of primitive spaces in a polynomial algebra

$$p(t) = \sum_{n \geq 0} \dim P_n t^n,$$

find the generating function for the sequence of dimensions of the entire algebra

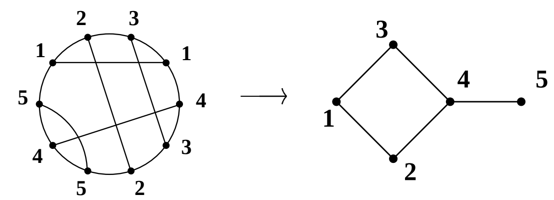
$$a(t) = \sum_{n \geq 0} \dim A_n t^n.$$

Note that primitive elements of $\mathcal{A}$ are represented by rather complicated linear combinations of chord diagrams. A more concise and clear representation can be obtained through connected Feynman diagrams (see Lecture 5).

4.3 Intersection graphs

Definition 4.2 *The intersection graph* $\Gamma(D)$ *of a chord diagram* D *is a graph whose vertices correspond to the chords of* D *and two vertices are connected by an edge iff the corresponding chords intersect. (Two chords,* a *and* b*, are said to intersect if their endpoints* a_1*,* a_2 *and* b_1*,* b_2 *appear in the interlacing order* a_1*,* b_1*,* a_2*,* b_2 *along the circle.)*

For example,



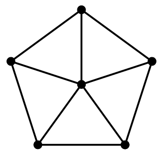
chord $\longrightarrow$ vertex

intersection $\longrightarrow$ edge

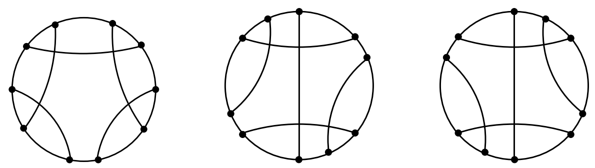
Note that not every graph can be obtained as the intersection graph of a chord diagram.

Exercises. (1) Prove that all graphs with no more than 5 vertices are intersection graphs. (2) Find a graph of degree 6 which is not an intersection graph.

Answer to (2):



Different diagrams may have one and the same intersection graph. For example, there are three different diagrams



with the same intersection graph

Intersection graph conjecture. *If* D_1 *and* D_2 *are two chord diagrams whose intersection graphs are equal,* $\Gamma(D_1) = \Gamma(D_2)$*, then* $D_1 = D_2$ *as elements of* $\mathcal{A}$ (*i.e. modulo four-term relations*).

This conjecture, announced in [CDL1], has been verified in the following situations:

1) for all diagrams D_1, D_2 with n chords up to $n = 8$ (a direct check by computer);

2) for Vassiliev invariants ν coming from standard representations of Lie algebras $\mathfrak{gl}_N$ or $\mathfrak{so}_N$ (see [BN1] and Lecture 10) (this means that $\Gamma(D_1) = \Gamma(D_2)$ implies $\nu(D_1) = \nu(D_2)$);

3) when $\Gamma(D_1) = \Gamma(D_2)$ is a tree (see [CDL2]) or, more generally, D_1, D_2 belong to the forest subalgebra (see [CDL3]).

However, it is not true in general (see section 9.4.4).

4.4 Lando's graph algebra

It turns out that the natural mapping that assigns to every chord diagram its intersection graph, can be converted into a homomorphism of bialgebras $\gamma : \mathcal{A} \to \mathcal{L}$, where $\mathcal{A}$ is the algebra of chord diagrams and $\mathcal{L}$ is an algebra generated by graphs modulo certain relations which was introduced by S. Lando [Lan]. Here is his construction.

Let $\mathcal{G}$ be the graded vector space (over a field $\mathbb{F}$) spanned by all simple graphs (without loops nor multiple edges) as free generators:

$$\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \dots,$$

The grading is by degree (the number of vertices). This space is easily turned into a Hopf algebra:

1. Multiplication is defined as the disjoint union of graphs, then extended by linearity. The empty graph plays the role of the unit in this algebra.

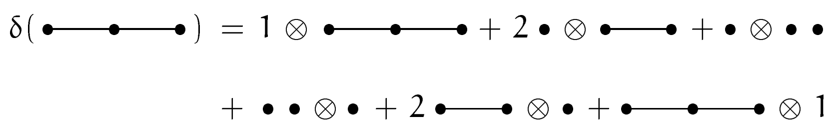
2. Comultiplication is defined similarly to the comultiplication in the bialgebra of chord diagrams. If G is a graph, let $V = V(G)$ be the set ot its vertices. For any subset $U \subset V$ denote by $G(U)$ the graph with the set of vertices U and those vertices of the graph G whose both ednpoints belong to G. We put by definition:

$$\delta(G) = \sum_{U \subseteq V(G)} G(U) \otimes G(V \setminus U), \tag{4.2}$$

and extend by linearity to the whole of $\mathcal{G}$.

The sum in 4.2 is taken over all subsets $U \subset V$ and contains as many as $2^{\#(V)}$ summands.

Example of coproduct.



Exercise. Check the bialgebra axioms.

The mapping from chord diagrams to intersection graphs does not extend to a linear operator $\mathcal{A} \to \mathcal{G}$, because the combinations of graphs that correspond to 4-term relations for chord diagrams, do not vanish in $\mathcal{G}$. To obtain a linear mapping, it is necessary to mod out the space $\mathcal{G}$ by relations, consistent with the 4 term relations. Here is the appropriate definition invented by S. Lando [Lan].

Let G be an arbitrary graph and u, v an ordered pair of its vertices. The pair u, v defines two transformations of the graph G: $G \mapsto G'_{uv}$ and $G \mapsto \widetilde{G}_{uv}$. Both graphs G'_{uv} and $\widetilde{G}_{uv}$ have the same set of vertices as G. They are obtained as follows.

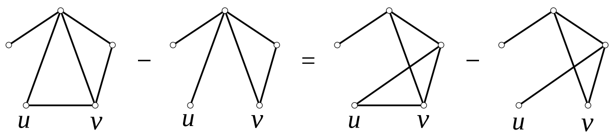
If uv is an edge in G, then the graph G'_{uv} is obtained from G by deleting the edge uv; otherwise this edge should be added (thus, $G \mapsto G'_{uv}$ is toggling the adjacency of u and v).

The graph $\widetilde{G}_{uv}$ is obtained from G in a more tricky way. Consider all vertices $w \in V(G) \setminus \{u, v\}$ which are adjacent in G with v. Then in the graph $\widetilde{G}_{uv}$ vertices u and w are joined by an edge if and only if they are not joined in G. For all other pairs of vertices their adjacency in G and in $\widetilde{G}_{uv}$ is the same.

Definition 4.3 *A four-term relation for graphs is*

$$G - G'_{uv} = \widetilde{G}_{uv} - \widetilde{G}'_{uv} \tag{4.3}$$

Example.



Exercises. 1. Check that, passing to intersection graphs, the four-term relation for chord diagram carries over exactly into this four-term relation for graphs. 2. Find the four-term relation of chord diagrams which is the preimage of the relation shown in the previous example.

Definition 4.4 *The graph algebra of Lando $\mathcal{L}$ is the quotient of the graph algebra $\mathcal{G}$ over the ideal generated by all 4-term relations 4.3.*

Theorem 4.2 *The multiplication and the comultiplication defined above induce a bialgebra structure in the quotient space $\mathcal{L}$.*

Proof. The only thing that needs checking is that the multiplication and the comultiplication both respect the 4-term relation 4.3. For the multiplication (disjoint union of graphs) this statement is obvious. In order to verify it for the comultiplication, it is sufficient to consider two different cases. Namely, let $u, v \in V(G)$ be two distinct vertices of a graph G. The right hand side summands in the comultiplication formula 4.2 split into two groups: those where both vertices u and v belong either to the subset $U \subset V(G)$ or to its complement $V(G) \setminus U$; and those where u and u belong to different subsets. By cleverly grouping the terms of the first kind for the coproduct $\delta(G - G'_{uv} - \widetilde{G}_{uv} + \widetilde{G}'_{uv})$ they all cancel out in pairs. The terms of the second kind can be paired to mutually cancel already for each of the two summands $\delta(G - G'_{uv})$ and $\delta(\widetilde{G}_{uv} - \widetilde{G}'_{uv})$. The theorem is proved.

Relations 4.3 are homogeneous with respect to the number of vertices, therefore $\mathcal{L}$ is a graded algebra. By Theorem 4.1 (p. 27) the algebra $\mathcal{L}$ is polynomial over its space of primitive elements.

Now we have a well-defined bialgebra homomorphism

$$\gamma : \mathcal{A} \to \mathcal{L}$$

that extends the assignment of the intersection graph to a chord diagram. It is defined by the linear mapping between the corresponding primitive spaces $P(\mathcal{A}) \to P(\mathcal{L})$.

According to S. Lando [Lan], the dimensions of the homogeneous components of $P(\mathcal{L})$ are known up to degree 7. It turns out that the homomorphism γ is an isomorphism in degrees up to 6, while the mapping $\gamma : P_7(\mathcal{A}) \to P_7(\mathcal{L})$ has a 1-dimensional kernel. See [Lan] for further details and open problems related to the algebra $\mathcal{L}$.

Lecture 5

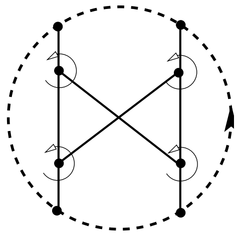
Algebra of Feynman diagrams

Feynman diagrams generalize chord diagrams. Modulo appropriate relations, the space of Feynman diagrams can be turned into an algebra which is isomorphic to the algebra of chord diagrams $\mathcal{A}$.

5.1 Construction

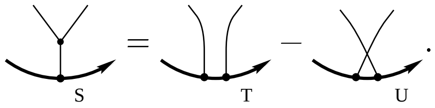
Definition 5.1 *A Feynman diagram of degree* n *is a connected three-valent graph on* $2n$ *vertices with a distinguished oriented cycle, called Wilson loop, and a fixed rotation (cyclic order of edges) in all the vertices that do not belong to the Wilson loop.*

Example. Here is a Feynman diagram of degree 4:



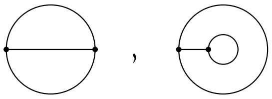
The Wilson loop is shown here by a dashed line. Its orientation, as well as the rotation of the inner vertices, is indicated by arrows. In the pictures below, we will always suppose that the diagram is contained in its Wilson loop, which is oriented counterclockwise. Inner trivalent vertice will be also supposed to be oriented countercloskwise. (This convention is referred to as *blackboard orientation*). Note that the intersection of two edges in the center of the diagram is not actually an intersection: there is no vertex at this imaginary point.

Definition 5.2 *The space of Feynman diagrams* $\mathcal{F}_n$ *is the linear space generated by all Feynman diagrams of degree* n *modulo STU relations:*

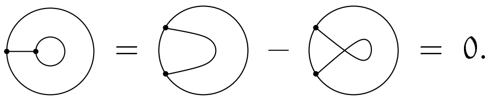


The choice of a specific STU relation is determined by the diagram on the left and one of its vertices on the Wilson loop.

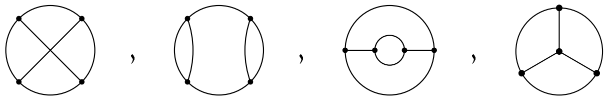
Examples. Up to a sign, there exist two different Feynman diagrams of order 1:



one of which vanishes due to STU relation:

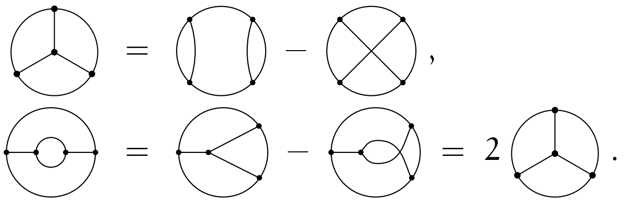


There are four non-zero Feynman diagrams of degree 2:



whereas the STU relations ensure that $\dim \mathcal{F}_2 \leq 2$.

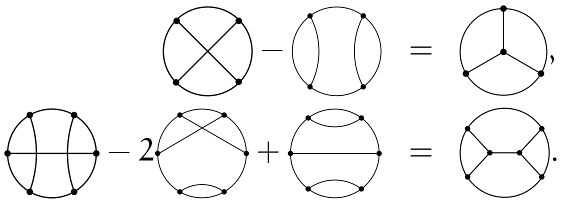
Indeed, by STU relations:



Note, moreover, that there are four more Feynman diagrams of degree 2, which are 0 due to the AS relations (see Lemma 6.2 below).

Exercise. Using STU relations, rewrite the basic elements of $\mathcal{P}_2$ and $\mathcal{P}_3$ in a concise way.

Answer.



Chord diagrams are a particular case of Feynman diagrams, namely those without inner trivalent vertices. Using STU relations, one can rewrite any Feynman diagram as a linear combination of chord diagrams. (Examples were given just above.)

A vertex of a Feynman diagram that belongs to the Wilson loop is called *external*; otherwise it is called *internal*. In the space $\mathcal{F}_n$ there is an increasing filtration by subspaces $\mathcal{F}_n^m$ generated by diagrams with at most m external vertices:

$$\mathcal{F}_n^1 \subset \mathcal{F}_n^2 \subset \dots \subset \mathcal{F}_n^{2n}.$$

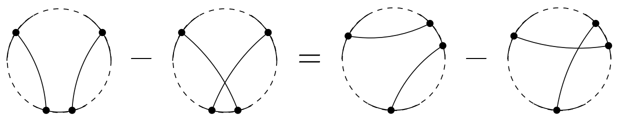
Exercise. Prove that $\mathcal{F}_n^1 = 0$.

Hint. This easily follows from the antisymmetry relation (see page 41 in Lecture 6).

5.2 Isomorphism $\mathcal{A} \simeq \mathcal{F}$

Lemma 5.1 *STU relations imply 4T relations for chord diagrams.*

Proof. Indeed, if we write the four-term relation under the form



and apply the STU relations to the left and to the right part of this equation, we will get the same Feynman diagrams.

Let CD_n be the set of chord diagrams of order n and FD the set of Feynman diagrams of the same order. We have a natural inclusion $\lambda : \mathrm{CD}_n \to \mathrm{FD}_n$.

Theorem 5.1 *The inclusion λ gives rise to an isomorphism of linear spaces $\hat{\lambda}$: $\mathcal{A}_n \to \mathcal{F}_n$.*

Proof. Indeed, we must check the following things:

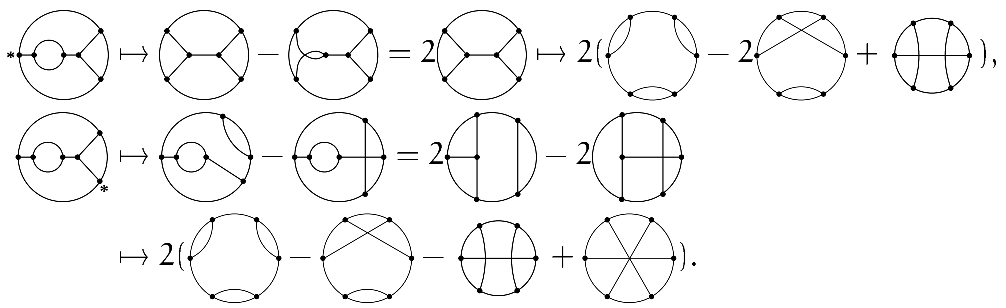
(A) That λ leads to a well-defined linear mapping from $\mathcal{A}_n$ to $\mathcal{F}_n$.

(B) That this mapping is a linear isomorphism.

Well, let's see.

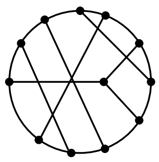
(A) Indeed, $\mathcal{A}_n = \langle \mathrm{CD}_n \rangle / \langle 4\mathrm{T} \rangle$, $\mathcal{F}_n = \langle \mathrm{FD}_n \rangle / \langle \mathrm{STU} \rangle$. (Here, as everywhere in these lectures, angular brackets denote linear span.) Lemma 5.1 implies that $\lambda(\langle 4\mathrm{T} \rangle) \subset \langle \mathrm{STU} \rangle)$, therefore the map of the quotient spaces is well-defined.

(B) We will construct a linear mapping $\mathcal{F}_n \to \mathcal{A}_n$ and prove that it is inverse to $\hat{\lambda}$.
As we noticed before, any Feynman diagram by iterative use of STU relations can be transformed into a combination of chord diagrams. This gives rise to a mapping $\rho : \mathrm{FD}_n \to \langle \mathrm{CD}_n \rangle$ which is, however, multivalued, because the result may depend on the specific sequence of resolutions used. Here is an example where the place where the STU relation is applied is marked:



However, the combination $\rho(f)$ is well-defined as an element of $\mathcal{A}_n$, i. e. modulo the 4T relations. The proof of this fact proceeds by induction on the number k of inner points in the diagram f.

If $k = 1$, then the diagram f consists of one tripod and several chords and may look something like this:

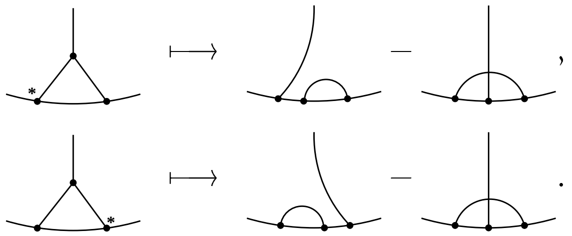


There are 3 ways to resolve the inner triple point, and the fact that the results are the same in $\mathcal{A}_n$ is exactly the definition of the 4T relation.

Suppose that ρ is well-defined on Feynman diagrams with $< k$ inner vertices. Pick a diagram $f \in \mathrm{FD}_n^k$. The process of resolving the triple points starts with some 2 out of $2n - k$ outer vertices. Let us prove, modulo the inductive hypothesis, that if we change the order of these two points, then the final result will be the same.

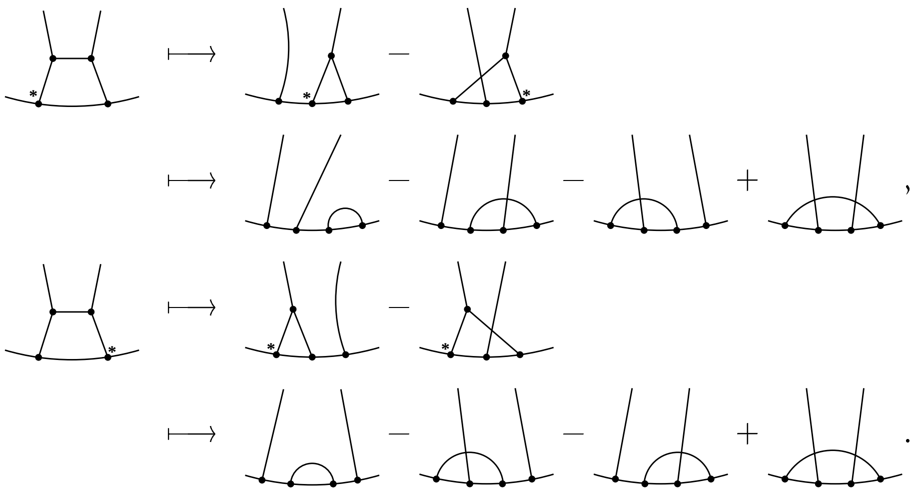
There are 3 cases to consider: the two chosen points on the Wilson loop are (1) adjacent to a common inner vertex, (2) adjacent to neighbouring inner vertices, (3) adjacent to non-neighbouring inner vertices. The proof for cases (1) and (2) is shown in the pictures that follow.

(1)



The position of an isolated chord does not matter, because, as we know, the multiplication in $\mathcal{A}$ is well-defined.

(2)



After the first resolution, we can choose the sequence of further resolutions arbitrarily, by the inductive hypothesis.

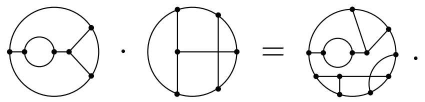
Exercise. Find a similar proof in case (3).

We thus have a well-defined linear mapping $\hat{\rho} : \mathcal{F}_n \to \mathcal{A}_n$. The fact that it is two-sided inverse to $\hat{\lambda}$ is fairly evident. The theorem is proved.

5.3 Product and coproduct in the space $\mathcal{F}$

Now we will define a bialgebra structure in the space $\mathcal{F}$.

Definition 5.3 *The product of two Feynman diagrams is defined in the same way as for chord diagrams: the two Wilson loops are cut in arbitrary places and then glued into one loop, taking care of the orientations:*



Proposition 5.1 *This multiplication is well-defined, i. e. does not depend on the place of cuts.*

Proof. Due to 4T relations in $\mathcal{A}$ and STU relations in $\mathcal{F}$, we see straight away that this product in $\mathcal{F}$ agrees with the product in $\mathcal{A}$ along the isomorphism $\mathcal{A} \simeq \mathcal{F}$ constructed in Theorem 5.1 above. Now the fact that the multiplication is well-defined in $\mathcal{A}$ implies the fact that it is well-defined in $\mathcal{F}$. $\square$

To define the *coproduct* in the space $\mathcal{F}$, we will use connected components of a Feynman diagram.

By definition, the underlying graph of a Feynman diagram is always connected. We will, however, introduce a different notion of connectivity and distinguish between connect and disconnected Feynman diagrams.

Definition 5.4 *A Feynman diagram is said to be connected, if its underlying graph remains connected after stripping the Wilson loop. The connected components of a Feynman diagram are defined as connected components of the graph obtained when the Wilson loop is removed.*

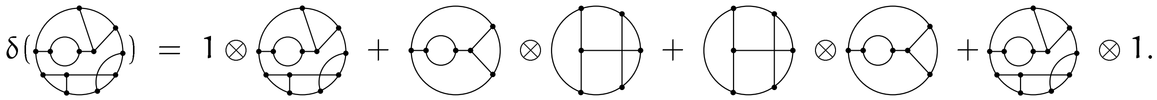
In the sense of this definition, any chord diagram of order n consists of n connected components (chords).

Now, the construction of coproduct proceeds in the same way as for chord diagrams.

Definition 5.5 *Let* D *be a Feynman diagram and* $[D]$ *the set of its connected components. For any subset* $J \subset [D]$ *denote by* D_J *the Feynman diagram with only those components that belong to* J *and by* D'_J *the 'complementary' diagram. We set by definition*

$$\delta(D) = \sum_{J \subset [D]} D_J \otimes D'_J.$$

Example:



An easy technical check shows that the product is consistent with the coproduct and that the isomorphisms $\hat{\lambda}$, $\hat{\rho}$ are in fact isomorphisms of bialgebras.

5.4 Primitive subspace of $\mathcal{F}$

It is evident that connected Feynman diagrams are primitive with respect to the coproduct δ. Taking into account the situation with chord diagrams, it may sound surprising that the converse is also true:

Theorem 5.2 *The primitive space of the bialgebra $\mathcal{F}$ coincides with the linear span of connected Feynman diagrams.*

Proof. This fact will follow from the isomorphism of coalgebras $\mathcal{F}$ and $\mathcal{B}$ (see next lecture), because in $\mathcal{B}$ the coproduct has a simpler description.

Lecture 6

Algebra of Chinese characters

The subject of this lecture is the combinatorial bialgebra $\mathcal{B}$ which is isomorphic to the previously defined algebras $\mathcal{A}$ and $\mathcal{F}$ as linear space and as coalgebra, but has a different natural multiplication. This leads to a remarkable fact that in our algebra $\mathcal{A} \simeq \mathcal{F} \simeq \mathcal{B}$ there are two multiplications both compatible with one and the same comultiplication.

6.1 Construction

Definition 6.1 *A Chinese character (CC) is a graph with 1- and 3-valent vertices, cyclic order of (half)edges at every 3-valent vertex and with at least one 1-valent vertex in every connected component.*

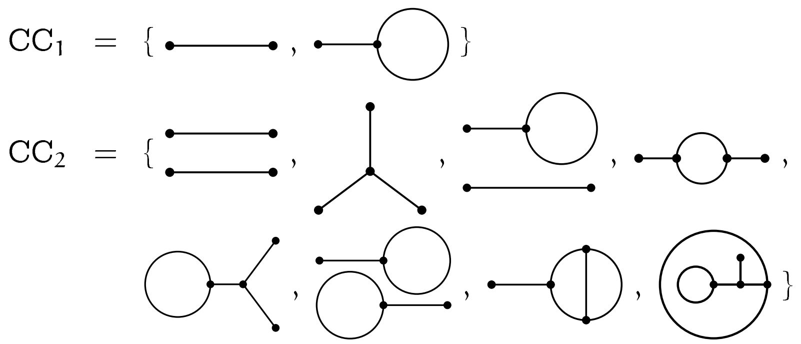
Remark 1. A CC is not required to be connected. It may have loops and multiple edges.

Remark 2. Ordering halfedges, not whole edges is important, because of possible loops.

Remark 3. The total number of vertices of a CC is even. Half of this number is called the degree (or order) of a CC.

Remark 4. In the literature, CC's are also referred to as "1-3-valent diagrams", "open (Feynman) diagrams", "web diagrams" and "Jacobi diagrams".

Examples. Below is the list of all Chinese characters of degree 1 and 2 up to a sign (I can only hope that it is complete!)



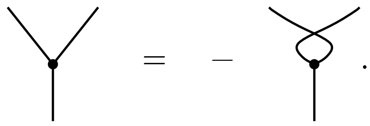
Now you will see that most of the elements listed above are not important: they are killed by the following definition.

Definition 6.2 *The space of Chinese characters of degree* n *is the quotient space*

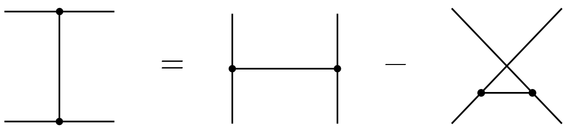
$$\mathcal{B}_n = \langle CC_n \rangle / \langle AS,\ IHX \rangle,$$

where $\langle CC_n \rangle$ *is the linear space formally generated by all Chinese characters of degree* n *and* $\langle AS,\ IHX \rangle$ *stands for the subspace spanned by all AS and IHX relations.*

Definition 6.3 *An AS (=antisymmetry) relation is:*



Definition 6.4 *An IHX relation is:*



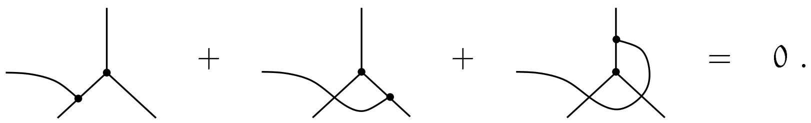
(You may notice that the three fragments involved resemble the capital letters I, H and X, respectively).

As always, the unfinished fragments of the pictures denote graphs that are identical (and arbitrary) everywhere but in this explicitly shown fragment.

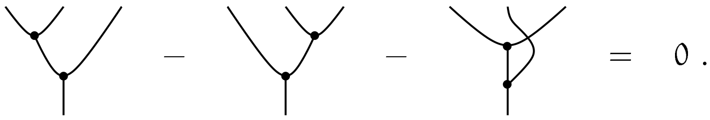
Exercise. Prove (directly from definitions) that $\dim \mathcal{B}_1 = 1$, $\dim \mathcal{B}_2 = 2$.

The IHX relation can be redrawn under several different forms, for example:

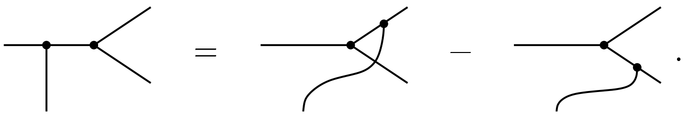
- (rotationally symmetric form)



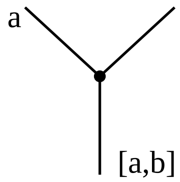
- (Jacobi form)



- (Kirchhof form)



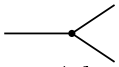
The Jacobi presentation of the IHX relation can be interpreted as follows. Suppose that to the upper 3 endpoints of each diagram we assign 3 elements of a Lie algebra, x, y and z, while every trivalent vertex, traversed downwards, corresponds to taking the commutator:

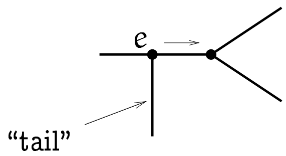


Then the IHX relation means that

$$[[x, y], z] - [x, [y, z]] - [[x, z], y] = 0,$$

which is the classical Jacobi identity.

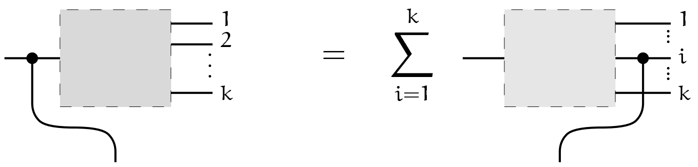
The Kirchhof presentation is related to the Kirchhof's law in electricity. Let us view the portion  of the given graph as a piece of electrical circuit, and the variable vertex as an 'electron' e with a 'tail' whose endpoint is fixed. Suppose that the electron moves towards a node of the circuit:



Then the IHX relation expresses the well-known Kirchhof rule: *the sum of currents entering a node is equal to the sum of currents going out of it.* This electrotechnical analogy is very useful, e. g. in the proof of the generalized Kirchhof law below.

The IHX relation can be generalized as follows:

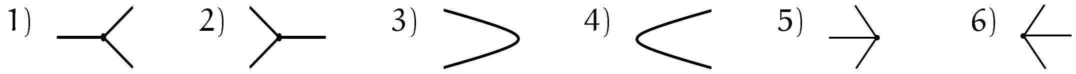
Lemma 6.1 *(Generalized Kirchhof rule). In the algebra $\mathcal{B}$ the following identity holds:*



where the grey box is an arbitrary subgraph which has only 3-valent vertices.

Proof. Fix a horizontal line in the plane and consider an immersion of the given graph into the plane with smooth edges, generic with respect to the projection onto this line. More exactly, we assume that (1) the projections of all vertices onto the horizontal line are distinct, (2) when restricted to an arbitrary edge, the projection has only Morse critical points, and (3) the images of all critical points are distinct and different from the images of vertices.

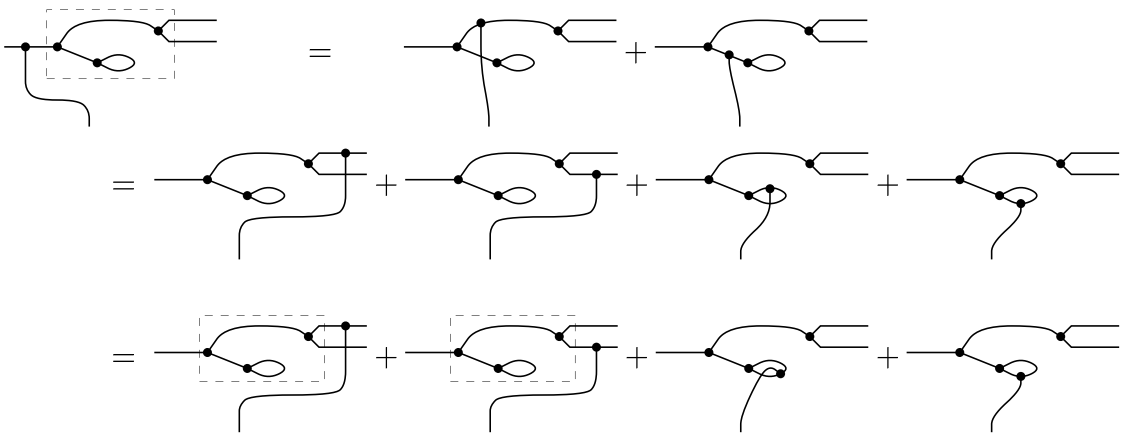
Bifurcation points are the images of vertices and critical points of the projection. The proof proceeds by induction on the number of bifurcation points. If there is only one bifurcation points, it must be the image of a vertex, in which case the equality to be proved coincides with the IHX relation. Suppose that the lemma is established for s bifurcation points. The $(s+1)$-th bifurcation point falls under one of the six categories:



In the first two cases the lemma follows from the IHX relation, in cases 3 and 4 — from the AS relation. Cases 5 and 6 by a deformation of the immersion are reduced to a combination of the previous cases.

This completes the proof of the lemma.

Example.



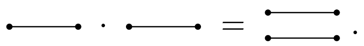
The two last summands cancel out because of the antisymmetry relation.

6.2 Product and coproduct in $\mathcal{B}$

Both the product and the coproduct in the linear space $\mathcal{B}$ are defined in a quite straightforward way. We first define the product and coproduct of Chinese characters, then extend it by linearity to the free space geberated by CC's, then note that it is compatible wit the AS and IHX relations and thus descends to the quotient space $\mathcal{B}$.

Definition 6.5 *The product of two Chinese characters is their disjoint union.*

Example.

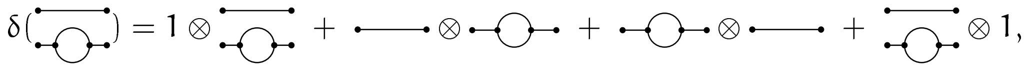


Definition 6.6 *Let* D *be a Chinese character and* $[D]$ *the set of its connected components. For a subset* $J \subset [D]$ *denote by* D_J *the union of the components that*

belong to J *and by* D'_J *the union of the components that do not belong to* J*. We set:*

$$\delta(D) = \sum_{J \subset [D]} D_J \otimes D'_J.$$

Example.



Now, it is easy to see that the product of an AS or IHX combination of diagrams by an arbitrary Chinese character belongs to the linear span of the relations of the same types. Also, the coproduct of any AS or IHX relation is also 0 modulo these relations. Therefore, we have well-defined algebraic operations in the space $\mathcal{B}$ which thus becomes a Hopf algebra.

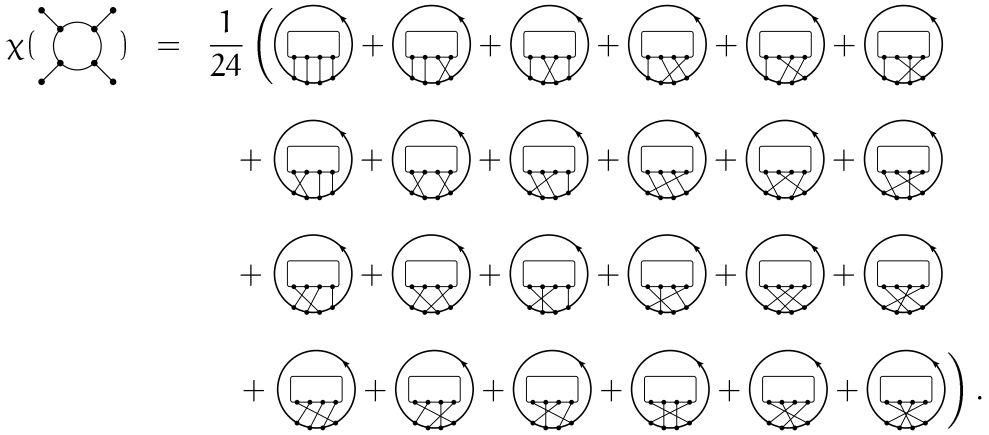
6.3 Isomorphism $\mathcal{B} \simeq \mathcal{F}$

Here, we will construct a linear isomorphism between the spaces $\mathcal{B}_n$ and $\mathcal{F}_n$.

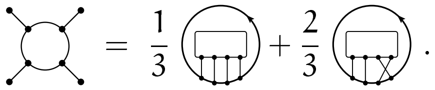
To convert a Chinese character into a Feynman diagram, we will join all of its univalent vertices (legs) by a Wilson loop. For a Chinese character with k legs $D \in CC^k_n$ there are $k!$ ways[1] of doing that, and we set $\chi(D)$ to be equal to the arithmetic mean of all the resulting Feynman diagrams. Thus we get a mapping

$$\chi : CC \to \mathcal{F}.$$

For example,



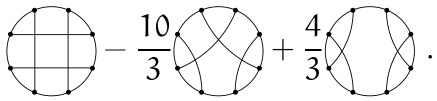
Scrutinizing these pictures, one can see that 16 out of 24 summands are equivalent to the first diagram, while the remaining 8 are equivalent to the second one. Therefore,



[1]Well, actually $(k-1)!$, not $k!$, because the external vertices of a Feynman diagram have a cyclic, rather than linear, ordering. But it is somewhat easier to enumerate all linear orderings.

Exercise. Express this element through chord diagrams, using the isomorphism $\mathcal{F} \simeq \mathcal{A}$.

Answer.



Theorem 6.1 *The mapping $\chi : \mathrm{CC} \to \mathcal{F}$ descends to a linear operator $\hat{\chi} : \mathcal{B} \to \mathcal{F}$ which is a graded isomorphism between the linear spaces $\mathcal{B}$ and $\mathcal{F}$.*

Proof. The theorem consists of two parts:

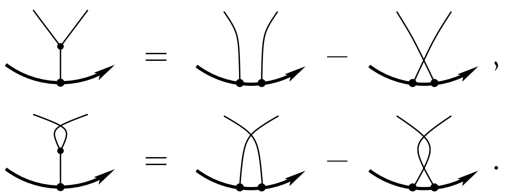
- Easy part: the linear mapping $\hat{\chi}$ is well-defined.
- Difficult part: dhe mapping $\hat{\chi}$ is bijective.

Easy part of the theorem. To prove the easy part, we must show that the AS and IHX combinations of Chinese Characters go into 0 in the space $\mathcal{F}$. This follows from the two lemmas.

Lemma 6.2 *STU relations imply AS relations for the internal vertices of a Feynman diagram.*

Proof. Induction on the distance (in edges) of the vertex in question from the Wilson loop.

Induction base. If the vertex is adjacent to an external vertex, then the assertion follows by one application of the STU relation:

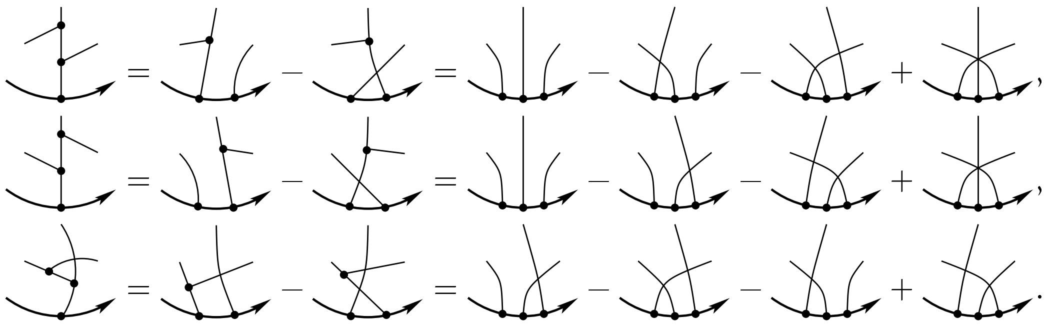


Induction step. Take two Feynman diagrams f_1 and f_2 that differ only by the rotation of one internal vertex v. Apply STU relations to both diagrams *in the same way* so that v gets closer to the Wilson loop.

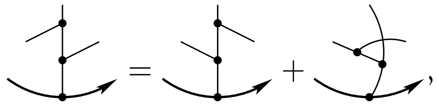
The lemma is proved.

Lemma 6.3 *STU relations imply IHX relations for the internal vertices of a Feynman diagram.*

Proof. The argument is similar to the one used in the previous proof. We take an IHX relation somewhere inside a Feynman diagram and, applying the same sequence of STU moves to each of the three diagrams, move the IHX fragment closer to the Wilson loop. The proof of the induction base is shown in these pictures:



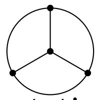
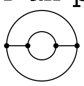
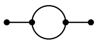
Therefore,



which is the IHX relation that we wanted to prove.

Difficult part of the theorem. To prove the difficult part, we will construct a linear mapping from $\mathcal{F}$ to $\mathcal{B}$, inverse to $\hat{\chi}$.

As usual, we first define this mapping on generators. If an element $f \in \mathrm{FD}$ is symmetric, that is $F = \chi(b)$ for a certain combination of Chinese Characters $b \in \mathcal{B}$, then, of course, we can just set $\tau(f) = b$.

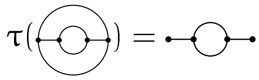
For example, the element  is *not* symmetric (although it looks so), because symmetrizing it over all permutations of its external vertices, we get 0. On the other hand, the element  *is* symmetric, as it is equal to the image of the Chinese Character  under χ. This is true about any Feynman diagram with just two legs, because they can be swapped by a rotation of the Wilson loop, which is by definition identical on $\mathcal{F}$. Therefore, the idea of constructing the mapping τ inductively by the number of external vertices naturally arises.

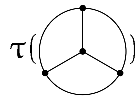
Note that the difference of any Feynman diagram and the same diagram whose legs are permuted by an arbitrary permutation, is equivalent, modulo STU relations, to a combination of diagrams with a smaller number of external vertices. Assuming that the mapping τ is defined for Feynman diagrams having less than k legs, we define it for a diagram f with exactly k legs by the formula:

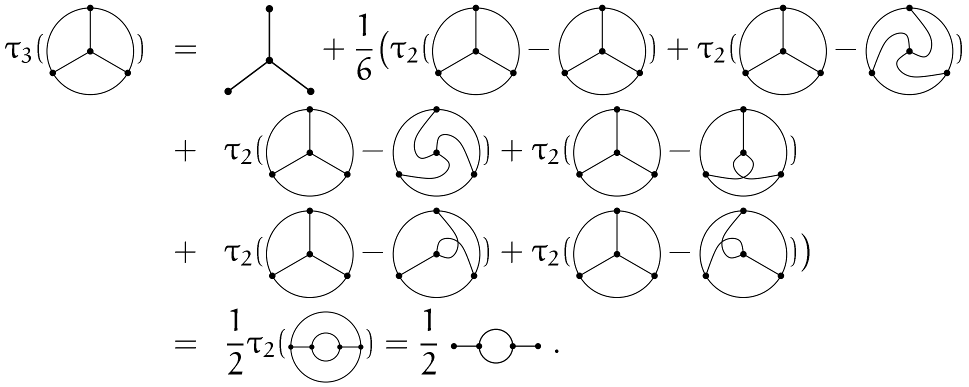
$$\tau_k(f) = \widetilde{f} + \frac{1}{k!}\sum_{\pi \in S_k} \tau_{k-1}(f - \pi(f)),$$

where $\widetilde{f}$ is the Chinese character that you get by stripping the Wilson loop off f and $\pi(f)$ denotes the action of the symmetric group S_k on Feynman diagrams by permutations of external vertices (of course, the action requires a numbering of these vertices, but the sum does not depend on a particular numbering).

For example, we know that



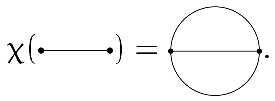
and we want to find . By the above formula, we have:

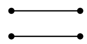


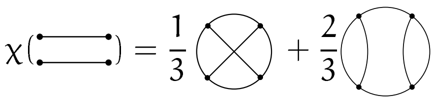
The proof that the mapping τ is well-defined and inverse to χ is rather intricate, and we refer to [BN1] for details.

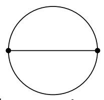
6.4 Two products compatible with one coproduct

It is easy to check that the isomorphism χ is compatible with the coproduct in the algebras $\mathcal{F}$ and $\mathcal{B}$. However, it is *not* compatible with the product. For example,



The square of the element in $\mathcal{B}$ is . However, the corresponding element of $\mathcal{F}$



is not equal to the square of .

We can, of course, carry the natural multiplication of the algebra $\mathcal{F}$ onto the algebra $\mathcal{B}$ along the isomorphism $\hat{\chi}$, thus obtaining a Hopf algebra with two different products, both compatible with one and the same coproduct.

The coproduct in $\mathcal{B}$ is compatible with its product, which is the simple disjoint union of graphs. This immediately implies the following fact.

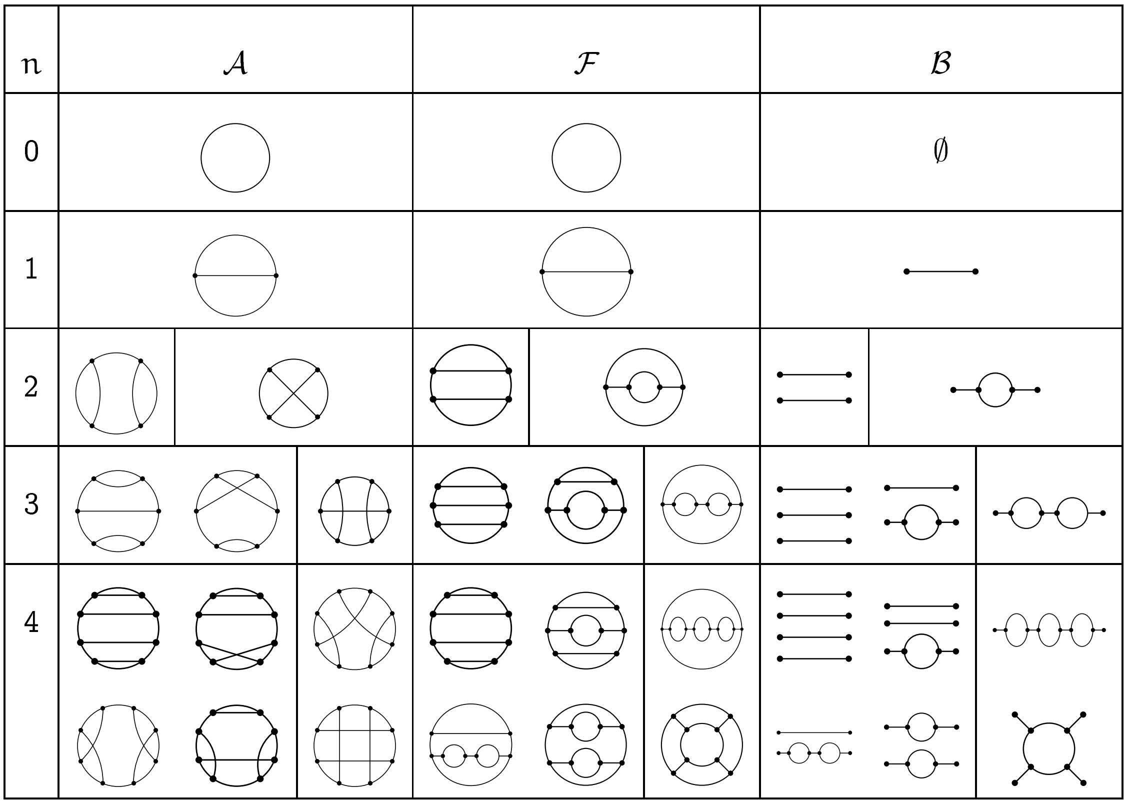
Proposition 6.1 *The primitive space of the bialgebra $\mathcal{B}$ is spanned by connected Chinese characters.*

Now, we know that the coproduct in $\mathcal{F}$ corresponds to the coproduct in $\mathcal{B}$ via the isomorphism $\hat{\chi}$. Therefore, the corresponding primitive spaces are carried by $\hat{\chi}$ one into another. This isomorphism preserves the class of connected diagrams.

Corollary. The primitive space of the bialgebra $\mathcal{F}$ is spanned by connected Feynman diagrams. (This is the assertion of Theorem 5.2 announced above).

6.5 A glimpse of the three algebras

Here is a comparative table which shows the structure of the algebras $\mathcal{A}$, $\mathcal{F}$ and $\mathcal{B}$ in small dimensions.



In every grading up to 4, for each of the three algebras, this table displays a basis of the corresponding homogeneous component. Starting from grading 2, decomposable elements (products of elements of smaller degree) appear on the left, while the new indecomposable elements appear on the right.

Lecture 7

The Kontsevich integral

Kontsevich's integral of a knot is a far-reaching generalization of the Gauss integral formula for the linking number. It provides a tool to construct the universal Vassiliev invariant. Any Vassiliev knot invariant can be derived from it.

The Kontsevich integral was invented by M. Kontsevich [Ko1]. Detailed expositions can be found in [BN1, CD3, Les]. Other relevant papers of primary importance are [Car], [LM1], [LM2].

7.1 The construction

Represent the three-dimensional space $\mathbb{R}^3$ as a direct product of a complex line $\mathbb{C}$ with coordinate z and a real line $\mathbb{R}$ with coordinate t. The integral is defined for Morse knots, i. e. knots K embedded in $\mathbb{R}^3 = \mathbb{C}_z \times \mathbb{R}_t$ in such a way that the coordinate t is a Morse function on K (all critical points are nondegenerate and all critical levels are different). Its values belong to the graded completion $\overline{\mathcal{A}}'$ of the algebra of chord diagrams with 1-term relations $\mathcal{A}' = \mathcal{A}/(\Theta)$. (By definition, the elements of a graded algebra are finite linear combinations of homogeneous elements. The *graded completion* consists of all infinite combinations of such elements.)

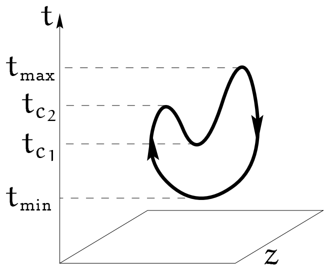
Definition 7.1 *The Kontsevich integral* $Z(K)$ *of the knot* K *is defined as follows:*

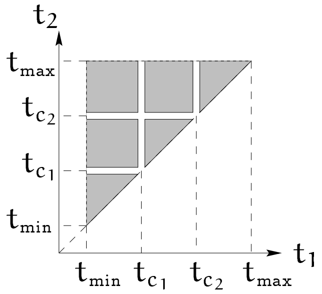
$$Z(K) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int\limits_{\substack{t_{min}<t_1<\cdots<t_m<t_{max} \\ t_j \text{ are noncritical}}} \sum_{P=\{(z_j,z_j')\}} (-1)^{\downarrow} D_P \bigwedge_{j=1}^{m} \frac{dz_j - dz_j'}{z_j - z_j'} \ .$$

The ingredients of this formula have the following meaning.

The real numbers $t_{\min}$ and $t_{\max}$ are the minimum and the maximum of the function t on K.

The integration domain is the m-dimensional simplex $t_{\min} < t_1 < \cdots < t_m < t_{\max}$ divided by the critical values into a certain number of *connected components*. For example, for the following embedding of the unknot and $m = 2$ the corresponding integration domain has six connected components and looks like

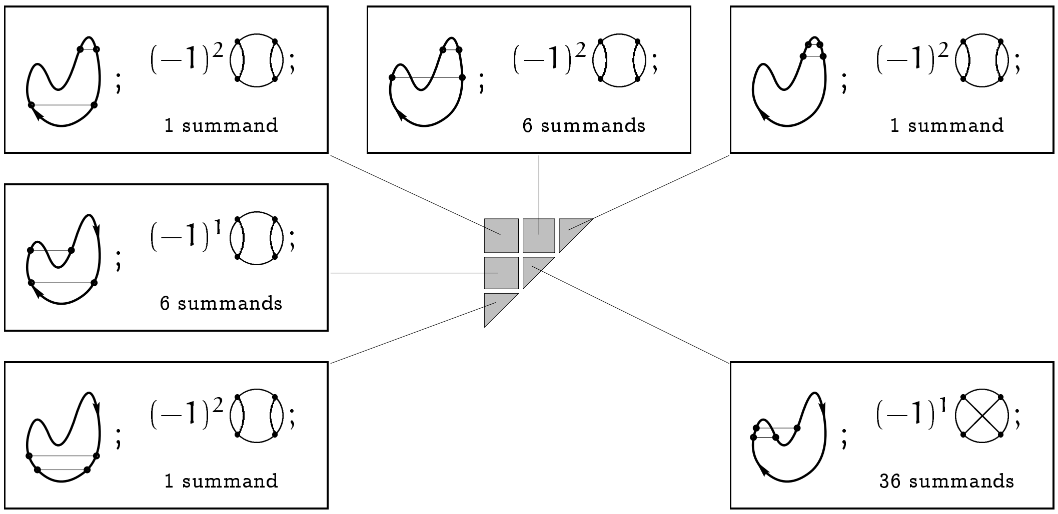




The number of summands in the integrand is constant in each connected component of the integration domain, but can be different for different components. In each plane $\{t = t_j\} \subset \mathbb{R}^3$ choose an unordered pair of distinct points (z_j, t_j) and (z'_j, t_j) on K, so that $z_j(t_j)$ and $z'_j(t_j)$ are continuous functions. We denote by $P = \{(z_j, z'_j)\}$ the set of such pairs for $j = 1, \ldots, m$. The integrand is the sum over all choices of the pairing P. In the example above for the component $\{t_{min} < t_1 < t_{c_1},\ t_{c_2} < t_2 < t_{max}\}$ we have only one possible pair of points on the levels $\{t = t_1\}$ and $\{t = t_2\}$. Therefore, the sum over P for this component consists of only one summand. Unlike this, in the component $\{t_{min} < t_1 < t_{c_1},\ t_{c_1} < t_2 < t_{c_2}\}$ we still have only one possibility for the level $\{t = t_1\}$, but the plane $\{t = t_2\}$ intersects our knot K in four points. So we have $\binom{4}{2} = 6$ possible pairs (z_2, z'_2) and the total number of summands is six (see the picture below).

For a pairing P the symbol '$\downarrow$' denotes the number of points (z_j, t_j) or (z'_j, t_j) in P where the coordinate t decreases along the orientation of K.

Fix a pairing P. Consider the knot K as an oriented circle and connect the points (z_j, t_j) and (z'_j, t_j) by a chord. We obtain a chord diagram with m chords. The corresponding element of the algebra $\mathcal{A}'$ is denoted by D_P. In the picture below, for each connected component in our example, we show one of the possible pairings, the corresponding chord diagram with the sign $(-1)^{\downarrow}$ and the number of summands of the integrand (some of which are equal to zero in $\mathcal{A}'$ due to one-term relation).



Over each connected component z_j and z_j' are smooth functions in t_j. By $\bigwedge\limits_{j=1}^{m} \dfrac{dz_j - dz_j'}{z_j - z_j'}$ we mean the pullback of this form to the integration domain of variables $t_1, \ldots, t_m$. The integration domain is considered with the orientation of the space $\mathbb{R}^m$ defined by the natural order of the coordinates t_1, ..., t_m.

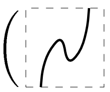
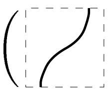
By convention, the term in the Kontsevich integral corresponding to $m = 0$ is the (only) chord diagram of order 0 with coefficient one. It represents the unit of the algebra $\mathcal{A}'$.

7.2 Basic properties

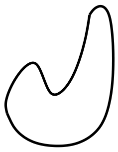
The Kontsevich integral has the following basic properties (the proofs will be given in the next lectures):

1. $Z(K)$ converges for any Morse knot K.
2. It is invariant under deformations of the knot in the class of Morse knots.
3. It behaves in a simple way under the deformations that add a pair of new critical points to a Morse knot.

Let us explain the last item in more detail. Unfortunately, the Kontsevich integral is not invariant under deformations that change the number of critical points of the function t, therefore it does not define a knot invariant. However, the following formula shows how the integral changes under such deformations:

$Z\Big($  $\Big) = Z(H) \cdot Z\Big($  $\Big)$. (7.1)

Here the first and the third pictures depict two embeddings of an arbitrary knot, differing only in the shown fragment,

$H =$ 

is the *hump* (unknot embedded in $\mathbb{R}^3$ in the specified way), and the product is the product in the completed algebra $\overline{\mathcal{A}}'$ of chord diagrams. The last equality allows to define a genuine knot invariant by the formula

$$I(K) = \frac{Z(K)}{Z(H)^{c/2}},$$

where c denotes the number of critical points of K and quotient means division in the algebra $\overline{\mathcal{A}}'$ according to the rule $(1+a)^{-1} = 1 - a + a^2 - a^3 + \ldots$

The expression $I(K)$ is sometimes referred to as 'final' Kontsevich's integral. as opposed to the 'preliminary' Kontsevich's integral $Z(K)$. It is invariant under arbitrary deformations of K and in fact represents a universal Vassiliev invariant in the following sense.

Consider a function w on the set of chord diagrams with m chords statisfying one- and four-term relations (a weight system). Applying this function to the series $I(K)$, we get a numerical knot invariant $w(I(K))$. This invariant is a Vassiliev invariant of order m and any Vassiliev invariant can be obtained in this way. A proof of this will be given in Lecture 9.

The Kontsevich integral behaves in a nice way with respect to the natural operations on knots, such as mirror reflection, changing the orientation of the knot, mutation of knots. In a different normalization it is multiplicative under the connected sum of knots:

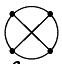
$$I'(K_1 \# K_2) = I'(K_1) I'(K_2),$$

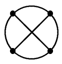
where

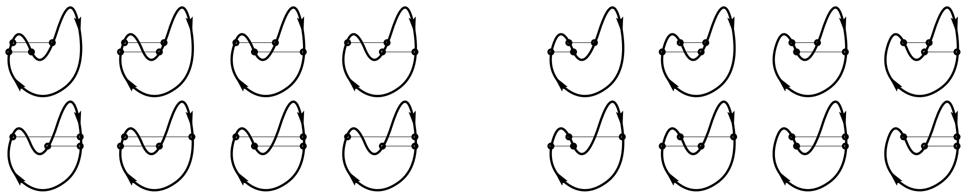
$$I'(K) = Z(H) I(K) = \frac{Z(K)}{Z(H)^{c/2-1}}.$$

For any knot K the coefficients in the expansion of $Z(K)$ over an arbitrary basis consisting of chord diagrams are rational [Ko1, LM2].

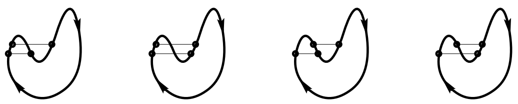
7.3 Example of calculation

We will calculate the coefficient of the chord diagram  in $Z(H)$, where H is the hump (unknot with 4 critical points, see p. 51) directly from the definition of the Kontsevich integral.

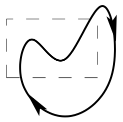
First of all, note that out of the total number of 51 pairings shown in the picture on p. 50, the following 16 contribute to the coefficient of :



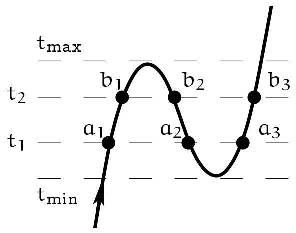
It luckily turns out we can leave only 4 out of these 16 summands:



The fact that the Kontsevich's integral is equal to the sum of only these 4 terms, follows from lemma 9.1 (see p. 64), because the remaining 12 parings involve 'long' chords with respect to the following distinguished portion of our knot:



The next picture shows the relevant fragment of the knot under magnifying glass. We denote by a_1, a_2, a_3 (resp. b_1, b_2, b_3) the three points of intersection of the knot with the level $\{t = t_1\}$ (resp. $\{t = t_2\}$):



.

We have four possibilities for chosing (z_1, z_1') and (z_2, z_2'):

(z_2, z_2')	(b_1, b_3)	(b_2, b_3)	(b_1, b_3)	(b_2, b_3)
(z_1, z_1')	(a_1, a_2)	(a_1, a_2)	(a_1, a_3)	(a_1, a_3)

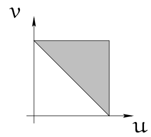
.

Put $a_{ij} = a_j - a_i$ and $b_{ij} = b_j - b_i$. Then the integral to compute is

$$\begin{aligned}
&\frac{1}{(2\pi i)^2} \int\limits_{t_{min}<t_1<t_2<t_{max}} (-d\ln a_{12} \wedge d\ln b_{13} + d\ln a_{12} \wedge d\ln b_{23} \\
&\qquad\qquad + d\ln a_{13} \wedge d\ln b_{13} - d\ln a_{13} \wedge d\ln b_{23}) \\
&= -\frac{1}{4\pi^2} \int\limits_{t_{min}<t_1<t_2<t_{max}} (-d\ln\left(\frac{a_{12}}{a_{13}}\right) \wedge d\ln b_{13} + d\ln\left(\frac{a_{12}}{a_{13}}\right) \wedge d\ln b_{23}) \\
&= -\frac{1}{4\pi^2} \int\limits_{t_{min}<t_1<t_2<t_{max}} d\ln\left(\frac{a_{12}}{a_{13}}\right) \wedge d\ln\left(\frac{b_{23}}{b_{13}}\right) .
\end{aligned}$$

The mapping $u(t_1, t_2) = \dfrac{a_{12}}{a_{13}}$, $v(t_1, t_2) = \dfrac{b_{23}}{b_{13}}$ transforms the integration domain $t_{min} < t_1 < t_2 < t_{max}$ to the standard triangle:

$$u \le 1; \quad v \le 1; \quad u + v \ge 1$$



.

It is easy to see that the mapping $(t_1, t_2) \mapsto (u, v)$ is a diffeomorphism with a negative Jacobian (we assume that a_{12} is a decreasing function in t, while a_{23} is an increasing function). Therefore after the change of variables the integral becomes:

$$\frac{1}{4\pi^2} \int\limits_{\substack{u\le 1;\ v\le 1;\\ u+v\ge 1}} d\ln u \wedge d\ln v = \frac{1}{4\pi^2} \int_0^1 \left(\int_{1-u}^1 d\ln v \right) \frac{du}{u} = -\frac{1}{4\pi^2} \int_0^1 \ln(1-u) \frac{du}{u} .$$

Taylorising the logarithm we get

$$-\frac{1}{4\pi^2}\sum_{k=1}^{\infty}\int_0^1 \frac{u^k}{k}\frac{du}{u} \;=\; -\frac{1}{4\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2} \;=\; -\frac{1}{4\pi^2}\zeta(2) \;=\; -\frac{1}{24}.$$

Coefficients of other chord diagrams in $Z(H)$ can be expressed through the values of D. Zagier's multivariate generalization of ζ-function:

$$\zeta(a_1,\dots,a_n) \;=\; \sum_{0<k_1<k_2<\cdots<k_n} k_1^{-a_1}\dots k_n^{-a_n}$$

(see [LM1]).

A complete formula for $Z(H)$ was found in [BGRT] (see section 9.3). Up to now, this is the only fully computed Kontsevich integral.

Lecture 8

Properties of the Kontsevich integral

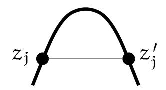
In this lecture we prove the basic properties of the Kontsevich integral announced earlier.

8.1 Convergence of the integral

Proposition 8.1 *For any Morse knot* K*, the Kontsevich integral* $Z(K)$ *is convergent.*

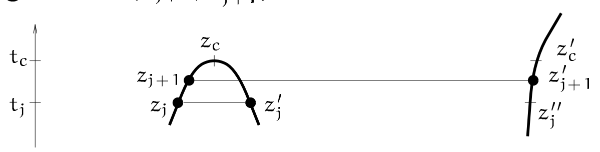
Proof. The integrand of the Kontsevich integral may have singularities near the boundaries of the connected components. This happens only in the following two cases.

1) An isolated chord (z_j, z_j') tends to zero:



In this case the corresponding chord diagram D_P is equal to zero in $\mathcal{A}'$ by the one-term relation.

2) A chord (z_j, z_j') tends to zero near a critical point but is separated from that point by another, 'long' chord (z_{j+1}, z_{j+1}'):



In this case the integral for $(j+1)$-th chord can be estimated as follows:

$$\left| \int_{t_j}^{t_c} \frac{dz_{j+1} - dz_{j+1}'}{z_{j+1} - z_{j+1}'} \right| \leq C \left| \int_{t_j}^{t_c} d(z_{j+1} - z_{j+1}') \right| = C \left| (z_c - z_j) - (z_c' - z_j'') \right| \leq C' |z_j - z_j'|$$

for some positive constants C and C'. So the integral for $(j+1)$-th chord is as small as $z_j - z'_j$ and this smallness compensates the smallness of the denominator corresponding to j-th chord.

This implies the convergence of the Kontsevich integral.

8.2 Invariance of the integral

Theorem 8.1 *The Kontsevich integral is invariant under the deformations in the class of knots all of whose critical points are nondegenerate (but critical values are allowed to coincide).*

A *deformation* is understood as a smooth family of diffeomorphisms of $\mathbb{R}^3$ which starts with the identity.

The proof of this theorem spans the whole of this section. It is based on two propositions (8.3 and 8.4) that we prove below.

Any deformation that keeps a given Morse knot within the class of knots with nondegenerate critical points and hence preserves the number of critical points can be decomposed into *horizontal deformations* and *movements of critical points.*

A *horizontal deformation* is a deformation of $\mathbb{R}^3$ which preserves all horizontal planes $\{t = \text{const}\}$ and leaves all the critical points fixed. The invariance under horizontal deformations is the most essential point of the theory. We prove it in the next subsection.

A *movement of the critical point* C is a deformation which is identical everywhere outside a small neighbourhood of C and does not introduce new critical points on the knot. In subsection 8.2.3 we consider invariance under the movements of critical points.

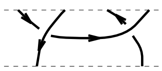
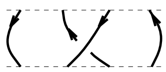
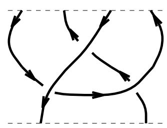
As we mentioned before, the Kontsevich integral is not invariant under deformations that change the number of critical points. Its behaviour under such deformations will be discussed in section 9.1.

8.2.1 Tangles

To prove the horizontal invariance of the Kontsevich integral, we will use tangles and their properties.

Definition 8.1 *A tangle is a one-dimensional compact submanifold of the slice $[a,b] \times \mathbb{C} \subset \mathbb{R}^3$ whose boundary points belong to the boundary of the slice, considered up to an isotopy of the slice, identical on the boundary. We also assume that in the points where the tangle meets the boundary planes (the top and the bottom), it is not tangent to these planes.*

In the case when the top of one tangle T_1 coincides with the bottom of another tangle T_2, one can define the *product* $T_1 \cdot T_2$ by putting T_2 on top of T_1:

$T_1 =$  ; $T_2 =$  ; $T_1 \cdot T_2 =$  .

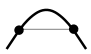
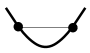
Any Morse knot is cut into tangles by horizontal planes and is almost equal to the product of all such tangles (the critical levels of the function t have to be considered with more attention, because of the tangency to the horizontal plane).

Definition 8.2 *A tangled chord diagram (T-chord diagram) is a tangle T supplied with a set of horizontal chords considered up to a diffeomorphism of the tangle that preserves the horizontal fibration.*

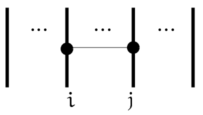
The product of tangles induces a multiplication of tangled chord diagrams.

Definition 8.3 *Let T be a tangle. The space of T-chord diagrams $\mathcal{A}'_T$ is a vector space generated by all chord diagrams on T modulo the set of tangled one- and four-term relations, defined below.*

Tangled one-term relation: A tangled chord diagram with an isolated chord is equal to zero in $\mathcal{A}'_T$:

 $= 0$ or  $= 0$.

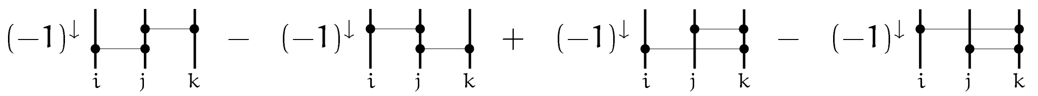
Tangled four-term relation: Denote by t_{ij} the following tangled chord diagram consisting of n parallel vertical strings and one chord between the i-th and j-th strings:

$t_{ij} = (-1)^{\downarrow}$  ,

where $\downarrow$ stands for the number of endpoints of the chords in which the orientation is directed downwards. Then the tangled four-term relation reads:

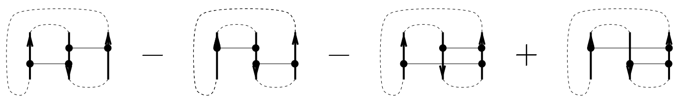
$$[t_{ij} + t_{ik}, t_{jk}] = 0$$

or, in pictures:



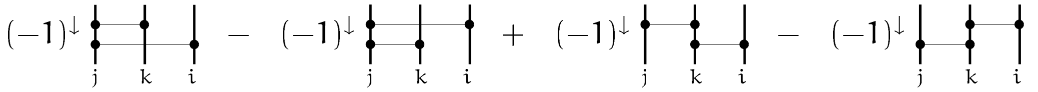
$= 0$. (8.1)

Here is an example:



$= 0$.

Remark. By a permutation of the indices i, j, k we obtain another form of the tangled four-term relation:



$= 0$. (8.2)

Exercise. If a tangle is closed into a circle in agreement with orientations of the strings, the tangled four-term relation carries over into the ordinary four-term relation of Lecture 2.

If (top T_1) $=$ (bottom T_2) then the multiplication of tangled chord diagrams descends to the multiplication of quotient spaces

$$\mathcal{A}'_{T_1} \otimes \mathcal{A}'_{T_2} \to \mathcal{A}'_{T_1 \cdot T_2}.$$

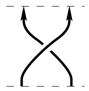
Now we can consider the Kontsevich integral for tangles, taking values in the space of tangled chord diagrams. It is defined by the same formula as the Kontsevich integral for knots (see sec. 7.1), with $t_{\min}$ and $t_{\max}$ corresponding to the bottom and the top of the tangle, respectively.

Proposition 8.2 *The Kontsevich integral for tangles is multiplicative:*

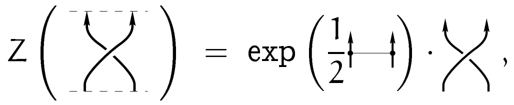
$$Z(T_1) \cdot Z(T_2) = Z(T_1 \cdot T_2)$$

whenever the product $T_1 \cdot T_2$ *is defined.*

This important fact is a direct consequence of the Fubini theorem about multiple integrals. In particular it implies the *vertical multiplicativity* of the Kontsevich integral for Morse knots: if one Morse knot is put on top of the other (with respect to the coordinate t in $\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$), then the Kontsevich integral of their connected sum is equal to the product (in the sense of algebra $\mathcal{A}'$) of Kontsevich integrals of the summands.

Exercise. Compute the Kontsevich integral of the tangle  (this can be done in a straightforward way, directly from the definition).

Answer:



where the last factor means a tangled chord diagram without chords.

8.2.2 Invariance under horizontal deformations

Let us decompose the given knot into a product of tangles without critical points of the function t and (very-very thin) tangles containing the critical levels. Due to multiplicativity, it is enough to prove two facts:

1) the Kontsevich integral for a tangle without critical points is invariant under horizontal deformations that preserve the top and the bottom pointwise,

2) the Kontsevich integral for a tangle with a unique critical point is invariant under horizontal deformations that preserve the critical point.

These two facts are proved in the following two propositions.

Proposition 8.3 *Let* T_0 *be a tangle without critical points and* T_λ*, a horizontal deformation of* T_0 *to* T_1 *(preserving the top and the bottom of the tangle). Then* $Z(T_0) = Z(T_1)$.

Proof. Denote by ω the integrand form in the m-th term of the Kontsevich integral:

$$\omega = \sum_{P=\{(z_j,z_j')\}} (-1)^{\downarrow} D_P \bigwedge_{j=1}^{m} \frac{dz_j - dz_j'}{z_j - z_j'} .$$

Here the functions z_j, z_j' depend not only on t_1, ..., t_m, but also on λ, and *all differentials are understood as complete differentials with respect to all these variables.* This means that the form ω is not exactly the form which appears in the Kontsevich's integral (it has some additional $d\lambda$'s), but this does not change the integrals over the simplices

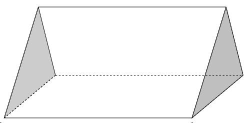
$$\Delta_\lambda = \{t_{min} < t_1 < \cdots < t_m < t_{max}\} \times \{\lambda\},$$

because the value of λ on such a simplex is fixed.

We must prove that the integral of ω over Δ_0 is equal to its integral over Δ_1.

Consider the product polytope

$$\Delta = \Delta_0 \times [0,1] =$$



Δ_0 Δ_1 .

By Stokes' theorem we have $\displaystyle\int_{\partial\Delta} \omega = \int_{\Delta} d\omega$.

The form ω is exact: $d\omega = 0$. The boundary of the integration domain is $\partial\Delta = \Delta_0 - \Delta_1 + \sum\{faces\}$. The theorem will follow from the fact that $\omega|_{\{face\}} = 0$. To show this consider two types of faces.

The first type corresponds to $t_j = t_{min}$ or $t_j = t_{max}$ for some j (of course, this j might be either 1 or m). In this situation $dz_j = dz_j' = 0$ because z_j and z_j' do not depend on λ.

The second type is that where we have $t_k = t_{k+1}$ for some k. In this case we have to choose the k-th and $(k+1)$-th chords on the same level $\{t = t_k\}$. In general, the endpoints of these chords may coincide and we don't get a chord diagram at all. Strictly speaking, we have to define the prolongation of ω and D_P to such a face more carefully. It is natural to prolong D_P to this face as a locally constant function. This means that for the case in which some endpoints of k-th and $(k+1)$-th chords belong to the same string (and therefore coincide) we place k-th chord a little lower than $(k+1)$-th chord, so that its endpoint differs from the endpoint of $(k+1)$-th chord. This trick yields a well-defined prolongation of D_P and ω to the face, and we use it here.

All summands of ω are divided into three parts:

1) k-th and $(k+1)$-th chords connect the same two strings;

2) k-th and $(k+1)$-th chords are chosen in such a way that their endpoints belong to four different strings;

3) k-th and $(k+1)$-th chords are chosen in such a way that there exist exactly three different strings containing their endpoints.

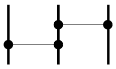
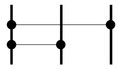
Consider all these cases one by one.

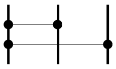
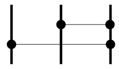
1) We have $z_k = z_{k+1}$ and $z'_k = z'_{k+1}$ or vice versa. So $d(z_k - z'_k) \wedge d(z_{k+1} - z'_{k+1}) = 0$ and therefore the restriction of ω to the face is zero.

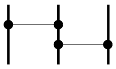
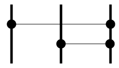
2) All choices of chords in this part of ω appear in mutually cancelling pairs. Fix four strings and number them by 1, 2, 3, 4. Suppose that for a certain choice of the pairing, the k-th chord connects the first two strings and $(k+1)$-th chord connects the last two strings. Then there exists another choice for which on the contrary the k-th chord connects the last two strings and $(k+1)$-th chord connects the first two strings. These two choices give two summands of ω differing by a sign:

$$\cdots d(z_k - z'_k) \wedge d(z_{k+1} - z'_{k+1}) \cdots + \cdots d(z_{k+1} - z'_{k+1}) \wedge d(z_k - z'_k) \cdots = 0.$$

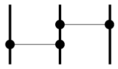
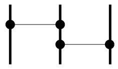
3) This is the most difficult case. The endpoints of k-th and $(k+1)$-th chords have exactly one common string. Let the three strings be 1, 2, 3 and denote by ω_{ij} the form $\dfrac{dz_i - dz_j}{z_i - z_j}$. Then ω is the product of a certain $(m-2)$-form and the sum of the following six 2-forms:

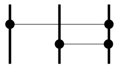
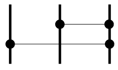
$(-1)^{\downarrow}$  $\omega_{12} \wedge \omega_{23} \quad + \quad (-1)^{\downarrow}$  $\omega_{12} \wedge \omega_{13}$

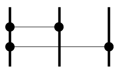
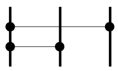
$+(-1)^{\downarrow}$  $\omega_{13} \wedge \omega_{12} \quad + \quad (-1)^{\downarrow}$  $\omega_{13} \wedge \omega_{23}$

$+(-1)^{\downarrow}$  $\omega_{23} \wedge \omega_{12} \quad + \quad (-1)^{\downarrow}$  $\omega_{23} \wedge \omega_{13}$.

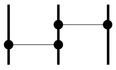
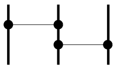
Using the fact that $\omega_{ij} = \omega_{ji}$, we can rewrite this as follows:

$\Big((-1)^{\downarrow}$  $- \quad (-1)^{\downarrow}$  $\Big)\ \omega_{12} \wedge \omega_{23}$

$+\Big((-1)^{\downarrow}$  $- \quad (-1)^{\downarrow}$  $\Big)\ \omega_{23} \wedge \omega_{31}$

$+\Big((-1)^{\downarrow}$  $- \quad (-1)^{\downarrow}$  $\Big)\ \omega_{31} \wedge \omega_{12}$.

The tangled four-term relations (8.1), (8.2) say that the expressions in parentheses are one and the same element of $\mathcal{A}'_T$, hence the whole sum is equal to

$\Big((-1)^{\downarrow}$  $- \quad (-1)^{\downarrow}$  $\Big)\ (\omega_{12} \wedge \omega_{23} + \omega_{23} \wedge \omega_{31} + \omega_{31} \wedge \omega_{12}).$

The 2-form that appears here is actually zero! This remarkable fact, known as *Arnold's identity*, can be put under the following form:

$$f + g + h = 0 \implies \frac{df}{f} \wedge \frac{dg}{g} + \frac{dg}{g} \wedge \frac{dh}{h} + \frac{dh}{h} \wedge \frac{df}{f} = 0$$

(in our case $f = z_1 - z_2$, $g = z_2 - z_3$, $h = z_3 - z_1$) and verified by a direct computation.

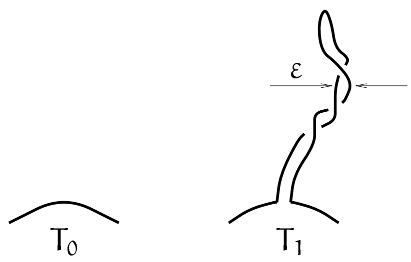
Proposition 8.3 is proved.

Proposition 8.4 *Let T_0 and T_1 be two tangles, each with exactly one critical point, differing from each other by a horizontal deformation preserving the top and the bottom of the tangles and the critical point. Then $Z(T_0) = Z(T_1)$.*

Proof. Assume for simplicity that the values of t at the bottom, the critical point and the top are -1, 0 and 1, respectively. Now consider a special horizontal deformation of T_0 into T_1 via a family of tangles T_λ, $0 \le \lambda \le 1$, such that T_λ coincides with T_0 for $|t| < \lambda^2$ and coincides with T_0 for $|t| > \lambda$. Of course, we require that all T_λ be smooth tangles with only one critical point which does not depend on λ. By proposition 8.3 we have that $Z(T_\lambda) = Z(T_0)$ for any λ between 0 and 1. But the limit of the tangle T_λ, as λ goes to 1, is T_1. This proves proposition 8.4 and the horizontal invariance of the Kontsevich integral (theorem 8.1).

8.2.3 Moving the critical points

Let T_0 and T_1 be two tangles which are identical except a sharp tail of width ε, which may be twisted:



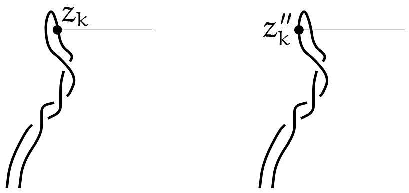
We assume that each level $\{t = \text{const}\}$ intersects the tail at most in two points, distance between which is no greater than ε.

Theorem 8.2 $Z(T_0) = Z(T_1)$.

Proof. The difference between $Z(T_0)$ and $Z(T_1)$ can come only from the terms with a chord ending on the tail.

If the highest of such chords connects the two sides of the tail, then the corresponding tangled chord diagram is zero by a one-term relation.

So we can assume that the highest, k-th, chord is a 'long' chord, which means that it connects the tail with another part of T_1. Suppose the endpoint of the chord belonging to the tail is (z_k, t_k). Then there exists another choice for k-th chord which is almost the same but touches the tail at another point (z''_k, t_k) of the same horizontal level:



The corresponding two terms appear in $Z(T_1)$ with the opposite signs due to $(-1)^{\downarrow}$.

Let us estimate the difference of the integrals corresponding to such k-th chords:

$$\left| \int_{t_{k-1}}^{t_c} d(\ln(z_k - z'_k)) - \int_{t_{k-1}}^{t_c} d(\ln(z''_k - z'_k)) \right| = \left| \ln\left(\frac{z''_{k-1} - z'_{k-1}}{z_{k-1} - z'_{k-1}} \right) \right|$$

$$= \left| \ln\left(1 + \frac{z''_{k-1} - z_{k-1}}{z_{k-1} - z'_{k-1}} \right) \right| \sim |z''_{k-1} - z_{k-1}| \leq \varepsilon$$

(here t_c is the value of t at the uppermost point of the tail).

Now, if the next $(k-1)$-th chord is also long, then, similarly, it can be paired with another long one so that they give a contribution to the integral proportional to $\left|z''_{k-2} - z_{k-2}\right| \leq \varepsilon$.

In the case the $(k-1)$-th chord is short (i.e. connects two points z''_{k-1}, z_{k-1} of the tail) we have the following estimate for the double integral corresponding to k-th and $(k-1)$-th chords:

$$\left| \int_{t_{k-2}}^{t_c} \left(\int_{t_{k-1}}^{t_c} d(\ln(z_k - z'_k)) - \int_{t_{k-1}}^{t_c} d(\ln(z''_k - z'_k)) \right) \frac{dz''_{k-1} - dz_{k-1}}{z''_{k-1} - z_{k-1}} \right|$$

$$\leq \text{const} \cdot \left| \int_{t_{k-2}}^{t_c} |z''_{k-1} - z_{k-1}| \frac{dz''_{k-1} - dz_{k-1}}{\left|z''_{k-1} - z_{k-1}\right|} \right|$$

$$= \text{const} \cdot \left| \int_{t_{k-2}}^{t_c} d(z''_{k-1} - z_{k-1}) \right| \sim |z''_{k-2} - z_{k-2}| \leq \varepsilon.$$

Continuing such considerations, we see that the difference between $Z(T_0)$ and $Z(T_1)$ is $O(\varepsilon)$. Now by horizontal deformations we can make ε tend to zero. This proves the theorem and completes the proof of the Kontsevich integral's invariance in the class of knots with nondegerate critical points.

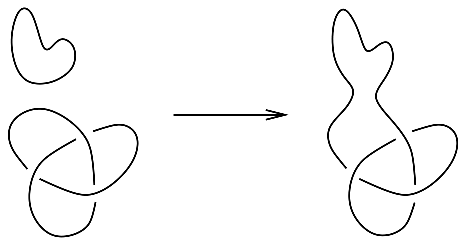
Lecture 9

The universal Vassiliev invariant

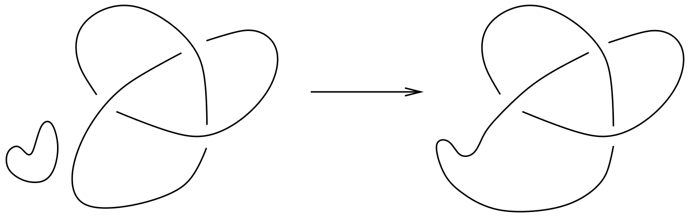
Here we will prove formula 7.1 (page 51) which describes the behaviour of the Kontsevich integral under the addition of a pair of critical points; using this formula, we will construct the universal Vassiliev invariant (aka *final* Kontsevich's integral) and discuss its various properties.

9.1 Horizontal multiplicativity

If two Morse knots are added together *vertically*, like this:

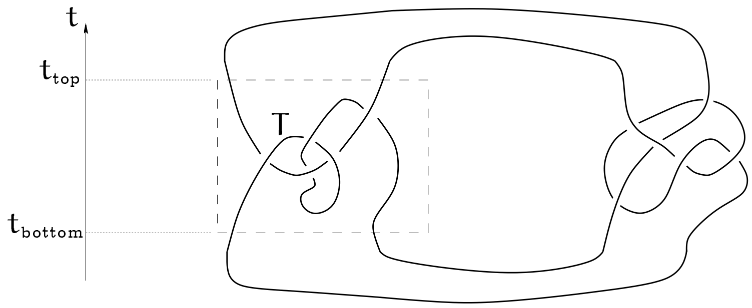


then the Kontsevich integral of the connected sum is the product of the Kontsevich integrals of the summands — simply by the Fubini theorem of standard calculus. To prove formula 7.1, we must be able, however, to add up Morse knots horizontally, like this:



In the proof, we will use the relation between the Kontsevich integral of knots, of tangles and of the complements to tangles.

Suppose we have a Morse knot K with a distinguished tangle T.



Let t_{top} and t_{bottom} be the maximal and minimal values of t on the tangle T. Consider the integral $Z_T(K)$ defined by the same formula as the Kontsevich integral $Z(K)$ but with a reduced number of summands in the integrand for those components of the integration domain whose projection on the coordinate axis t_j is entirely contained in the segment $[t_{bottom}, t_{top}]$. Namely, in $Z_T(K)$ we take into account only those pairings $P = \{(z_j, z'_j)\}$ on the level t_j ($t_{bottom} < t_j < t_{top}$) for which either both points (z_j, t_j) and (z'_j, t_j) belong to the tangle T or neither of them does. This means that we omit the summands of $Z(K)$ which contain a 'long' chord (connecting T with the remaining part of K).

Definition 9.1 *The integral* $Z_T(K)$ *will be referred to as the Kontsevich integral of the knot* K *reduced with respect to the tangle* T.

Lemma 9.1 *Reduction with respect to any tangle does not change the value of the Kontsevich integral:* $Z_T(K) = Z(K)$.

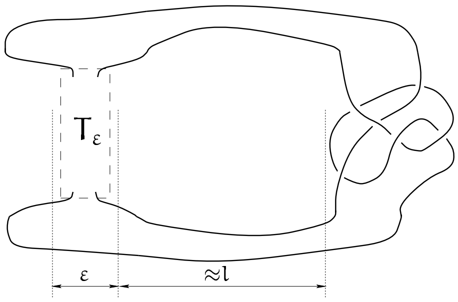
Warning. The Lemma does not say anything about the integrals over a particular component of the integration domain. It only claims the coincidence of the total integrals.

To prove the Lemma, we need the following observation which follows from the same arguments that we used in the proof of the horizontal invariance of the Kontsevich's integral in section 8.2.2:

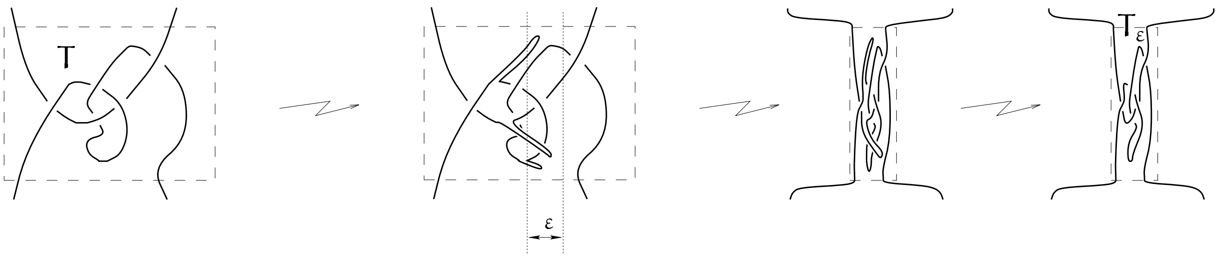
Let (K_s, T_s) *be a one-parameter deformation of the pair* $(K = K_0, T = T_0)$ *of a knot* K *with a distinguished tangle* T *in the class of Morse knots. Then the reduced integral is invariant:* $Z_{T_s}(K_s) = Z_T(K)$.

Proof of the lemma. We must prove that the difference $Z(K) - Z_T(K)$ is equal to zero.

Let us deform the knot K shrinking the tangle T sideways into a narrow rectangle box of width ε:



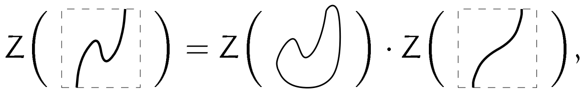
We will prove that the difference $Z(K_\varepsilon) - Z_{T_\varepsilon}(K_\varepsilon)$ tends to zero as $\varepsilon \to 0$. Since both integrals are invariant under the deformation, this immediately implies the lemma. Unfortunately this shrinking cannot be done by horizontal deformations alone because we have to drag critical points into the narrow box. To perform the shrinking, we will first move all critical points of T into the box, using special movements of critical points. Then, by a horizontal deformation, we shrink the whole tangle to the box. After that return the critical points to their initial levels:



Applying a similar procedure to the portion of $K_\varepsilon \backslash T_\varepsilon$ that lies between the horizontal planes $t = t_{\text{bottom}}$ and $t = t_{\text{top}}$, we can squeeze it into a parallelepiped of width ε, too. (*Remark:* The process under study involves moving the critical values, hence it makes no sense to consider the integral over one particular component. This is why the lemma is about the integrals over the whole integration domain.)

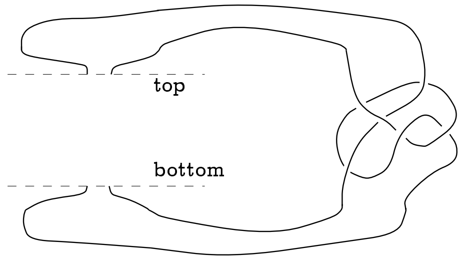
Now the difference $Z(K_\varepsilon) - Z_{T_\varepsilon}(K_\varepsilon)$ consists of summands of $Z(K_\varepsilon)$ with 'long' chords. So it is enough to prove that the integral corresponding to one 'long' chord tends to zero. For a long chord (z_j, z_j') connecting T_ε with the remaining part of K_ε, the corresponding integral is equal to the logarithm of the quotient of the differences $z_j - z_j'$ at the upper and lower limits of integration, which obviously tends to zero as $\varepsilon \to 0$. The lemma is proved.

Theorem 9.1 *We have*



where the first and the third pictures depict an arbitrary knot, differing only in the shown fragment, while the second picture represents an unknot embedded in $\mathbb{R}^3$ *in the specified way and the product is the product in the completed algebra* $\overline{\mathcal{A}}'$ *of chord diagrams.*

To prove the theorem we will extend the construction of the Kontsevich integral for a tangle $Z(T) \in \overline{\mathcal{A}}'_T$ to the complement of the tangle. For example, the complement $K \setminus T$ for the pair K, T shown on p. 64 looks like



We consider chord diagrams on $K \setminus T$ made up of horizontal chords modulo one- and four-term relations which look exactly like tangled ones. The Kontsevich integral $Z(K \setminus T) \in \overline{\mathcal{A}}'_{K\setminus T}$ is defined in the same way as for the tangles.

If T' be a tangle whose top and bottom coincide with top and bottom of $K\setminus T$, then one can obviously define their product $T' \cdot (K \setminus T)$ and the product of chord diagrams on them (*concatenation product*):

$$\overline{\mathcal{A}}'_{T'} \times \overline{\mathcal{A}}'_{T\setminus K} \to \overline{\mathcal{A}}'.$$

By definition, for any tangle T in the knot K

$$Z_T(K) = Z(T) \cdot Z(K \setminus T),$$

and lemma 9.1 implies

Lemma 9.2 *(Tangled multiplication formula for the Konstevich integral)*

$$Z(K) = Z(T) \cdot Z(K \setminus T).$$

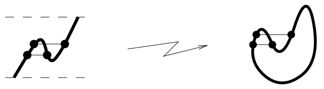
We will investigate a little further the properties of the Kontsevich integral for *one-string* tangles and their complements and then use them to prove the theorem.

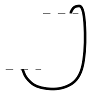
A *one-string tangle* is a connected tangle that has only one string at the top and at the bottom. For such tangles, as well as for their complements, we have two naturally defined *closure* operations:

- For the tangle.
  1. Closure of the tangle $T \mapsto \overline{T}$. The tangle is converted into a knot by adding an arc with two nondegenerate critical points.
  2. Closure of chord diagrams: the mapping $\overline{\mathcal{A}}'_T \to \overline{\mathcal{A}}'$ defined by the identification of the two endpoints of T. Let $\overline{Z}(T) \in \overline{\mathcal{A}}'$ stand for the Kontsevich integral of the tangle closed in this way.
- For the complement.

1. Closure of the complement $K \setminus T \mapsto \overline{K \setminus T}$. The complement is converted into a knot by adding an arc without critical points.
2. Closure of chord diagrams: the mapping $\overline{\mathcal{A}}'_{K\setminus T} \to \overline{\mathcal{A}}'$ defined by the identification of the two endpoints of $K \setminus T$. Let $\overline{Z}(K \setminus T) \in \overline{\mathcal{A}}'$ stand for the Kontsevich integral of the complement closed in this way.

Here is a figure illustrating the closure operation for tangles:

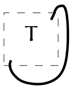
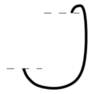
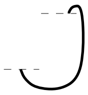
.

As you can see, the closure is carried out by adding the arc .

Lemma 9.3 *For a one-string tangle* T*, the following identities hold:*

1. $\overline{Z}(T) = Z(\overline{T})$.
2. $\overline{Z}(K \setminus T) = Z(\overline{K \setminus T})$.

Proof.

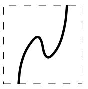
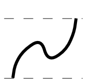
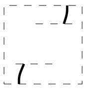
1. Indeed, by Lemma 9.2, $Z(\overline{T}) = Z\Big($  $\Big) = Z(T) \cdot Z\Big($  $\Big)$. But each chord diagram on  is equal to zero by one-term relation except for the diagram of order 0 (without any chords) which closes the chord diagrams on T.
2. The same argument holds for the complements, too, but it is even simpler, because in this case on the additional arc there is only one (empty) chord diagram.

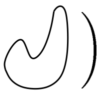
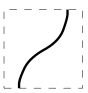
Lemma 9.4 *For a one-string tangle* T *and its complement* $K \setminus T$ *the concatenation product* $\overline{\mathcal{A}}'_T \times \overline{\mathcal{A}}'_{K\setminus T} \to \overline{\mathcal{A}}'$ *agrees with the product in the algebra* $\overline{\mathcal{A}}'$ *in the sense that the following identity holds:*

$$Z(T) \cdot Z(K \setminus T) = \overline{Z}(T) \cdot \overline{Z}(K \setminus T).$$

Proof. This follows directly from the definitions.

Proof of the theorem. By Lemmas 9.3 and 9.4 we have:

$Z\Big($  $\Big) = \overline{Z}\Big($  $\Big) \cdot \overline{Z}\Big($  $\Big)$

$= Z\Big($  $\Big) \cdot Z\Big($  $\Big)$.

The theorem is proved.

Definition 9.2 *The theorem allows to define the universal Vassiliev invariant by the formula*

$$I(K) = \frac{Z(K)}{Z(H)^{c/2}},$$

where c *denotes the number of critical points of* K *in an arbitrary Morse representation, and the quotient means division in the algebra* $\overline{\mathcal{A}}'$*:* $(1+a)^{-1} = 1 - a + a^2 - a^3 + \ldots$.

The invariance of $Z(K)$ in the class of Morse knots and the theorem above imply that $I(K)$ is invariant under an arbitrary deformation of K.

9.2 Proof of the Kontsevich Theorem

First of all we reformulate the Kontsevich theorem (or, more exactly, the Kontsevich part of the Vassiliev–Kontsevich theorem 2.1) as follows.

Theorem 9.2 *Let* w *be a weight system of order* n *(i. e. vanishing on chord diagrams whose number of chords is different from* n*). Then there exists a Vassiliev invariant of order* $\leq n$ *whose symbol is* w*.*

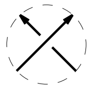
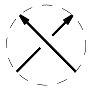
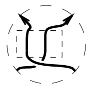
Proof. The desired knot invariant is given by the formula

$$K \longmapsto w(I(K)).$$

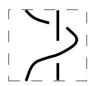
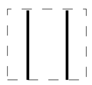
Let D be a chord diagram of order n and K_D be a singular knot with chord diagram D. The theorem follows from the fact that $I(K_D) = D + (\text{terms of order} > n)$. Since the denominator of $I(K)$ always starts with the unit of the algebra $\mathcal{A}'$, it is sufficient to prove that

$$Z(K_D) = D + (\text{terms of order} > n)$$

To see what happens at a single double point let us consider the difference of Z on two knots K_+ and K_- where an overcrossing in K_+ is changed to an undercrossing in K_-. By a horizontal deformation we have:

$Z(K_+) - Z(K_-) = Z\Big($  $\Big) - Z\Big($  $\Big) = Z\Big($  $\Big) - Z\Big($  $\Big)$.

The two knots differ only in that the first knot has one string that makes a small curl around another string. We assume that this curl is very thin and contained in a slice of thickness δ. Enclose the curl in a box of height δ and consider it as a tangle T_+. Let T_- be the similar tangle for K_- consisting of two parallel vertical strings:

$T_+ =$  $\qquad T_- =$  .

By Lemma 9.2 we have

$$Z(K_+) = Z(T_+) \cdot Z(K_+ \setminus T_+) \qquad\qquad Z(K_-) = Z(T_-) \cdot Z(K_- \setminus T_-) \ .$$

But $K_+ \setminus T_+$ coincides with $K_- \setminus T_-$, therefore,

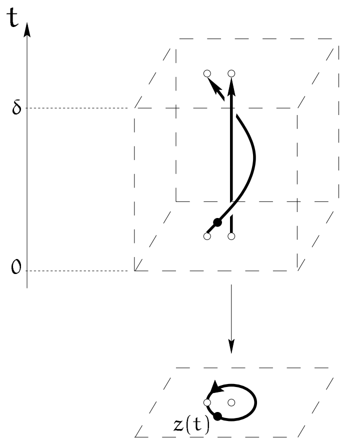
$$Z(K_+) - Z(K_-) = (Z(T_+) - Z(T_-)) \cdot Z(K_+ \setminus T_+) \ .$$

Now, for the tangle T_- both functions $z(t)$ and $z'(t)$ are constant, hence $Z(T_-)$ consists of only one summand: the trivial tangled chord diagram $D_-^{(0)}$ on T_- (here 'trivial' means with an empty set of chords) with coefficient one. The zero degree term $D_+^{(0)}$ of $Z(T_+)$ is also the trivial chord diagram on T_+ with coefficient one. But $D_-^{(0)} = D_+^{(0)}$ as chord diagrams. Thus the difference $Z(T_+) - Z(T_-)$, and therefore the difference $Z(K_+) - Z(K_-)$, starts with a first degree term, which comes from the chord connecting the two strings of T_+.

Now consider the n double points $p_1, \ldots, p_n$ of the knot K_D. Let $\varepsilon_1, \ldots, \varepsilon_n$ be a sequence of signs '+' or '−'. Denote by $K_{\varepsilon_1,\ldots,\varepsilon_n}$ the knot obtained from K_D by a positive (like in K_+ above) resolution of p_j if $\varepsilon_j = +$ or by a negative (like in K_- above) resolution of p_j if $\varepsilon_j = -$. Deforming all knots $K_{\varepsilon_1,\ldots,\varepsilon_n}$ near the points $p_1, \ldots, p_n$ in the same manner as above for K_+ and K_-, we get, for every $j = 1, \ldots, n$, two tangles $T_{j,+}$ and $T_{j,-}$ contained in a box of height δ near the point p_j. The tangle $T_{j,-}$ always consists of two vertical strings. Then we can write

$$\begin{aligned}
Z(K_D) &= \sum_{\varepsilon_1,\ldots,\varepsilon_n} \varepsilon_1 \cdot \ldots \cdot \varepsilon_n \cdot Z(K_{\varepsilon_1,\ldots,\varepsilon_n}) \\
&= \sum_{\varepsilon_1,\ldots,\varepsilon_{n-1}} \varepsilon_1 \cdot \ldots \cdot \varepsilon_{n-1} \cdot (Z(T_{n,+}) - Z(T_{n,-})) \cdot Z(K_{\varepsilon_1,\ldots,\varepsilon_{n-1},+} \setminus T_{n,+}) \\
&= \sum_{\varepsilon_1,\ldots,\varepsilon_{n-2}} \varepsilon_1 \cdot \ldots \cdot \varepsilon_{n-2} \cdot (Z(T_{n,+}) - Z(T_{n,-})) \cdot (Z(T_{n-1,+}) - Z(T_{n-1,-})) \\
&\qquad\qquad \cdot Z(K_{\varepsilon_1,\ldots,\varepsilon_{n-2},+,+} \setminus (T_{n,+} \cup T_{n-1,+})) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&= (Z(T_{1,+}) - Z(T_{1,-})) \cdot \ldots \cdot (Z(T_{n,+}) - Z(T_{n,-})) \cdot Z(K_{+,\ldots,+} \setminus (\bigcup_j T_{j,+})) \ .
\end{aligned}$$

As above, each difference $Z(T_{j,+}) - Z(T_{j,-})$ starts with a term of degree 1. Hence the product $Z(K_D)$ starts from degree n. Moreover, the term of degree n is proportional to a chord diagram whose j-th chord connects the two strings of the tangle $T_{j,+}$. Since the two strings of $T_{j,+}$ correspond to the double point p_j in K_D, this chord diagram is precisely D. Now to calculate the order n part of $Z(K_D)$ we must compute the coefficient of D. It is equal to the product of coefficients of one chord terms in $Z(T_{j,+})$. We will show that all of them are equal to 1. Indeed, it is possible to choose the coordinates z and t is such a way that $z'(t) \equiv 0$ and when t varies from 0 to δ the point $z(t)$ makes one complete turn around zero:



So we have

$$\frac{1}{2\pi i}\int_0^{\delta}\frac{dz - dz'}{z - z'} = \frac{1}{2\pi i}\oint_{|z|=1}\frac{dz}{z} = 1$$

by the Cauchy theorem. The Kontsevich theorem is proved.

We will now state some properties of the universal Vassiliev invariant $I(K)$ which reveal the precise meaning of the word "universal". The first of these properties follows from the argument used in the proof of the Kontsevich's theorem above.

Proposition 9.1 *If* $I_n(K)$ *denotes the* n*-th graded component of the series* $I(K)$*, then the function* $K \mapsto I_n(K)$ *is a Vassiliev invariant of order* n*.*

The converse assertion is contained in the following

Proposition 9.2 *Any Vassiliev invariant can be factored through* I*: for any* $v \in \mathcal{V}$ *there exists a function* f *on* $\overline{\mathcal{A}}'$ *such that* $v = f \circ I$*.*

Proof. Let $v \in \mathcal{V}_n$. By Kontsevich's theorem we know that there is a function f_0 such that v and $f_0 \circ I_n$ have the same symbol. Therefore, the difference $v - f_0 \circ I_n$ belongs to $\mathcal{V}_{n-1}$ and is thus representable as $f_1 \circ I_{n-1}$. Proceeding in this way, we will finally obtain:

$$v = \sum_{i=1}^{n} f_i \circ I_{n-i}.$$

Remark. The construction of the previous proof shows that the universal Vassiliev invariant induces a splitting of the filtred space $\mathcal{V}$ into a direct sum with summands isomorphic to the factors $\mathcal{V}_n/\mathcal{V}_{n-1}$.

The two last propositions, taken together, imply the following theorem.

Theorem 9.3 *The universal Vassiliev invariant* I *is exactly as strong as the set of all Vassiliev invariants: for any two knots* K_1 *and* K_2 *we have*

$$I(K_1) = I(K_2) \quad \Longleftrightarrow \quad \forall v \in \mathcal{V} \quad v(K_1) = v(K_2).$$

9.3 Framed version of the Kontsevich integral

For framed knots (and links), the Kontsevich's integral was defined in [LM2]. We do not give the details of their construction, only noting that it is equivalent to the following formula from [BGRT]:

$$\widetilde{I}(K) = e^{\frac{w(K)}{2}\Theta} \cdot I(\overline{K}) \in \overline{\mathcal{A}},$$

where K is a framed knot, $\overline{K}$ is the same knot without framing, $w(K)$ is the writhe of K (the linking number of K and a copy of K shifted slightly in the direction of the framing), Θ is the chord diagram with one chord, and the exponent means the usual power series in the algebra $\overline{\mathcal{A}}$. The universal Vassiliev invariant $I(\overline{K}) \in \overline{\mathcal{A}}'$ can be understood as an element of the completed algebra $\overline{\mathcal{A}}$ (without 1-term relations), because there is a natural inclusion $\mathcal{A}' \to \mathcal{A}$ defined as identity on the primitive subspace of $\mathcal{A}'$.

In [BGRT] the following formula for the framed Kontsevich integral of the unknot was found. Let O be the unknot with zero framing.

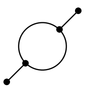
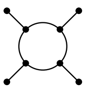
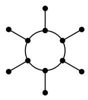
Theorem 9.4

$$\widetilde{I}(O) = \exp \sum_{n=1}^{\infty} b_{2n} w_{2n} = 1 + (\sum_{n=1}^{\infty} b_{2n} w_{2n}) + \frac{1}{2}(\sum_{n=1}^{\infty} b_{2n} w_{2n})^2 + \dots \qquad (9.1)$$

Here b_{2n} are modified Bernoulli numbers, i.e. the coefficients of the Taylor series

$$\sum_{n=0}^{\infty} b_{2n} x^{2n} = \frac{1}{2} \ln \frac{e^{x/2} - e^{-x/2}}{x},$$

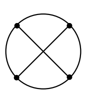
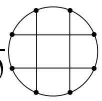
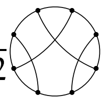
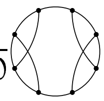
($b_2 = 1/48$, $b_4 = -1/5760$, $b_6 = 1/362880$, ...), and w_{2n} are the *wheels*, i. e. Chinese characters of the form

$w_2 =$  , $\quad w_4 =$  , $\quad w_6 =$  , ...

The sums and products are understood as operations in the algebra of Chinese characters $\mathcal{B}$, which is isomorphic to the algebra of chord diagrams $\mathcal{A}$.

Exercise. Prove that, after being carried over from $\mathcal{B}$ to $\mathcal{A}$, the right hand part of Equation 9.1 belongs in fact to the subalgebra $\mathcal{A}' \subset \mathcal{A}$. Find an explicit expression of the series through some basis of $\mathcal{A}'$ up to degree 2.

Answer. The infinite series giving the Kontsevich integral of the unknot, begins as follows:

$I(O) = 1 - \frac{1}{24}$  $- \frac{1}{5760}$  $+ \frac{1}{1152}$  $+ \frac{1}{2880}$  $+ \dots$

Note that the coefficients of this series are rational. This is a manifestation of the general rationality theorem due to Le and Murakami [LM1].

9.4 Operations on knots and Kontsevich's integral

We saw in Sec. 8.2.2 that the Kontsevich integral is multiplicative with respect to the connected sum of knots. In this section we consider three other operations on knots and describe the effect of these operations on the Kontsevich integral. The operations in question are:

- R — mirror reflection (changing the orientation of the ambient space $\mathbb{R}^3$),
- S — changing the orientation of a knot,
- M — mutation of a knot with respect to a distinguished tangle T.

The effect of the operation R is closely related to the fact that the Kontsevich integral is real, proved in Sec. 9.4.1 below. Roughly speaking, this result means that odd order Vassiliev invariants detect the amphicheirality of knots, while even order invariants do not.

The operation S allows to reformulate the invertibility problem for knots in combinatorial terms, as a property of chord diagrams (Sec. 9.4.3).

Using the nice behaviour of the Kontsevich integral under mutations of knots, we describe (Sec. 9.4.4) a counterexample to the intersection graph conjecture from [CDL1] belonging to T. Q. T. Le [Le].

Detailed proofs of the results described here can be found in [CD3].

9.4.1 Reality of the integral

Choose a countable family of chord diagrams $\{D_i\}$ that constitute a basis in the linear space $\mathcal{A}'$. The Kontsevich integral of a knot K can be written as an infinite series $Z(K) = \sum c_i D_i$ whose coefficients c_i depend on the knot K. A priori these coefficients are complex numbers.

Theorem 9.5 *All coefficients c_i of the Kontsevich integral $Z(K) = \sum c_i D_i$ are real numbers.*

Remark. Of course, this fact is a consequence of the Le–Murakami theorem [LM1] that these coefficients are rational. We stated the previous theorem for two reasons: (1) it has a simple independent proof (see [CD3]) and (2) it is used below in relation to the mirror symmetry of the Kontsevich integral.

9.4.2 Mirror symmetry

Let R be the operation sending a knot to its mirror image. Define the corresponding operation R on chord diagrams to be identical on the diagrams of even order and to be multiplication by (-1) on the diagrams of odd order.

Theorem 9.6 *The Kontsevich integral commutes with the operation R:*

$$Z(R(K)) = R(Z(K)) \ ,$$

where by $\mathsf{R}(\mathsf{Z}(\mathsf{K}))$ *we mean simultaneous application of* R *to all the chord diagrams participating in* $\mathsf{Z}(\mathsf{K})$.

A knot K is called *amphicheiral*, if it is equivalent to its mirror image: $\mathsf{K} = \mathsf{R}(\mathsf{K})$.

Corollary. The Kontsevich integral $\mathsf{Z}(\mathsf{K})$ and the universal Vassiliev invariant $I(\mathsf{K})$ of an amphicheiral knot K consist only of even order terms.

Proof. Indeed, the theorem implies that $\mathsf{Z}(\mathsf{K}) = \mathsf{R}(\mathsf{Z}(\mathsf{K}))$, hence all the odd order terms in the series $\mathsf{Z}(\mathsf{K})$ vanish. The quotient of two even series in the graded completion $\overline{\mathcal{A}}'$ is obviously even, therefore the same property holds for $I(\mathsf{K}) = \mathsf{Z}(\mathsf{K})/\mathsf{Z}(\mathsf{H})^{c/2}$, too.

9.4.3 Orientations of knots

Let S be the operation on knots which inverts their orientation. The same letter will also denote the analogous operation on chord diagrams (inverting the orientation of the outer circle or, which is the same thing, axial symmetry of the diagram).

Theorem 9.7 *The Kontsevich integral commutes with the operation* S *of inverting the orientation:*

$$\mathsf{Z}(\mathsf{S}(\mathsf{K})) = \mathsf{S}(\mathsf{Z}(\mathsf{K})).$$

Corollary. Vassiliev invariants do not distinguish the orientation of knots if and only if all chord diagrams are symmetric $\mathsf{D} = \mathsf{S}(\mathsf{D})$ modulo one- and four-term relations.

The calculations of [Kn0] show that up to order 12 all chord diagrams are symmetric. For bigger orders the problem is still open.

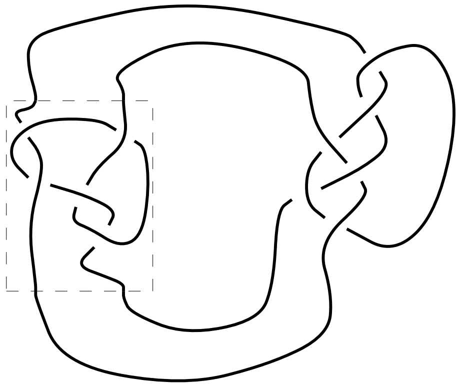
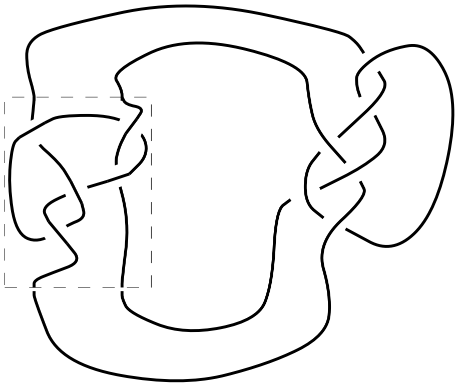
9.4.4 Mutation

The purpose of this subsection is to show that the Kontsevich integral commutes with the operation of mutation (this fact was first noticed by T. Le). As an application, we will construct a counterexample to the intersection graph conjecture (p. 31). Finally we will formulate a conjecture about the combinatorial description of Vassiliev invariants that do not distinguish mutant knots.

Mutation of knots

Definition 9.3 *Suppose we have a knot* K *with a distinguished tangle* T *which has two strings at the bottom and two strings at the top. Let us cut out the tangle, rotate it through* $180°$ *around a vertical axis and insert it back. This operation* M *is called mutation and the obtained knot* $\mathsf{M}(\mathsf{K})$ *is called a mutant of* K.

Here is a widely known pair of mutant knots, the *Conway* and *Kinoshita–Teresaka* knots:

$C =$  $KT =$ 

Theorem 9.8 *[MC]. There exists a Vassiliev invariant* ν *of order 11 such that* $\nu(C) \neq \nu(KT)$.

Morton and Cromwell produced the invariant ν using the Lie algebra $\mathfrak{gl}_N$ with a nonstandard representation.

Mutation of the Kontsevich integral

Let K be a knot with a distinguished tangle T which has two strings at the bottom and two strings at the top. Let $M(K)$ be the mutant of K obtained by the rotation of T. We shall define the mutation $M(Z(K))$ of the Kontsevich integral as the simultaneous mutation of all chord diagram participating in $Z(K)$. Then the following theorem holds.

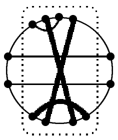
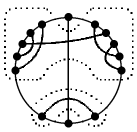
Theorem 9.9 *[Le]. The Kontsevich integral commutes with the mutation operation* M*:*

$$Z(M(K)) = M(Z(K)).$$

By lemma 9.1 $Z(K)$ does not contain any chord diagram with a chord connecting T with the remaining part of K. This means that all chord diagrams which appear in $Z(K)$ are products (in the sense of section 9.1) of tangled chord diagrams on T and chord diagrams on $K \setminus T$. A chord diagram on the tangle, considered as a part of the whole chord diagram for the knot, will be called a *share*. In purely combinatorial terms, this definition can be stated as follows.

Definition 9.4 *A share is a part of the chord diagram consisting of two arcs of the outer circle and all chords whose endpoints belong to these arcs. Moreover, we require that the whole chord diagram contains no chords that connect these two arcs with the remaining part of the circle.*

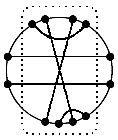
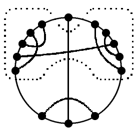
Examples. In the left picture below the collection of four chords inside the dotted oval is a share. But the collection of three fat chords is not a share because there exists a fourth chord that separates their ends. In the right picture there are two shares shown by dotted arcs.

The choice of a tangle T in a knot K distinguishes the T-shares (which correspond to the tangled chord diagrams of $Z(T)$) in all chord diagrams that appear in $Z(K)$.

Definition 9.5 *The mutation* $M(Z(K))$ *of* $Z(K)$ *is the simultaneous rotation of the* T*-shares in all diagrams of* $Z(K)$.

Mutants of the chord diagrams of last two examples look as follows

Counterexample to the Intersection Graph Conjecture

It is easy to see that the mutation of chord diagrams does not change the intersection graph. So, if the intersection graph conjecture (see 4.3) were true, the Kontsevich integrals of mutant knots would coincide. Hence all Vassiliev invariants would take the same value on mutant knots. But this contradicts Theorem 9.8. Therefore the intersection graph conjecture is false.

Lecture 10

Lie algebra weight systems

Here we explain the construction due to M. Kontsevich [Ko1], but with basic ideas already appearing in [Pen]. Given a Lie algebra with some additional structure, one can define a weight system with values in the universal enveloping algebra.

10.1 Lie algebra weight systems for the algebra $\mathcal{A}$

Suppose that $\mathfrak{g}$ is a finite-dimensional Lie algebra over $\mathbb{C}$ equipped with an ad-invariant symmetric non-degenerate bilinear form $\langle\cdot,\cdot\rangle : \mathfrak{g}\otimes\mathfrak{g}\to\mathbb{C}$. The ad-invariance means that the correspoding element of $\mathfrak{g}^*\otimes\mathfrak{g}^*$ is preserved by the adjoint action of the Lie algebra $\mathfrak{g}$, which is equivalent to the identity

$$\langle [x,y],z\rangle = \langle x,[y,z]\rangle \tag{10.1}$$

for all $x,y,z\in\mathfrak{g}$. Lie algebras with such a form are referred to as *metrized Lie algebras.* This class contains semi-simple algebras (because their Killing form is non-degenerate and ad-invariant). It also contains Abelian Lie algebras (because in this case any bilinear form is ad-invariant).

We will need the following lemma.

Lemma 10.1 *Let c_{ijk} be the structure constants of a metrized Lie algebra in a basis $\{e_i\}$, orthonormal with respect to the ad-invariant bilinear form $\langle\cdot,\cdot\rangle$:*

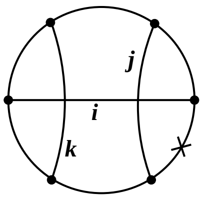
$$\langle e_i,e_j\rangle = \delta_{ij},$$
$$[e_j,e_j] = \sum_{k=1}^{m} c_{ijk}e_k.$$

Then the constants c_{ijk} are antisymmetric with respect to the permutations of the indices i, j and k.

Proof. Equality $c_{ijk}=-c_{jik}$ is the coordinate expression of the fact that the commutator is anisymmetric: $[x,y]=-[y,x]$. It remains to prove that $c_{ijk}=-c_{ikj}$. This follows immediately from equation 10.1, if we put $x=e_i$, $y=e_j$, $z=e_k$. $\square$

Kontsevich's construction proceeds as follows. Choose a basis e_i, $i = 1, ..., m$, of the Lie algebra $\mathfrak{g}$ and denote by e_i^* the dual basis with respect to the bilinear form $\langle \cdot, \cdot \rangle$. Given a chord diagram D with n chords, assign to each chord an independent integer variable (supposed to range from 1 to m, the dimension of the Lie algebra). Now, choose a base point on the circle of the diagram D, different from the chord endpoints, then, starting from this point and going in the positive direction, write out an indexed monomial by the following rule: if you meet an endpoint of a chord for the first time, write e_i, if you meet the second endpoint of the same chord, write e_i^*, where i is the index assigned to that chord. Then take the sum of m^n terms like this, making every index run from 1 to m independently. Denote by $\varphi_{\mathfrak{g}}(D)$ the resulting element of the universal enveloping algebra $U(\mathfrak{g})$. indexAlgebra!universal enveloping

For example, if

$D =$  ,

then

$$\varphi_{\mathfrak{g}}(D) = \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} e_i e_j e_k e_i^* e_k^* e_j^*.$$

Theorem 10.1 *The above construction has the following properties:*

1. *the element $\varphi_{\mathfrak{g}}(D)$ does not depend on the choice of the base point on the diagram,*
2. *it does not depend on the choice of the basis $\{e_i\}$ of the Lie algebra,*
3. *it belongs to the centre $ZU(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$,*
4. *the function $D \mapsto \varphi_{\mathfrak{g}}(D)$ satisfies 4-term relations,*
5. *the resulting mapping $\varphi : \mathcal{A} \to ZU(\mathfrak{g})$ is a homomorphism of algebras (but not of coalgebras).*

Proof.

(1) This property is closely related, and proved similarly to the fact that $\mathcal{A}$, the algebra of chord diagrams on the circle, is isomorphic to the algebra of chord diagrams on the line $\mathcal{A}^l$, and the isomorphism does not depend on the place of the cut. The proof uses property (4), which makes sense, because the diagrams that appear in a 4T relation, admit a coherent labelling of their chords (although there is no such coherent labelling for all diagrams in CD_n).

(2) Proved by a direct computation involving two different bases $\{e_i\}$ and $\{f_j\}$ of $\mathfrak{g}$.

(3) It is enough to prove that $\varphi_{\mathfrak{g}}(D)$ commutes with any basic element e_r. By property (2), we can choose the basis to be orthonormal with respect to the ad-invariant form $\langle \cdot, \cdot \rangle$, so that $e_i^* = e_i$ for all i. Now, the commutator of e_r and $\varphi_{\mathfrak{g}}(D)$ can be

expanded into the sum of $2n$ expressions, similar to $\varphi_{\mathfrak{g}}(D)$, only with one of the e_i's replaced by the corresponding commutator with e_r. Due to antisymmetry of the structure constants c_{ijk} (Lemma 10.1), these expressions cancel in pairs that correspond to the ends of each chord.

To take a concrete example,

$$
\begin{aligned}
&[e_r, \sum_{ij} e_i e_j e_i e_j] \\
&= \sum_{ij} [e_r, e_i] e_j e_i e_j + \sum_{ij} e_i [e_r, e_j] e_i e_j + \sum_{ij} e_i e_j [e_r, e_i] e_j + \sum_{ij} e_i e_j e_i [e_r, e_j] \\
&= \sum_{ijk} c_{rik} e_k e_j e_i e_j + \sum_{ijk} c_{rjk} e_i e_k e_i e_j + \sum_{ijk} c_{rik} e_i e_j e_k e_j + \sum_{ijk} c_{rjk} e_i e_j e_i e_k \\
&= \sum_{ijk} c_{rik} e_k e_j e_i e_j + \sum_{ijk} c_{rjk} e_i e_k e_i e_j + \sum_{ijk} c_{rki} e_k e_j e_i e_j + \sum_{ijk} c_{rkj} e_i e_k e_i e_j .
\end{aligned}
$$

Here the first and the second sums cancel with the third and the fourth sums, respectively.

(4) Any of the 3 pairwise differences of chord diagrams that constitute the 4 term relation as in equation 2.3 (page 17) goes to

$$\sum c_{ijk} \dots e_i \dots e_j \dots e_k \dots$$

where c_{ijk} are structure constants of the algebra $\mathfrak{g}$.

(5) If the base point in the product diagram $D_1 \cdot D_2$ is chosen between D_1 and D_2, then the identity $\varphi_{\mathfrak{g}}(D_1 \cdot D_2) = \varphi_{\mathfrak{g}}(D_1)\varphi_{\mathfrak{g}}(D_2)$ becomes evident. $\square$

The centre $ZU(\mathfrak{g})$ of the universal enveloping algebra can also be denoted by $U^{\mathfrak{g}}(\mathfrak{g})$ (meaning the $\mathfrak{g}$-invariant part of $U(\mathfrak{g})$), because an element of $U(\mathfrak{g})$ that commutes with all elements of $\mathfrak{g}$ belongs to the centre.

10.2 Lie algebra weight systems for the algebra $\mathcal{B}$

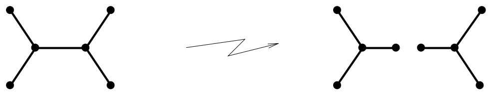
In this section for a metrized Lie algebra $\mathfrak{g}$ we construct a weight system $\psi : \mathcal{B} \to S(\mathfrak{g})$, defined on the space of Chinese characters $\mathcal{B}$ and taking values in the complete symmetric algebra of the vector space $\mathfrak{g}$ (in fact, even in its $\mathfrak{g}$-invariant subspace $S^{\mathfrak{g}}(\mathfrak{g})$).

Let $C \in CC$ be a Chinese character. Denote by U the set of its legs (univalent vertices). First, we will construct a tensor

$$T_{\mathfrak{g}}(C) \in \bigotimes_{\nu \in U} \mathfrak{g},$$

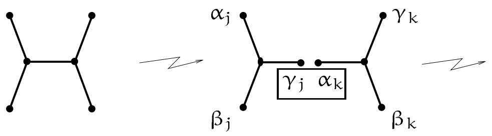
where the symbol $\bigotimes_{\nu\in U} \mathfrak{g}$ means the tensor product whose factors correspond to the elements of the set U: it is defined like the usual tensor product, only instead of linearly ordered arrays $(g_1, ..., g_m)$, $g_i \in \mathfrak{g}$ we must use mappings from U to $\mathfrak{g}$.

The construction proceeds as follows. Cut all the edges between the trivalent vertices of C. This splits C into a union of elementary pieces (tripods), each consisting of one trivalent vertex and three univalent vertices. Here is an example:



With each elementary piece we associate a copy of the tensor $J \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ whose factors correspond to the univalent vertices in agreement with the cyclic ordering of the edges. This tensor is defined as follows. Consider the Lie bracket $[\cdot,\cdot]$ as an element of $\mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$. Identification of $\mathfrak{g}^*$ and $\mathfrak{g}$ via $\langle \cdot,\cdot \rangle$ provides the tensor $J \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$. Due to the properties of $[\cdot,\cdot]$ and $\langle \cdot,\cdot \rangle$ this tensor J is skew-symmetric under the permutations of the three tensor factors which ensures that the previous construction makes sense.

The tensor $T_{\mathfrak{g}}(C)$ corresponding to C is combined from these elementary ones in the following way. Let us restore C from the tripod pieces. Each time two univalent vertices are put together, we contract the two tensor factors corresponding to these vertices by taking the form $\langle \cdot,\cdot \rangle$ on the factors. For example, if $J = \sum \alpha_j \otimes \beta_j \otimes \gamma_j$, then the tensor we relate to the union of two pieces will be:



$$\sum_{i,j,k} \langle \gamma_j, \alpha_k \rangle \alpha_j \otimes \beta_j \otimes \beta_k \otimes \gamma_k.$$

Gluing and contracting in such a way over all the edges previously cut we obtain an element of the tensor product of several copies of $\mathfrak{g}$ whose factors correspond to the univalent vertices of C. This is $T_{\mathfrak{g}}(C)$.

A numbering ν of the set of univalent vertices U yields an isomorphism between $\bigotimes_{v \in U} \mathfrak{g}$ and the conventional tensor power $\bigotimes_{i=1}^{m} \mathfrak{g} = \mathfrak{g}^{\otimes m}$ (with linear order of the factors). This isomorphism takes $T_{\mathfrak{g}}(C)$ into $T_{\mathfrak{g}}^{\nu}(C)$. The average over all numberings ν is the element of the symmetric power that we wanted to construct:

$$\psi_{\mathfrak{g}}(C) = \frac{1}{m!} \sum_{\nu \in S_m} T_{\mathfrak{g}}^{\nu}(C) \in S^m(\mathfrak{g}).$$

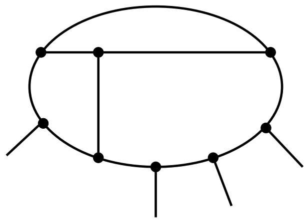
Algorithmically, this step means that we forget the symbols of the tensor product and view the result as an ordinary commutative polynomial in the variables that correspond to a basis of $\mathfrak{g}$.

Example. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{so}_3$. It has a basis $\{a, b, c\}$ which is orthonormal with respect to the Killing form $\langle \cdot,\cdot \rangle$ and with the commutators $[a, b] = c$, $[b, c] = a$, $[c, a] = b$. Therefore, the tensor that we put in every 3-valent vertex in the previous construction, in this case is

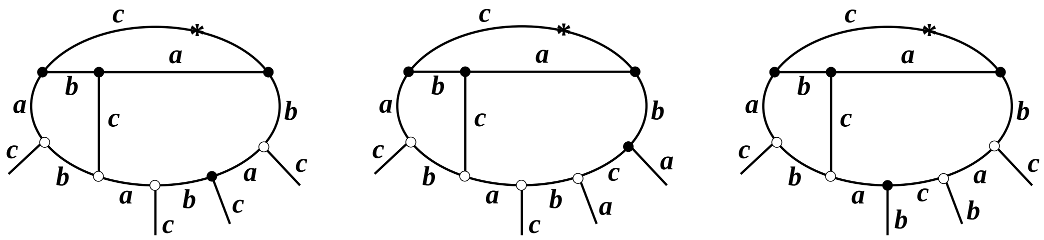
$$\begin{aligned} J &= a \wedge b \wedge c \\ &= a \otimes b \otimes c + b \otimes c \otimes a + c \otimes a \otimes b - b \otimes a \otimes c - c \otimes b \otimes a - a \otimes c \otimes b. \end{aligned}$$

Since the basis is orthonormal, the only way to get a non-zero element in the process of contraction along the edges is to choose the same basic element on either end of the edge. On the other hand, the formula for J shows that in every vertex we must choose a summand with different basic elements corresponding to the 3 edges. This leads to the following algorithm for finding the tensor $T_{\mathfrak{so}_3}(C)$: one must list all 3-colourings of the edges of the graph by 3 colours a, b, c such that the 3 colours in every vertex are always different, then sum the tensor products of the elements written on the legs with the sign $(-1)^s$, where s is the number of *negative* vertices (i. e. vertices where the colours, read counterclockwise, come in the order a, c, b).

For example, consider the diagram (*Pont-Neuf diagram with parameters* $(1,3)$ in the terminology of O. Dasbach [Da3]):

$C =$ 

It has 18 edge 3-colourings, which can be obtained from the following three by permutations of (a,b,c):



In these pictures, negative vertices are marked by small empty circles. Writing the tensors in the counterclockwise order, starting from the marked point, we get:

$$\begin{aligned}
&2(a\otimes a\otimes a\otimes a+b\otimes b\otimes b\otimes b+c\otimes c\otimes c\otimes c)\\
&+a\otimes b\otimes b\otimes a+a\otimes c\otimes c\otimes a+b\otimes a\otimes a\otimes b\\
&+b\otimes c\otimes c\otimes b+c\otimes a\otimes a\otimes c+c\otimes b\otimes b\otimes c\\
&+a\otimes a\otimes b\otimes b+a\otimes a\otimes c\otimes c+b\otimes b\otimes a\otimes a\\
&+b\otimes b\otimes c\otimes c+c\otimes c\otimes a\otimes a+c\otimes c\otimes b\otimes b.
\end{aligned}$$

This tensor is not symmetric with respect to permutations of the 4 tensor factors (although — we note this in passing — it is symmetric with respect to the permutations of the 3 letters a, b, c). Symmetrizing, we get:

$$\psi_{\mathfrak{so}_3}(C) = 2(a^2+b^2+c^2)^2.$$

10.3 The $\mathfrak{gl}_N$ weight system for the algebra $\mathcal{B}$

In this section we will use the PBW (Poincaré–Birkhoff–Witt) isomorphism $S^n(\mathfrak{g}) \simeq U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})$ and often speak about the weigth system $\psi_{\mathfrak{g}}$ as taking values in the graded universal enveloping algebra rather than the symmetric algebra of the algebra $\mathfrak{g}$.

Consider the Lie algebra $\mathfrak{g} = \mathfrak{gl}_N$ of $N \times N$ matrices. Fix the trace of the product of matrices as the ad-invariant form: $\langle x, y\rangle = \mathrm{Tr}(xy)$.

The algebra $\mathfrak{gl}_N$ is generated by matrices e_{ij} with 1 on the intersection of i-th row with j-th column and zero elsewhere. We have $\langle e_{ij}, e_{kl}\rangle = \delta_i^l \delta_j^k$, where δ is the Kronecker delta. Therefore, duality between $\mathfrak{gl}_N$ and $(\mathfrak{gl}_N)^*$ defined by $\langle \cdot, \cdot \rangle$ is given by the formula $e_{ij}^* = e_{ji}$.

One can check that $[e_{ij}, e_{kl}] \neq 0$ only in the following cases:

- $[e_{ij}, e_{jk}] = e_{ik}$, if $i \neq k$,
- $[e_{ij}, e_{ki}] = -e_{kj}$, if $j \neq k$,
- $[e_{ij}, e_{ji}] = e_{ii} - e_{jj}$, if $i \neq j$,

This gives the following formula for the Lie bracket:

$$[\cdot, \cdot] = \sum_{i,j,k=1}^{N} (e_{ij}^* \otimes e_{jk}^* \otimes e_{ik} - e_{ij}^* \otimes e_{ki}^* \otimes e_{kj}).$$

Using the identification of $(\mathfrak{gl}_N)^*$ with $\mathfrak{gl}_N$, we obtain

$$J = \sum_{i,j,k=1}^{N} (e_{ji} \otimes e_{kj} \otimes e_{ik} - e_{ji} \otimes e_{ik} \otimes e_{kj}).$$

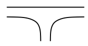
For the algebra $\mathfrak{gl}_N$ it is more convenient to use the tensor

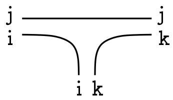
$$-J = \sum_{i,j,k=1}^{N} e_{ji} \otimes e_{ik} \otimes e_{kj}) - (e_{ji} \otimes e_{kj} \otimes e_{ik}$$

instead of J.

Graphical notation for tensors. Here we introduce a graphical notation for a special class of the elements of $(\mathfrak{gl}_N)^{\otimes n}$ and $U(\mathfrak{gl}_N)$ similar to that invented by R. Penrose in [Pen] (see also [BN1]). Consider a plane diagram (T-*diagram*) that consists of n pairs of mutually close points arranged along a circle and connected pairwise by n lines. Strictly speaking, a T-diagram is a set X of cardinality $2n$ endowed with three structures: a cyclic ordering and two involutions without fixed points (or, in other words, two partitions of X into n two-point subsets). The first partition should be consistent with the cyclic order in the sense that the two points of each pair must be neighbours with respect to that order. In the pictures below, X is the set of all protruding endpoints of the lines which indicate the second partition, while the pairs of the first partition are the endpoints that are drawn close to each other. The cyclic order is the one defined by going counterclockwise along an imagined circle that encompasses the whole picture.

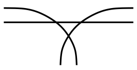
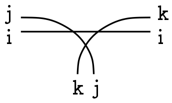
Given a T-diagram, we can write down an element of n-th tensor power of $\mathfrak{gl}_N$ in the following way. We put the same index at either end of each line, and then, travelling around the encompassing circle, write a factor e_{ij} each time we encounter a pair of neighbouring points, the first of which is assigned the index i and the second, the index j. Then, we consider the sum of all such tensors when all the indices range from 1 to N. In a sense, this procedure gives a graphical substitute for the formal Einstein summation rule in multi-index expressions. We must admit that in general the element of $\otimes^n \mathfrak{gl}_N$ thus obtained may depend on the choice of the starting point of the circle, — but the corresponding element of the symmetric power, obtained by symmetrization, will not depend on this choice.

For example, to find the tensor corresponding to the diagram  we take three indices i, j and k and write:



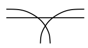
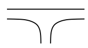
$$\rightsquigarrow \sum_{i,j,k=1}^{N} e_{ji} \otimes e_{ik} \otimes e_{kj}.$$

Here is another example:

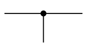
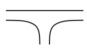
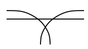
 $\rightsquigarrow$ 

$$\rightsquigarrow \sum_{i,j,k=1}^{N} e_{ji} \otimes e_{kj} \otimes e_{ik}.$$

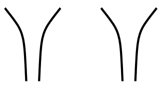
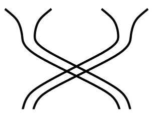
These two examples are sufficient to graphically express the tensor J:

$J =$  $-$  .

With every trivalent vertex of a Chinese character we associate a copy of the tensor $-J$:

 $\rightsquigarrow$  $-$  .

We thus consider two resolutions of each trivalent vertex — the *positive* resolution with the plus sign and the *negative* resolution with the minus sign. After all such resolutions we get a linear combination of 2^t diagrams, where t is the number of trivalent vertices, representing an element of $(\mathfrak{gl}_N)^{\otimes n}$. This linear combination is the graphical notation for $\psi_{\mathfrak{gl}_N}(C)$. Each diagram consists of n pairs of points and n lines connecting them. Permuting the tensor factors in ${\mathfrak{gl}_N}^{\otimes n}$ corresponds to interchanging the pairs of points in the diagram:

 $\rightsquigarrow$ 

Therefore, if we consider a T-diagram up to an arbitrary permutation of the pairs, it correctly defines an element of $U_n(\mathfrak{gl}_N)/U_{n-1}(\mathfrak{gl}_N) \simeq S^n(\mathfrak{gl}_N)$. On a purely combinatorial level, a T-diagram considered up to an arbitrary permutation of pairs of legs is a set of cardinality $2n$ with a splitting into n ordered pairs and a splitting into n unordered pairs. Thus, the linear combination of T-diagrams which is considered up to such permutations represents the element $\psi_{\mathfrak{gl}_N}(C)$.

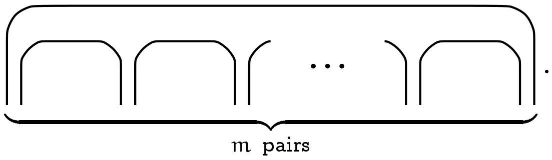
Multiplication in $S^*(\mathfrak{gl}_N)$ corresponds to the *circular union* of diagrams, defined similarly to the product of chord diagrams (the two circles containing the endpoints of the lines are glued into one circle).

The centre of $U(\mathfrak{gl}_N)$. It is known [Zh] that $ZU(\mathfrak{gl}_N)$ is generated by N variables x_m, $1 \le m \le N$ (*generalized Casimir elements*):

$$x_m = \sum_{i_1,\ldots,i_m=1}^{N} e_{i_1 i_2} e_{i_2 i_3} e_{i_3 i_4} \ldots e_{i_{m-1} i_m} e_{i_m i_1}$$

as a free commutative polynomial algebra.

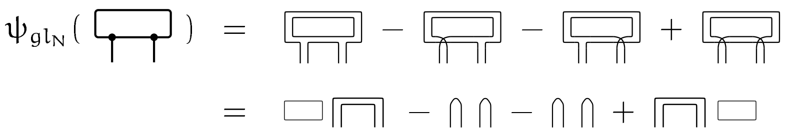
In the graphical notation

$x_m =$ 

In particular, $x_1 = \cap = \sum_{i=1}^{N} e_{ii}$ is the unit matrix (note that it is *not* the unit of the algebra $S(\mathfrak{g})$).

It is convenient to extend the list $x_1, \ldots, x_N$ of our variables by setting $x_0 = N$. The graphical notation for x_0 will be a circle. (Indeed, a circle has no legs, so it must correspond to an element of the 0-th symmetric power of $\mathfrak{g}$, which is the ground field. When the index written on the circle runs from 1 to N, we must take the sum of ones in the quantity N, thus getting N.)

Example.



$= 2(x_0 x_2 - x_1^2)$.

10.4 The formal PBW theorem

The relation between the Lie algebra weight system for chord diagrams and for Chinese characters is expressed by the following theorem.

Theorem 10.2 *For any metrized Lie algebra* $\mathfrak{g}$ *the diagram*

$$\begin{array}{ccc} B & \xrightarrow{\psi_{\mathfrak{g}}} & S(\mathfrak{g}) \\ \chi\downarrow & & \downarrow\beta_{\mathfrak{g}} \\ A & \xrightarrow[\varphi_{\mathfrak{g}}]{} & U(\mathfrak{g}) \end{array}$$

is commutative.

Here χ is the isomorphism between the combinatorial algebras $\mathcal{A}$ and $\mathcal{B}$ described in section 6.3, $\varphi_{\mathfrak{g}}$ and $\psi_{\mathfrak{g}}$ are the corresponding universal Lie-algebraic weight systems, defined in sections 10.1 and 10.2, and $\beta_{\mathfrak{g}}$ is the Poincaré–Birkhoff–Witt isomorphism taking an element $x_1x_2...x_n$ into the arithmetic mean of $x_{i_1}x_{i_2}...x_{i_n}$ over all permutations $(i_1, i_2, ..., i_n)$.

Lecture 11

The algebra of 3-graphs

The algebra of 3-graphs, introduced in [CDK], although not directly related to the Vassiliev invariants, is obtained by a construction which is very natural from an abstract point of view. The elements of this algebra differ from Feynman diagrams in that they do not have any distinguished cycle, like Wilson loop; they differ from Chinese characters in that these graphs are regular, without univalent vertices.

11.1 The space of 3-graphs

Definition 11.1 *A 3-graph is a regular 3-valent graph*[1] *with a fixed rotation. The rotation is the choice of a cyclic order of edges at every vertex, i. e. one of the two cyclic permutations in the set of three edges adjacent to this vertex.*

In this definition *graphs* are allowed to have multiple edges and loops. In the case of graphs with loops the notion of rotation requires the following refinement: the cyclic order is introduced not in the set of *edges* incident to a given vertex, but in the corresponding set of *half-edges*.

A topology free combinatorial definition of the set of half-edges can be given as follows. Let E, V be the sets of edges and vertices, respectively, of the graph under study. Then the set of half-edges is a 'double fibering' $E \overset{\alpha}{\leftarrow} H \overset{\beta}{\rightarrow} V$, that is a set H supplied with two projections $\alpha : H \to E$ and $\beta : H \to V$ such that (a) the inverse image of every edge $\alpha^{-1}(e)$ consists of two elements, (b) the cardinality of each set $\beta^{-1}(v)$ equals the valency of the vertex v, (c) for any point $h \in H$ the vertex $\beta(h)$ is one of the two endpoints of the edge $\alpha(h)$. Then the rotation is a cyclic permutation in the inverse image of each vertex $\beta^{-1}(v)$.

One can forget about edges and vertices altogether and adopt the following clear-cut, although less pictorial, definition of a 3-graph: this is *a set of cardinality* $6n$ *equipped with two permutations, one with the cyclic structure* $(3)(3)\dots(3)$*, another with cyclic structure* $(2)(2)\dots(2)$. The relation to the previous definition

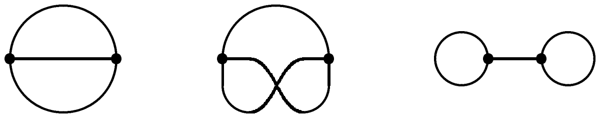
[1] Note that the number of vertices of such a graph is always even, and the number of edges a multiple of 3. Half the number of the vertices will be referred to as the *degree* (or *order* of a 3-graph.

consists in that H is the set of half-edges, the first permutation corresponds to the rotation in the vertices and the second, to the transition from one endpoint of an edge to another.

Definition 11.2 *Two 3-graphs are said to be isomorphic, if there is a one-to-one correspondence between their sets of half-edges which preserves the rotation and induces a usual graph isomorphism — or, in other words, which preserves both structural permutations.*

Remark. It is convenient to include the circle into the set of 3-graphs viewing it as a graph with a zero number of vertices.

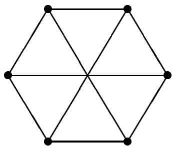
Example. Up to an isomorphism, there are three different 3-graphs of degree 1:



Definition 11.3 *The space Γ_n is the quotient space of the linear space over $\mathbb{Q}$ generated by connected 3-graphs of degree n (i. e. having $2n$ vertices), modulo the AS and IHX relations (see p. 41). By definition, Γ_0 is one-dimensional and generated by the circle.*

That is, the space of 3-graphs differs from the space of Chinese characters (p. 40) in that we consider only diagrams without univalent vertices.

Exercise. Check that the following graph is equal to zero as an element of the space Γ_3:



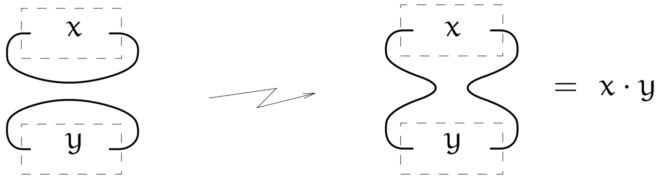
11.2 Multiplication

In the graded space

$$\Gamma = \Gamma_0 \oplus \Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3 \oplus \dots .$$

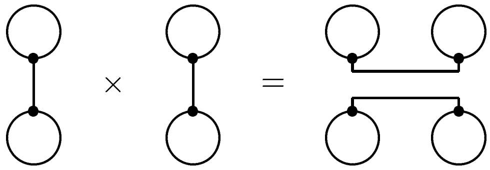
there is a natural structure of a commutative algebra.

Definition 11.4 *The product of two 3-graphs x and y is the 3-graph obtained in the following way. Choose arbitrarily an edge in x and an edge in y. Cut each of* *these two edges in the middle and attach the endpoints that appear on one edge, to the endpoints appearing on the other.*



The product of 3-graphs can also be understood as the result of insertion of one graph, say x, into an edge of y.

Remark. The product of two connected graphs may yield a disconnected graph, for example:



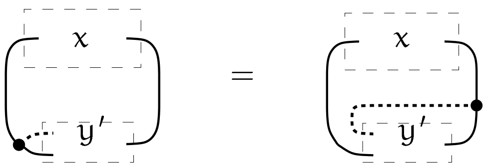
This happens, however, only in the case when each of the two graphs becomes disconnected after cutting the chosen edge, and in this case both graphs are 0 modulo AS and IHX relations (see Lemma 11.4(b)).

Theorem 11.1 *The product of 3-graphs, viewed as an element of the space Γ, is well-defined.*

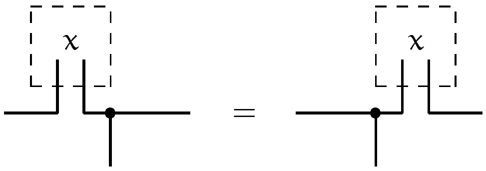
Note that, as soon as this assertion is proved one immediately sees that the multiplication is commutative.

The claim that the product is well-defined consists of two parts. First, that modulo the AS and IHX relations the product does not depend on the *choice* of the two edges x and y which are cut and pasted. Second, that it does not depend on the *way* they are put together (clearly, the two loose ends of one graph can be glued to the two loose ends of another graph in two different ways). These two facts are established in the following two lemmas.

Lemma 11.1 *(a) A vertex with an attached edge can be dragged through a subgraph that has two legs:*



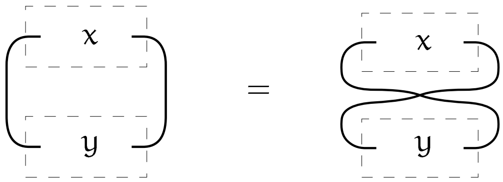
(b) A subgraph with two legs can be carried through a vertex:



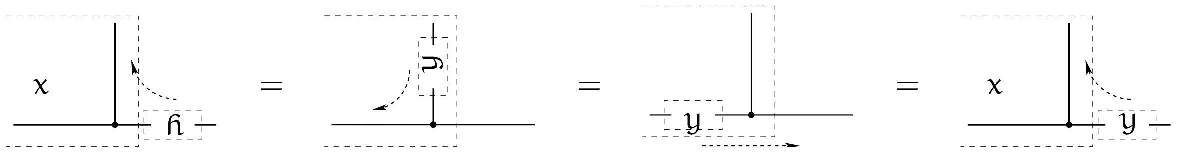
Proof. Taking a closer look at these pictures, the reader will understand that assertions (a) and (b) have exactly the same meaning and in fact are nothing but a particular case ($k = 1$) of the generalized Kirchhof law (see page 42). $\square$

Another wording for Lemma 11.1 is: *the results of insertion of a 3-graph* x *into the adjacent edges of the 3-graph* y, *are equal.* Since y is connected, this implies that the product does not depend on the choice of the edge in y.

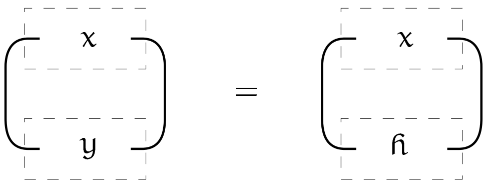
Lemma 11.2 *Two different ways to glue together two graphs with edges cut give the same result in the space* Γ*:*



Proof. Choosing the vertex of x which is nearest to the right exit of y, one can, by Lemma 11.1, effectuate the following manoeuvres:



Therefore,



The lemma is proved, and the correctness of multiplication of 3-graphs is thus established. $\square$

The product of 3-graphs is extended to the product of arbitrary elements of the space Γ in the usual way, by linearity.

Corollary. *The product of arbitrary elements of* Γ *is well-defined.*

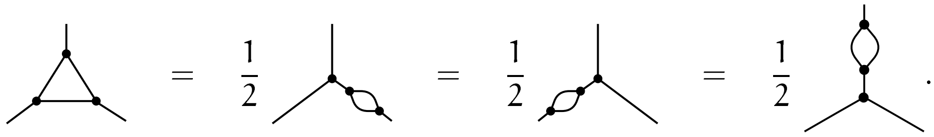
This follows from the fact that the product of 3-graphs is well-defined and a linear combination of either AS or IHX type, when multiplied by an arbitrary graph, is a combination of the same type.

11.3 Some identities

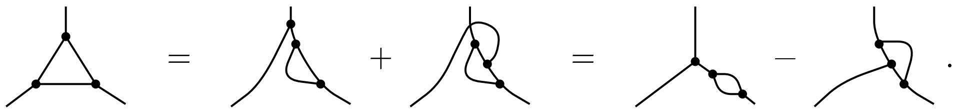
We will prove several identities elucidating the structure of the 3-graph algebra. They are easy consequences of the defining relations. Note that these identities make sense and hold also in the algebra of Chinese characters $\mathcal{B}$.

The first identity (Lemma 11.3) links the two operations defined in the set of 3-graphs: *the insertion of a triangle into a vertex* and *the insertion of a bubble into an edge*. To insert a bubble into an edge of a graph is the same thing as to multiply this graph by the element $\mathrm{b} = \ominus \in \Gamma_1$. The correctness of multiplication in Γ implies that the result of the bubble insertion does not depend on the edge chosen, then it follows from Lemma 11.3 that the result of the triangle insertion does not depend on the choice of the vertex.

Lemma 11.3 *A triangle is equal to one half of a bubble:*

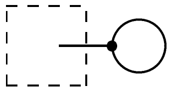


Proof

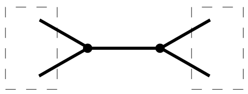


The second lemma describes two classes of 3-graphs which are equal to 0 in the algebra Γ, i. e. modulo the AS and IHX relations.

Lemma 11.4 *(a) A graph with a loop is* 0 *in* Γ*:*

 $= 0$

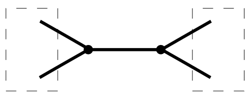
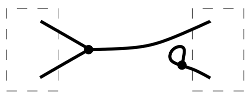
(b) If the edge connectivity of the graph γ *is* 1*, i. e. it becomes disconnected after the removal of an edge, then* $\gamma = 0$*:*

$\gamma =$  $= 0$.

Proof.

(a) A graph with a loop is zero because of antisymmetry. Indeed, changing the rotation in the vertex of the loop yields a graph which is, on one hand, isomorphic to the initial one, and on the other hand, differs from it by a sign.

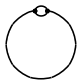
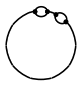
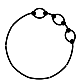
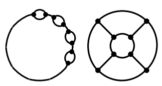
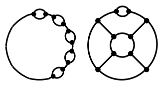
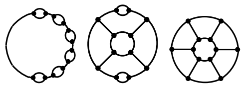
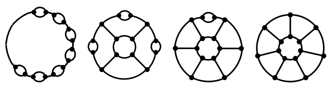
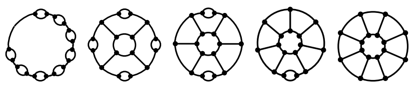
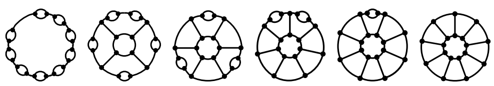
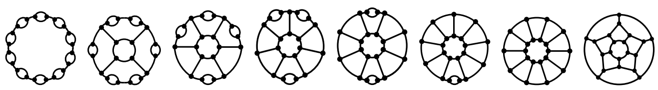
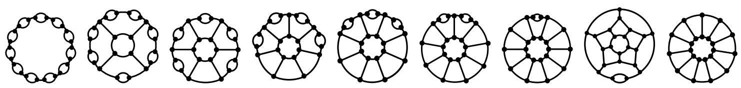
(b) By Lemma 11.3, γ is equal to a graph with a loop:

$\gamma =$  $=$  $= 0$,

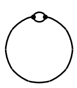
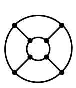
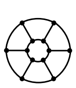
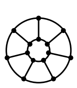
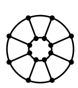
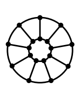
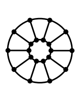
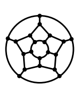
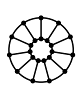
and the assertion follows from (a).

11.4 The Zoo

The following table shows the dimensions d_n and displays the bases of the linear spaces Γ_n for $n \leq 11$, obtained by computer calculations.

n	d_n	additive generators
1	1	
2	1	
3	1	
4	2	
5	2	
6	3	
7	4	
8	5	
9	6	
10	8	
11	9	

Therefore, the following elements can be chosen as multiplicative generators of the algebra Γ up to degree 11 (notations in the lower line mean: b — 'bubble', w_i — 'wheels', d — 'dodecahedron'):

1	4	6	7	8	9	10		11
								
b	w_4	w_6	w_7	w_8	w_9	w_{10}	d	w_{11}

Conjecture 1. The algebra Γ is generated by planar graphs.

The reader might have noted that the previous table of multiplicative generators does not contain the elements w_4^2 of degree 8 and $w_4 w_6$ of degree 10. This is not accidental. It turns out that the following relations hold in the algebra Γ:

$$\begin{aligned} w_4^2 &= \frac{5}{384}b^8 - \frac{5}{12}b^4 w_4 + \frac{5}{2}b^2 w_6 - \frac{3}{2}b w_7, \\ w_4 w_6 &= \frac{305}{27648}b^{10} - \frac{293}{864}b^6 w_4 + \frac{145}{72}b^4 w_6 - \frac{31}{12}b^3 w_7 + 2b^2 w_8 - \frac{3}{4}b w_9. \end{aligned}$$

In fact, as we shall see in Lecture 12 (p. 100), it is true in general that *the product of an arbitrary pair of homogeneous elements of* Γ *of positive degree belongs to the ideal generated by* b.

Since there are non-trivial relations between the generators, the algebra of 3-graphs, in contrast to the algebras $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{F}$, is not free and hence does not possess the structure of a Hopf algebra.

Conjecture 2. The mapping $\Gamma \to \Gamma$ of multiplication by the bubble ($x \mapsto bx$) has no kernel.

This is an old conjecture: in the context of Vassiliev invariants, it was stated several times by various mathematicians, beginning from 1992, and stood several unsuccessful attempts of proof. We generalize it as follows.

Conjecture 3. The algebra Γ does not have zero divisors.

11.5 Connected Feynman diagrams and 3-graphs

By construction, the space of primitive elements $\mathcal{P}$ of the algebra $\mathcal{F}$ (see sec. 5.4), considered by itself, does not possess any structure of an algebra. Primitive elements only generate this algebra in the same way as the variables $x_1, \dots, x_n$ generate the polynomial algebra $\mathbb{R}[x_1, \dots, x_n]$. However, the link between the space $\mathcal{P}$ and the algebra of 3-graphs Γ allows to introduce in $\mathcal{P}$ a natural (non-commutative) multiplication. We shall now describe how this is done.

As we know (see section 5.4), the primitive space of the algebra $\mathcal{F}$ coincides with the subspace spanned by the connected Feynman diagrams.

The relation between the linear spaces Γ and $\mathcal{P}$ is expressed by two facts:

1. There is a linear inclusion $i : \Gamma_n \to \mathcal{P}_{n+1}$, defined on the generators as follows: we cut an arbitrary edge of the graph and attach a Wilson loop to the two resulting endpoints. The orientation of the Wilson loop can be chosen arbitrarily.

2. There is a projection $\pi : \mathcal{P}_n \to \Gamma_n$, which consists in introducing the rotation in the vertices of the Wilson loop according to the rule 'forward–sideways–backwards' and then forgetting the fact that the Wilson loop was distinguished.

The composition homomorphism $\Gamma \to \Gamma$ in the sequence

$$\Gamma \xrightarrow{i} \mathcal{P} \xrightarrow{\pi} \Gamma$$

coincides with the multiplication by the bubble $g \mapsto bg$.

The multiplicative structure of the graph algebra implies that there is an action $\Gamma \times \mathcal{P} \to \mathcal{P}$ which gives rise to an operation $\mathcal{P} \times \mathcal{P} \to \mathcal{P}$.

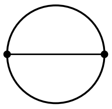
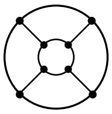
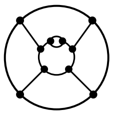
We first define the action $*' : \Gamma \times \mathcal{P} \to \mathcal{P}$. For two given elements $g \in \Gamma$ and $p \in \mathcal{P}$ we construct its product $g *' p$ using the same idea of the connected sum. More specifically, we cut an arbitrary edge of the graph g. Then we cut an edge in p not lying on the Wilson loop and attach the new endpoints of P to those of g, like we did for the algebra Γ itself. The graph that results is $g *' p$; it has one Wilson loop and is connected, hence it belongs to P.

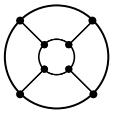
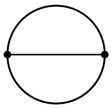
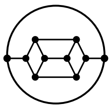
Now define the multiplication $*$ of connected Feynman diagrams $p, q \in \mathcal{P}$ by the rule

$$p * q = \pi(p) *' q,$$

where $\pi : \mathcal{P} \to \Gamma$ is the homomorphism of forgetting the Wilson loop defined above.

The operation $*$ is, in general, non-commutative:

 $*$  $=$  ;

 $*$  $=$  .

These two elements of the space $\mathcal{P}$ are different; they can be distinguished, e. g., by the $\mathfrak{sl}_2$-invariant (see [CV]).

Conjecture. The algebra of primitive elements $\mathcal{P}$ has no divisors of zero.

11.6 Lie algebra weight systems for the algebra of 3-graphs

11.6.1 Construction

The construction of the Lie algebra weight system for the algebra of 3-graphs proceeds in the same way as for the algebra $\mathcal{B}$, using the structure tensor J. Since 3-graphs have no univalent vertices, this weight system takes values in the ground field.

We can view a 3-graph G as a set H of cardinality $6n$ ('the set of half-edges'), endowed with two partitions: into 3-element subsets, each with a fixed cyclic order of its elements, and into 2-element subsets. The 3-element subsets correspond to the vertices of the graph and form the *vertex partition*, while the 2-element subsets correspond to its edges and form the *edge partition*.

Consider three linear spaces:

- $\bigotimes\limits_H \mathfrak{g}$ — the tensor product of $6n$ copies of the Lie algebra $\mathfrak{g}$, whose factors correspond to the half-edges of the graph,
- $\bigotimes\limits_V (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})$ — the tensor product of $2n$ copies of the space $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$, whose factors correspond to the vertices of the graph,
- $\bigotimes\limits_E (\mathfrak{g} \otimes \mathfrak{g})$ — the tensor product of $3n$ copies of the space $\mathfrak{g} \otimes \mathfrak{g}$, whose factors correspond to the edges of the graph.

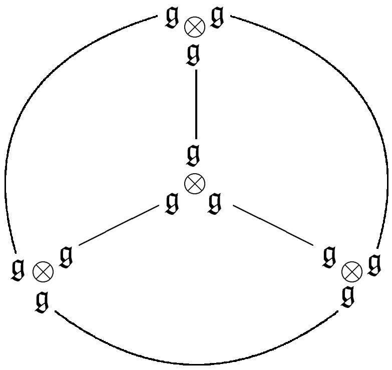
The vertex partition of the set H defines an isomorphism $\alpha : \bigotimes\limits_V (\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}) \to \bigotimes\limits_H \mathfrak{g}$. The edge partition defines an isomorphism $\beta : \bigotimes\limits_H \mathfrak{g} \to \bigotimes\limits_E (\mathfrak{g}\otimes\mathfrak{g})$. A particular choice of the isomorphism α is fixed by choosing a specific order of the three half-edges around each vertex — we will suppose that it is arbitrary, but agreeing with the rotation in G. In the same way, a particular choice of the isomorphism β depends on the preferred ordering of the two half-edges that correspond to one and the same edge of the graph — for our purpose this order will prove irrelevant, and we fix it arbitrarily.

Now consider the composition of these two isomorphisms and the mapping to the ground field induced by the bilinear form $B : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$:

$$\bigotimes_V (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \xrightarrow{\alpha} \bigotimes_H \mathfrak{g} \xrightarrow{\beta} \bigotimes_E (\mathfrak{g} \otimes \mathfrak{g}) \to \mathbb{C}$$

The value $L^B_{\mathfrak{g}}(G)$ of the function $L^B_{\mathfrak{g}}$ on a graph G, by definition, is equal to the value of the composition map on the element $J \otimes J \otimes \ldots \otimes J$. Despite the arbitrariness inherent in the construction of the isomorphisms α and β, the final result is uniquely defined due to the fact that the tensor J is invariant under even permutations, while the bilinear form B is symmetric.

Speaking pictorially, we suspend a copy of the tensor cube of $\mathfrak{g}$ with the distinguished element J living inside it at every vertex of the graph, then take the product over all vertices and compute the complete contraction of the resulting tensor of rank $6n$, applying the bilinear form B to every pair of factors that correspond to the endpoints of one edge, for example:



$\longrightarrow \mathbb{C}$.

Note that for every 3-graph of degree n we build one and the same element in the $6n$-th tensor power of the Lie algebra — but the mapping of this space into the ground field, defined by a specific pairing of the tensor factors, depends on the choice of a particular graph.

This assignment gives rise to a well-defined mapping $\Gamma \to \mathbb{C}$, because

1. the AS relation follows the fact that the tensor J changes sign under odd permutations of the three factors in $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$.

2. the IHX relation is a corollary of the Jacobi identity in $\mathfrak{g}$.

11.6.2 Changing the bilinear form

The bilinear form B preserves all the required properties when multiplied by a constant. A direct tracing of the above described construction shows that the corresponding function $L_\mathfrak{g}^B$ changes in the following way:

$$L_\mathfrak{g}^{\lambda B}(G) = \lambda^{-n} L_\mathfrak{g}^B(G)$$

for $G \in \Gamma_n$.

11.6.3 Multiplicativity

Proposition 11.1 *For a simple r-dimensional Lie algebra $\mathfrak{g}$ and any choice of the ad-invariant non-degenerate symmetric bilinear form B the function*

$$\frac{1}{r} L_\mathfrak{g}^B : \Gamma \to \mathbb{C}$$

is multiplicative.

Proof. This fact follows from the properties of the Casimir operator of a simple Lie algebra.

To prove it, we need some preliminary remarks. First of all, note that the construction of the function $L_\mathfrak{g}^B$ is consistent with the similar construction for Chinese characters (see section 10.2, where it was denoted by $\psi_\mathfrak{g}$). If the algebra is simple and if the Chinese character has only two univalent vertices, then the resulting tensor must be proportional to the quadratic Casimir element $c \in \mathfrak{g} \otimes \mathfrak{g}$. We recall that the Casimir element for a given form B is the element of $\mathfrak{g} \otimes \mathfrak{g}$ which corresponds to $B \in \mathfrak{g}^* \otimes \mathfrak{g}^*$ and to $\mathrm{id} \in \mathfrak{g}^* \otimes \mathfrak{g}$ under isomorphisms induced by the form B. One usually considers the Casimir element for the Killing form, but for us this is not important. What matters is that the value of the bilinear form $B : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ on the corresponding Casimir element is always equal to the trace of the identity operator, i. e. to the dimension of the Lie algebra.

We pass to the proof of the assertion. Cut an arbitrary edge of the graph G_1 and consider the tensor that corresponds to the obtained graph with two univalent vertices. This tensor is proportional to the Casimir element $c \in \mathfrak{g} \otimes \mathfrak{g}$ with coefficient

$\frac{1}{r}L_{\mathfrak{g}}^{B}(G_1)$. Similarly, for the second graph G_2 we get the tensor $\frac{1}{r}L_{\mathfrak{g}}^{B}(G_2)\cdot c$. Now, if we put together one pair of univalent vertices of the graphs G_1 and G_2 thus cut, then the partial contraction of the element $c\otimes c \in \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$ will give an element proportional to $c \in \mathfrak{g}\otimes\mathfrak{g}$, while the coefficients involved get multiplied. This proves the multiplicativity of the function $\frac{1}{r}L_{\mathfrak{g}}^{B}$.

11.6.4 $\mathfrak{sl}_N$- and $\mathfrak{so}_N$-polynomials

For the classical Lie algebras $\mathfrak{sl}$ and $\mathfrak{so}$ the invariant just constructed admits an elegant combinatorial interpretation, which we will now describe. The proof for $\mathfrak{sl}_N$ (or, which is the same thing, for $\mathfrak{gl}_N$) rests on the same argument that we used in Lecture 10. The proof for $\mathfrak{so}_N$ is similar.

Computing $L_{\mathfrak{g}}^{B}$ for $\mathfrak{g}=\mathfrak{sl}_N$ supplied with the bilinear form $B(x,y)=\mathrm{tr}(xy)$. A *vertex marking* of a 3-graph G is an assignment of numbers ± 1 to the vertices of G. If the number of vertices is $2n$, then the total number of vertex markings is 2^{2n}. For every marking consider a plane curve obtained from the graph immersion by *thickening* its edges, i. e. by replacing every edge by two parallel lines. In the vicinity of a vertex the behaviour of the line follows one of the two patterns, depending on the sign of this vertex (we recall that the rotation agrees with the orientation of the plane):

$\oplus$ $\ominus$

Let $|\mu|$ be the number of negative vertices in the marking μ and $b(\mu)$ the number of connected components of the curve obtained. Then

$$L_{\mathfrak{sl}_N}(G)=\sum_{\mu\in MV}(-1)^{|\mu|}N^{b(\mu)}$$

where MV is the set of all vertex markings. The sum consists of 2^{2n} terms.

If the parameter N is variable, then $L_{\mathfrak{sl}_N}(G)$ is a polynomial in N. We will call it sl-*polynomial* and denote by $sl(G)$.

The polynomial $sl(G)$ is divisible by the dimension of the Lie algebra N^2-1 and the quotient is a multiplicative function on the algebra of 3-graphs, which we call the *reduced* sl-*polynomial* and denote by $\widetilde{sl}(G)$.

Computing $L_{\mathfrak{g}}^{B}$ for $\mathfrak{g}=\mathfrak{so}_N$ supplied with the bilinear form $B(x,y)=\mathrm{tr}(xy)$. To compute the value of the so-polynomial on a given 3-graph, one has to perform the similar actions with edges instead of vertices. Here is the precise description.

An *edge marking* of a 3-graph is an assignment of numbers ± 1 to the edges of the graph. For every edge marking consider the plane curve which is obtained from the graph immersion by thickening the edges, with the edges marked by -1 twisted in the way shown below:

$\oplus$ $\ominus$

In the neighbourhood of the vertices we take the direct connection of the lines, like in the positive vertices for $\mathfrak{sl}_N$. Denoting by $|\mu|$ the number of negative edges in the edge marking μ and by $b(\mu)$ the number of connected components of the curve obtained, we can write the formula:

$$L_{\mathfrak{so}_N}(G) = \sum_{\mu \in ME} (-1)^{|\mu|} N^{b(\mu)}$$

where ME is the set of all edge markings. The sum consists of 2^{3n} summands.

Dividing the so-polynomial by the dimension of the algebra $N(N-1)/2$, we obtain the *reduced* $\mathfrak{so}$*-polynomial* $\widetilde{\mathfrak{so}}(G)$, which is an algebra homomorphism $\Gamma \to \mathbb{Z}[N]$.

Lecture 12

Vogel's algebra Λ

The algebra of 3-graphs Γ studied in the previous lecture, although it is defined in a very natural way, is not a fundamental object in the theory of finite type knot invariants. In fact, its structure can be best understood through the algebra Λ introduced by P. Vogel [Vo], which is the subject of the present lecture. This algebra is interesting both by itself and by its natural action on various spaces generated by diagrams.

12.1 Construction

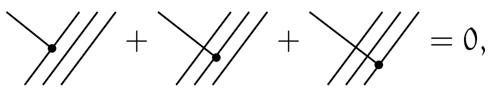
Diagrams with 1- and 3-valent vertices can be considered with different additional structures on the set of univalent vertices (legs). If there is no structure, then we get the notion of a *Chinese character*. If the legs are cyclically ordered, then we obtain *Feynman diagrams*. A diagram with a linear order (numbering) on the set of legs will be called a *fixed diagram*. We will denote the set of all such diagrams equipped, as usual, with rotation in the 3-valent vertices, by Y. By definition, speaking about the numbering means that an isomorphism between two such diagrams must preserve the numbering of the legs. In the set Y there are two gradings: by the number of legs (denoted by the superscript) and by half the total number of vertices (denoted by subscript).

Definition 12.1 *The linear space generated by connected fixed diagrams with* k *legs modulo the usual AS and IHX relations*

$$\mathcal{Y}^k = \langle \mathrm{Y}^k \rangle / \langle \mathrm{AS}, \mathrm{IHX} \rangle,$$

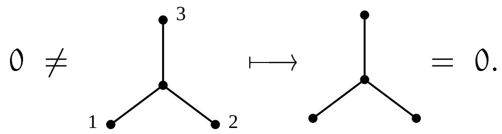
is called the space of fixed diagrams with k *legs. (Everything is over a field of characteristic 0.)*

Exercise. Explain the precise meaning of and prove the following equation:



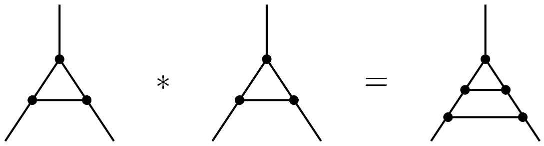
where the individual pictures designate three linear operators from $\mathcal{Y}^4$ to $\mathcal{Y}^3$.

The space of Chinese characters $\mathcal{B}^k$ studied in lecture 6 is the quotient of $\mathcal{Y}^k$ by the permutation group S_k that renumbers the legs. The mapping of taking the quotient $\mathcal{Y} \to \mathcal{B}$ has a nontrivial kernel, for example, a tripod which iz nonzero in $\mathcal{Y}^3$ becomes zero in $\mathcal{B}$:



Definition 12.2 *The algebra Λ is the subspace of elements of $\mathcal{Y}^3$, antisymmetric with respect to the permutations of legs. The multiplication is defined on the generators as follows: choose a vertex in the first diagram, remove it together with 3 adjacent halfedges and insert the second diagram instead.*

Example.



Conjecturally, the antisymmetry requirement in this definition is superfluous:

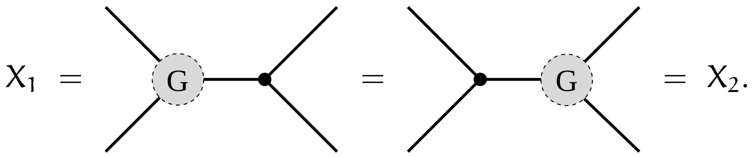
Conjecture. Any element of the space $\mathcal{Y}^k$ with an odd k is antisymmetric with respect to leg permutations. Any element of the space $\mathcal{Y}^k$ with an even k is symmetric with respect to leg permutations.

This conjecture is directly related to the problem of detecting the knot orientation by Vassiliev invaiants. Simple as it sounds, it is an open problem even for $k = 3$.

Theorem 12.1 *(P. Vogel) The above described multiplication in Λ is well-defined.*

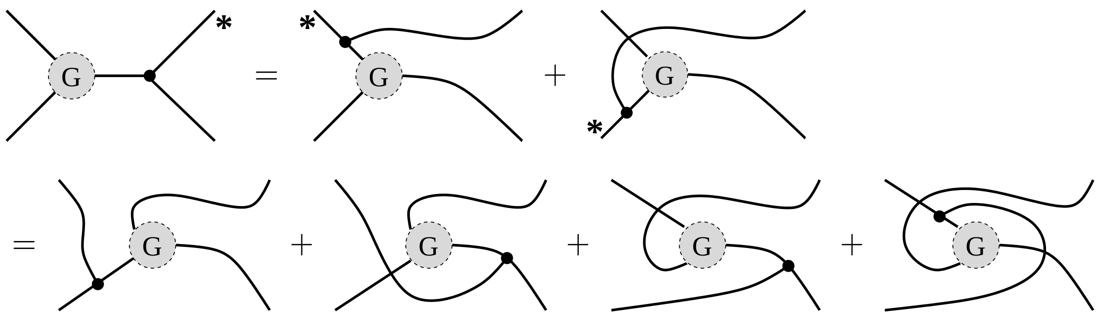
Proof. By definition, elements of Λ are skew-symmetric, therefore the insertion of such an element into a given vertex does not depend on the choice of one out of the six possible ways of doing this.

We must only prove that the following equality holds due to AS and IHX relations:



where G denotes an arbitrary 3-valent subgraph.

By the generalized Kirchhof law we have:



(the stars indicate the place where the tail of the 'moving electron' is fixed in the Kirchhof's relation). Now, in the last line the first and the fouth diagrams are equal to X_2, and the sum of the second and the third ones is equal to X_1 (again, by an application of the generalized Kirchhof's rule). We thus have $2X_1 = 2X_2$ and therefore $X_1 = X_2$. □

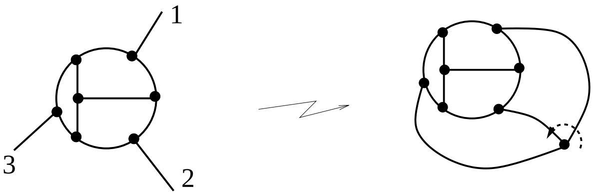
Multiplication in the algebra Λ naturally generalizes to the action of Λ on different spaces generated by 1- and 3-valent diagrams, e. g. the space of Chinese characters $\mathcal{B}$ and the space of 3-graphs Γ. The same argument as above leads to the following theorem.

Theorem 12.2 *The action of the algebra Λ on any space generated by diagrams modulo AS and IHX relations is well-defined.*

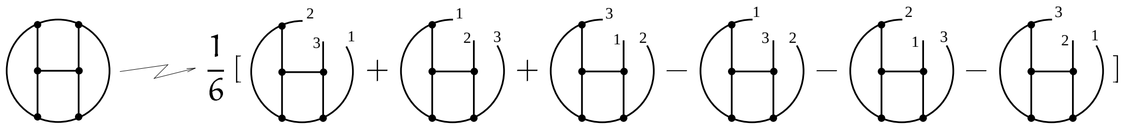
12.2 Relation between Λ and Γ

There are two natarully defined maps between Λ and Γ:

- $\Lambda \to \Gamma$. Add three half-edges and a vertex to an element of Λ in agreement with the leg numbering:



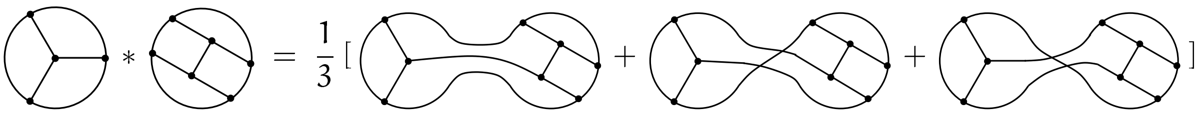
- $\Gamma \to \Lambda$. Choose an arbitrary vertex of a 3-graph, delete it together with the 3 adjacent half-edges and antisymmetrize:



It is fairly evident that these maps are mutually inverse (in particular, this implies that the map $\Gamma \to \Lambda$ is well-defined). We thus arrive at the following result.

Proposition 12.1 *Algebras Γ and Λ are isomorphic as linear spaces.*

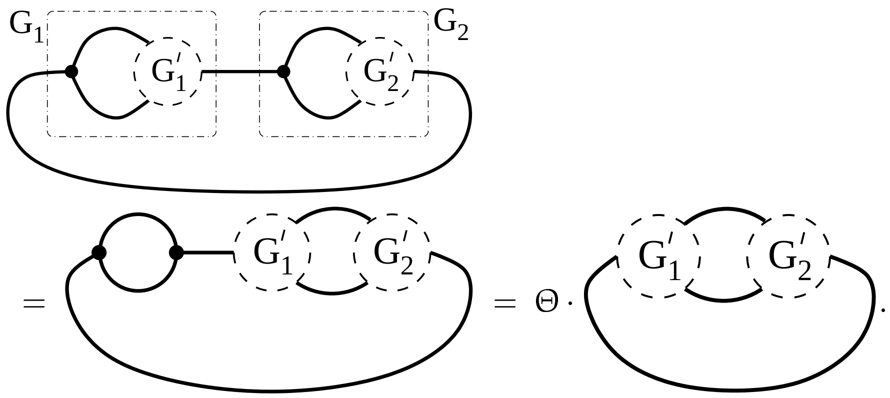
However, the product of 3-graphs introduced in Lecture 11, does not correspond to the Vogel product in the algebra Λ. If the latter is carried over to Γ via the isomorphism $\Lambda \to \Gamma$, then we arrive to a different product of 3-graphs which may be called 'vertex multiplication':



Proposition 12.2 *The usual ('edge') multiplication in the algebra of 3-graphs* Γ *is expressed through the 'vertex multiplication' as follows:*

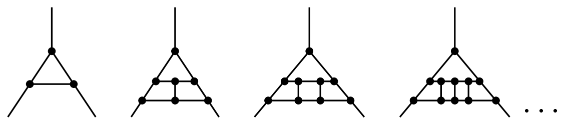
$$G_1 * G_2 = \Theta * (G_1 \vee G_2)$$

Proof. Choose a vertex in each of the given graphs G_1 and G_2 and call what remains G_1' and G_2', respectively. Then, by Theorem 12.2 we have:



12.3 Conjectures of J. Kneissler

Conjecture 1. The algebra Λ is generated by the elements t and x_k with odd $k = 3, 5, \ldots$:



t $\quad$ x_3 $\quad$ x_4 $\quad$ x_5 $\quad \ldots$

Conjecture 2. The linear operator $\Lambda \to \Lambda$ of multiplication by the element t has no kernel (cf. similar conjecture for the algebra of 3-graphs, p. 91).

If these two conjectures are true, then the structure of the algebra Λ is fully known: J. Kneissler's paper [Kn2] contains a complete list of all relations in terms of the generators t and x_k. (The relations are rather sophisticated.)

Lecture 13

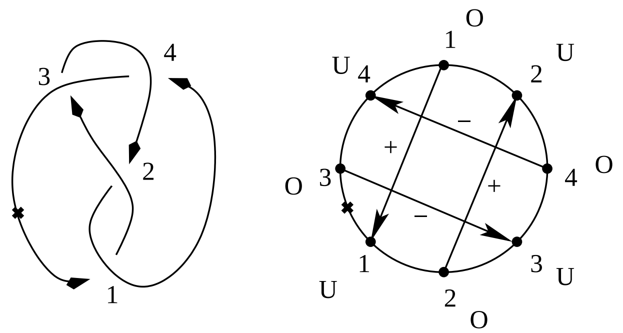
Arrow diagrams

In this lecture we will discuss some combinatorial objects related to Vassiliev invariants and differing from the objects studied before in that they contain oriented edges (chords).

13.1 Arrow diagram formulas for Vassiliev invariants

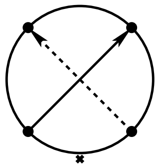
Definition 13.1 *A Gauss diagram is a chord diagram with oriented chords and with numbers* $+1$ *or* -1 *assigned to each chord. To a plane knot diagram there corresponds a Gauss diagram where each chord is oriented from the overpass to the underpass and the local writhe number is assigned to each double point (chord).*

Here is an example of an oriented plane knot diagram and the corresponding Gauss diagram:



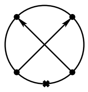
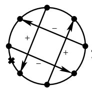
Definition 13.2 *An arrow diagram is a chord diagram with oriented chords supplied with multiplicities 1 or 2. (In the pictures, chords of multiplicity 2 are shown by dotted lines.)*

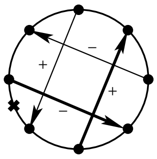
Here is an example of an arrow diagram:

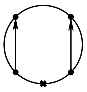
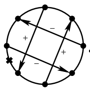


Both in this picture and in the previous one we have marked a base point on the circle which is needed in the constructions below.

Definition 13.3 *Let* A *be an arrow diagram and* G *a Gauss diagram, both with base points. Denote by* $[A]$, $[G]$ *the corresponding sets of chords and by* $E(A)$, $E(G)$ *the sets of their endpoints with base point added. Since every chord knows its beginning and its end, any mapping* $\phi : [A] \to [G]$ *induces a mapping* $\widetilde{\phi} : E(A) \to E(G)$*. An injective mapping* $\phi : [A] \to [G]$ *is called a homomorphism from* A *to* G *(written* $\phi \in \mathrm{Hom}(A, G)$*), if the corresponding* $\widetilde{\phi}$ *preserves the cyclic order of the points and the orientation of the circles.*

Examples. There is exactly one homomorphism from  to , shown by thick chords in this picture:



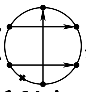
There are no homomorphisms from  to .

Definition 13.4 *The pairing between an arrow and a chord diagrams is defined by*

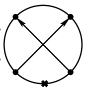
$$\langle A, G \rangle = \sum_{\phi \in \mathrm{Hom}(A,G)} \prod_{c \in [A]} \mathrm{sign}(\phi(c))^{\mu(c)},$$

where μ *denotes the multiplicity (1 or 2) of the chord.*

Now, there is bad news and good news.

The bad news is that the value $\langle A, G(K) \rangle$ is not always uniquely defined by the knot K. For example, $\langle$$, G(K)\rangle$ can change when a different base point or a different plane projection of K in used.

The good news is that there are many linear combinations of arrow diagrams that give knot invariants and, moreover, there is a general theorem due to M. Goussarov [G2] that any Vassiliev invariant can be obtained from a suitable linear combination of arrow diagrams.

Example. If G is a Gauss diagram of a knot K, then $\langle$$, G\rangle = v_2(K)$, where v_2 is the Vassiliev invariant defined in Lecture 2 (page 21).

We must verify three things:

(1) The value of $\nu_2(K)$ does not depend on the choice of the base point.

(2) The value of $\nu_2(K)$ does not depend on the choice of a plane diagram of the knot (i. e. it is preserved by Reidemeister moves).

(3) It is a Vassiliev invariant of order 2.

This can be done by a direct check which we leave to the reader.

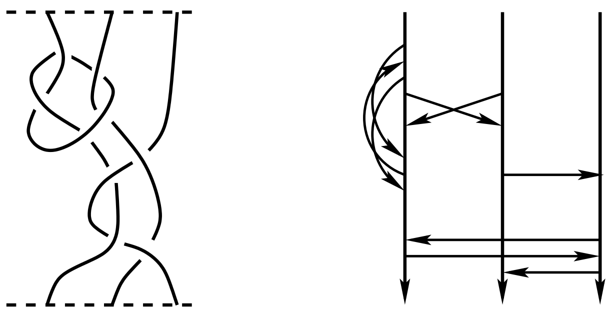
13.2 The algebra of arrow dagrams

The linear space spanned by all arrow diagrams can be made into an algebra in the following way. We will define it in a more general context of string links (the case of arrow diagrams for knots considered before is a particular case).

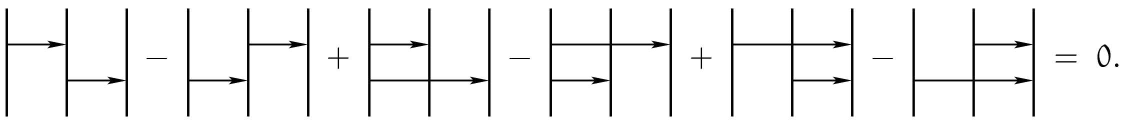
Definition 13.5 *Let D^2 be the disk in $\mathbb{R}^2$ and $p_1, ..., p_n \in D^2$. An n component string link is an embedding of the union of n intervals into the product $D^2 \times [0, 1]$ such that the endpoints of i-th interval go into $p_i \times 0$ and $p_i \times 1$, considered up to an isotopy, fixed on the boundary.*

Definition 13.6 *A (string link) arrow diagram is a collection of n oriented intervals with pairs of distinct internal points connected by oriented by oriented chords.*

With a plane diagram of a string link one can naturally associate an arrow diagram. Here is an example:



Arrow diagrams can be multiplied by putting one on top of the other, just like the tangled chord diagrams (see Lecture 8). The role of the 4 term relation is played by the 6 term relation defined as follows:



Definition 13.7 *The algebra of arrow diagrams on n strings $\vec{\mathcal{A}}_n$ is the quotient of the space generated by all arrow diagrams modulo the $6T$ relations.*

Setting $|\!\!-\!\!| = |\!\!\rightarrow\!\!| + |\!\!\leftarrow\!\!|$, one gets a mapping from the algebra of chord diagrams on n strings $\mathcal{A}_n$ into the algebra $\vec{\mathcal{A}}_n$, because 4T relations go into some linear combinations of 6T relations.

Conjecture. (1) This mapping is injective. (2) Its image coincides with the symmetric part of $\vec{\mathcal{A}}_n$.

The importance of the algebra of arrow diagrams is based on the following three facts:

1. The algebra $\vec{\mathcal{A}}_n$ provides a convenient means to work with Gauss diagram formulas for Vassiliev invariants.
2. It is the counterpart of the algebra of chord diagrams for *virtual links*.
3. The 6 term relation is closely connected with the classical Young-Baxter equation.

For the details we refer the reader to the paper of M. Polyak [Po].

Lecture 14

Estimates for the number of Vassiliev knot invariants

The knowledge of $\dim \mathcal{P}_i$ for $i \leq d$ is equivalent to the knowledge of $\dim \mathcal{A}_i$, $i \leq d$, or $\dim \mathcal{V}_i$, $i \leq d$. At present, the exact asymptotic behaviour of these numbers as d tends to infinity is not known. Below, we summarize the known results on the exact enumeration as well as on the asymptotic lower and upper bounds.

14.1 Historical remarks

14.1.1 Exact results

The exact dimensions of the spaces related to Vassiliev invariants are known up to $n = 12$. The results are shown in the table below. They were obtained by D. Bar-Natan (for $n \leq 9$) in 1993 [BN1] and by J. Kneissler (for $n = 10, 11, 12$) in 1997 [Kn0].

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{P}_n$	0	0	1	1	2	3	5	8	12	18	27	39	55
$\dim \mathcal{A}_n$	1	0	1	1	3	4	9	14	27	44	80	132	232
$\dim \mathcal{V}_n$	1	1	2	3	6	10	19	33	60	104	184	316	548

14.1.2 Upper bounds

A priori it was obvious that $\dim \mathcal{A}_d < (2d-1)!! = 1 \cdot 3 \cdots (2d-1)$, because this is the total number of chord diagrams before factorization over the rotations.

Then, there appeared 3 papers where this estimate was successively improved:

1. (1993) Chmutov and Duzhin [CD1] proved that $\dim \mathcal{A}_d < (d-1)!$
2. (1995) K. Ng in [Ng] replaced $(d-1)!$ by $(d-2)!/2$.
3. (1996) A. Stoimenow [St1] proved that $\dim \mathcal{A}_d$ grows slower than $d!/a^d$ for a constant $a > 1$.

14.1.3 Lower bounds

The story of lower bounds for the number of Vassiliev knot invariants contains a funny episode. The first paper by Kontsevich about Vassiliev invariants ([Ko1], section 3) contains the following passage:

"Using this construction [1], one can obtain the estimate

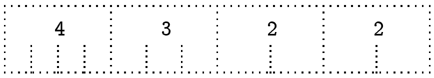
$$\dim(\mathcal{V}_n) > e^{c\sqrt{n}}, \quad n \to +\infty$$

for any positive constant $c < \pi\sqrt{2/3}$ (see [BN1a], Exercise 6.14)."

Here $\mathcal{V}_n$ is a slip of the pen, instead of $\mathcal{P}_n$, because of the reference to Exercise 6.14 where primitive elements are considered. Exercise 6.14 was present, however, only in the first edition of Bar-Natan's preprint and eliminated in the following editions as well as in the final published version of his text [BN1]. In [BN1a] it reads as follows (page 43):

"*Exercise* 6.14. (Kontsevich, [24]) Let $P_{\geq 2}(m)$ denote the number of partitions of an integer m into a sum of integers bigger than or equal to 2. Show that $\dim \mathcal{P}_m \geq P_{\geq 2}(m+1)$.

Hint 6.15. Use a correspondence like



$$\longleftrightarrow \qquad 10 + 1 = 4 + 3 + 2 + 2,$$

and ... "

The reference [24] was to "M. Kontsevich. Private communication."! Thus, both authors referred to each other, and none of them gave any proof. Later, however, Kontsevich explained what he had in mind (see item 5 below).

Arranged by time, the history of world records in asymptotic lower bounds for the dimension of the primitive space $\mathcal{P}_d$ looks as follows.

1. (1994) $\dim \mathcal{P}_d \geq 1$ ("forest elements" found by Chmutov, Duzhin and Lando [CDL3]).

2. (1995) $\dim \mathcal{P}_d \geq [d/2]$ (given by coloured Jones function — see Melvin–Morton [MeM] and Chmutov–Varchenko [CV]).

3. (1996) $\dim \mathcal{P}_d \gtrsim d^2/96$ (see Duzhin [Du]).

4. (1997) $\dim \mathcal{P}_d \gtrsim d^{\log d}$, i. e. the growth is faster than any polynomial (Chmutov–Duzhin [CD2]).

5. (1997) $\dim(\mathcal{P}_n) > e^{\pi\sqrt{n/3}}$ (Kontsevich [Ko2]).

6. (1997) $\dim(\mathcal{P}_n) > \frac{C}{n\sqrt{n}} e^{\pi\sqrt{2n/3}}$ (Dasbach [Da3]).

[1] Of Lie algebra weight systems.

14.2 Proof of the lower bound

We will sketch the proof of the lower bound for the number of Vassiliev knot invariants, following the paper of Chmutov and Duzhin [CD2] and then explain how O. Dasbach [Da3], using the same method, managed to improve the estimate and establish the world record.

The idea of the proof is simple: we construct a large family of Chinese characters whose linear independence in the algebra $\mathcal{B}$ follows from the linear independence of the values on these diagrams of a certain polynomial invariant P, which is obtained from the universal $\mathfrak{gl}_N$ invariant by means of a simplification.

As we know from Lecture 10, the $\mathfrak{gl}_N$ invariant $\psi_{\mathfrak{gl}_N}$, evaluated on a Chinese character C is a polynomial in the generalized Casimir elements x_0, x_1, ..., x_N. This polynomial is homogeneous in the sense of the grading defined by setting $\deg x_m = m$. However, it is in general not homogeneous, if x_m's are considered as simple variables of degree 1.

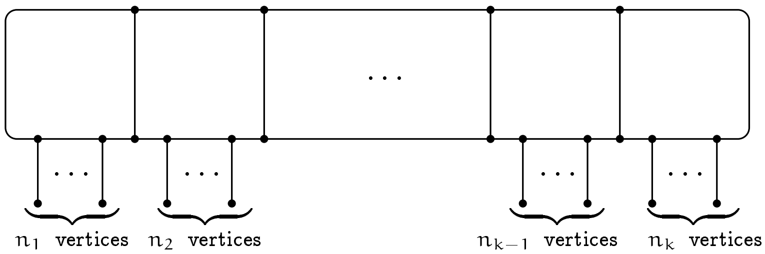
Definition 14.1 *The polynomial invariant* $P : \mathcal{B} \to \mathbb{Z}[x_0, ..., x_N]$ *is the highest homogeneous part of* $\psi_{\mathfrak{gl}_N}$*, if all the variables are taken with degree 1.*

For example, if we had $P(C) = x_0^2 x_2 - x_1^2$, then we would have $P(C) = x_0^2 x_2$.

Now we introduce the family of primitive Chinese characters whose linear independence we shall prove.

Definition 14.2 *The baguette diagram* $B_{n_1,...,n_k}$ *is*

$B_{n_1,...,n_k} \quad =$



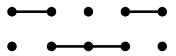
It has a total of $2(n_1 + \cdots + n_k + k - 1)$ *vertices, out of which* $n_1 + \cdots + n_k$ *are univalent.*

To write down the formula for the value $P(B_{n_1,...,n_k})$, we will need the following definitions.

Definition 14.3 *A scheme is a combinatorial object defined as follows. Consider* k *pairs of vertices arranged in two rows:* $:\,:\,\cdots\,:$ *. Choose one of* 2^{k-1} *subsets of the set* $\{1, \ldots, k-1\}$*. If* s *belongs to the chosen subset then we connect the lower vertices of* s*-th and* $(s+1)$*-th pairs, otherwise we connect the upper vertices.*

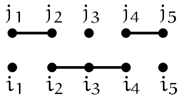
A weighted scheme is a scheme supplied with two sets of non-negative integers: $0 \le i_1 \le n_1, \ldots, 0 \le i_k \le n_k$.

Example. Here is a scheme corresponding to $k = 5$ and the subset $\{2, 3\}$:



With a weighted scheme, one can associate a monomial in the variables x_i as follows.

Definition 14.4 *Given a weighted scheme, we assign i_s to the lower vertex of the s-th pair of σ and $j_s = n_s - i_s$ — to the upper vertex. For example*



Then the corresponding monomial is $x_{\sigma_0} x_{\sigma_1} \dots x_{\sigma_k}$ where σ_t is the sum of integers assigned to the vertices of t-th connected component of σ.

Example. For the above scheme we get the monomial $x_{i_1} x_{j_1+j_2} x_{i_2+i_3+i_4} x_{j_3} x_{j_4+j_5} x_{i_5}$.
Now the formula for P can be stated as follows.

Proposition 14.1 *If $N > n_1 + \dots + n_k$ then*

$$P_{\mathfrak{gl}_N}(B_{n_1,\dots,n_k}) = \sum_{i_1,\dots,i_k} (-1)^{j_1+\dots+j_k} \binom{n_1}{i_1} \dots \binom{n_k}{i_k} \sum_{\sigma} x_{\sigma_0} x_{\sigma_1} \dots x_{\sigma_k},$$

where the external sum ranges over all integers $i_1, \dots, i_k$ such that $0 \le i_1 \le n_1, \dots, 0 \le i_k \le n_k$; the internal sum ranges over all 2^{k-1} schemes and $x_{\sigma_0} x_{\sigma_1} \dots x_{\sigma_k}$ is the monomial associated with the scheme σ and integers $i_1, \dots, i_k$.

Example. For the baguette diagram B_2 we have $k = 1$, $n_1 = 2$. There is only one scheme: . The corresponding monomial is $x_{i_1} x_{j_1}$, and

$$\begin{aligned} P(B_2) &= \sum_{i_1=0}^{2} (-1)^{j_1} \binom{2}{i_1} x_{i_1} x_{j_1} \\ &= x_0 x_2 - 2 x_1 x_1 + x_2 x_0 \quad = \quad 2(x_0 x_2 - x_1^2) \end{aligned}$$

which agrees with the example given in Lecture 10 on page 83.

Sketch of the proof of Proposition 14.1. The Chinese character $B_{n_1,\dots,n_k}$ has k parts separated by $k - 1$ walls. Each wall is an edge connecting trivalent vertices to which we will refer as *wall vertices.* The s-th part has n_s outgoing legs. We will refer to the corresponding trivalent vertices as *leg vertices.*

The proof consists of three steps.

At the first step we study the effect of resolutions of the wall vertices. We prove that the monomial obtained by certain resolutions of these vertices has the maximal possible degree if and only if for each wall both resolutions of its vertices have the same sign. These signs are related to the above defined schemes in the following way. If we take the positive resolutions at both endpoints of s-th wall, then we connect the lower vertices of s-th and $(s+1)$-th pairs in the scheme. If we take the negative resolutions, then we connect the upper vertices.

At the second step we study the effect of resolutions of leg vertices. We show that the result depends only on the numbers of positive resolutions of leg vertices in each part and does not depend on which vertices in a part were resolved positively and which negatively. We denote by i_s the number of positive resolutions in part s. This yields the binomial coefficients $\binom{n_s}{i_s}$ in the formula of Proposition 14.1. The total number $j_1 + \cdots + j_k$ of negative resolutions of leg vertices gives the sign $(-1)^{j_1+\cdots+j_k}$.

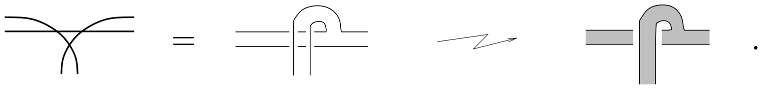
The first two steps allow us to consider only those cases where the resolutions of the left i_s leg vertices in the part s are positive, the rest j_s resolutions are negative and both resolutions at the ends of each wall have the same sign. At the third step we prove that such resolutions of wall vertices lead to monomials associated with corresponding schemes according to definition 14.4.

I will make some comments only about the first step, because it is exactly at this step where Dasbach found an improvement of the original argument of [CD2].

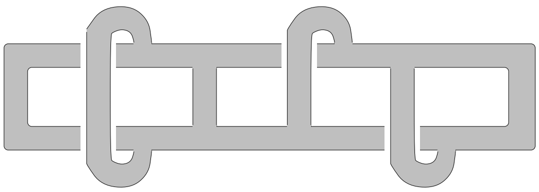
Let us fix certain resolutions of all trivalent vertices of $B_{n_1,\dots,n_k}$. We denote the obtained T-diagram (see p. 81) by T. It consists of $n = n_1 + \cdots + n_k$ pairs of points and a number of lines connecting them. After a suitable permutation of the pairs T will look like a disjoint union of certain x_m's. Hence it defines a monomial in x_m's which we denote by $m(T)$.

Let us close all lines in the diagram by connecting the two points in every pair with an additional short line. We obtain a number of closed curves, and we can draw them in such a way that they have 3 intersection points in the vicinity of each negative resolution and do not have other intersections. Each variable x_m gives precisely one closed curve. Thus the degree of $m(T)$ is equal to the number of these closed curves.

Consider an oriented surface S which has our family of curves as its boundary (the Seifert surface):



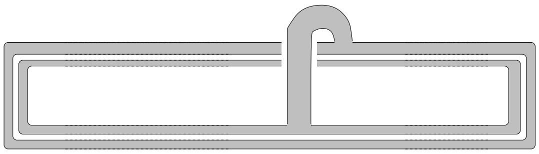
The degree of $m(T)$ is equal to the number of boundary components b of S. The whole surface S consists of an annulus corresponding to the big circle in $B_{n_1,\dots,n_k}$ and $k-1$ bands corresponding to the walls. Here is an example:



where each of the two walls on the left has the same resolutions at its endpoints, while the two walls on the right have different resolutions at their endpoints. The resolutions of the leg vertices do not influence the surface S.

The Euler characteristic χ of S can be easily computed. The surface S is contractible to a circle with $k-1$ chords, thus $\chi = 2(k-1) - (k-1+2k-2) = -k+1$. On the other hand $\chi = 2-2g-b$, where g and b are the genus and the number of boundary components of S. Hence $b = k+1-2g$. Therefore, the degree of $m(T)$, equal to b, attains its maximal value $k+1$ only for such resolutions which give a zero genus surface S.

We claim that if there exists a wall whose ends are resolved with the opposite signs then the genus of S is not zero. Indeed, in this case we can draw a closed curve in S which does not separate the surface (independently on the remaining resolutions):



So, the contribution to $P(B_{n_1,\dots,n_k})$ is given by only those monomials which come from equal resolutions at the ends of each wall.

Now, with Proposition 14.1 in hand, we can prove the following result.

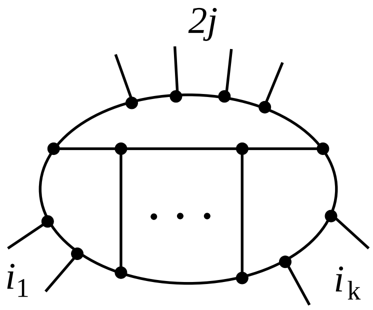
Theorem 14.1 *Let* $n = n_1 + \cdots + n_k$ *and* $d = n+k-1$. *Baguette diagrams* $B_{n_1,\dots,n_k}$ *are linearly independent in* $\mathcal{B}$ *if* $n_1,\dots,n_k$ *are all even and satisfy the following conditions:*

$$\begin{aligned}
&n_1 < n_2\\
&n_1 + n_2 < n_3\\
&n_1 + n_2 + n_3 < n_4\\
&\dots\dots\dots\dots\dots\dots\dots\\
&n_1 + n_2 + \cdots + n_{k-2} < n_{k-1}\\
&n_1 + n_2 + \cdots + n_{k-2} + n_{k-1} < n/3.
\end{aligned}$$

The proof is based on the study of the *supports* of polynomials $P(B_{n_1,\dots,n_k})$ — the subsets of $\mathbb{Z}^k$ corresponding to non-zero terms of the polynomial.

Counting the number of elements described by the theorem, one arrives at the lower bound $n^{\log(n)}$ for the dimension of the primitive subspace $\mathcal{P}_n$ of $\mathcal{B}$.

The main difficulty in the above proof is the necessity to consider 2^k resolutions for the wall vertices of a baguette diagram, which correspond to the zero genus Seifert surface. O. Dasbach in [Da3] avoided this difficulty by considering a different family of Chinese characters for which there are only two ways of resolution of the wall vertices leading to the surface of minimum genus. These are the *Pont Neuf diagrams*:

$PN_{i_1,\dots,i_k,j} =$ 

(the numbers i_1, ..., i_k, $2j$ refer to the number of legs attached to the corresponding edge of the inner diagram).

The reader may wish to check the above property of Pont Neuf diagrams by way of exercise. It is remarkable that Pont Neuf diagrams not only lead to a simpler considerations, but they are more numerous, too, and thus lead to a much better asymptotic estimate for $\dim \mathcal{P}_n$. The exact statement of Dasbach's theorem is as follows.

Theorem 14.2 *For fixed* n *and* k *the diagrams* $PN_{i_1,\dots,i_k,j}$ *with* $0 \le i_1 \le \dots \le j_k \le b$, $i_1 + \dots + i_k + 2j = 2n$ *are linearly independent.*

Counting the number of such partitions of $2n$, we obtain the estimate:

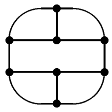
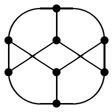
Corollary. $\dim \mathcal{P}_n$ is asymptotically greater than $e^{c\sqrt{n}}$ for any constant $c < \pi\sqrt{2/3}$.

Problems

Exercises

1.
   - Find some bases of the spaces $\mathcal{A}_n$, $\mathcal{F}_n$, $\mathcal{B}_n$ for $n = 4$.
   - Indicate an explicit form of the isomorphisms $\mathcal{A} \cong \mathcal{F} \cong \mathcal{B}$ in these bases.
   - Compile the multiplication table for $\mathcal{B}_m \times \mathcal{B}_n \to \mathcal{B}_{m+n}$, $m + n \leq 4$, for the second product in $\mathcal{B}$ (the one pulled back from $\mathcal{F}$ along the isomorphism $\mathcal{F} \cong \mathcal{B}$).

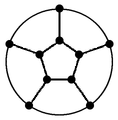
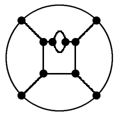
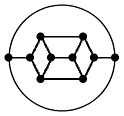
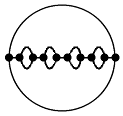
2. (Lecture 6). Check that the following linear combination is equal to a connected Feynman diagram:

 $-$ 

3. (Lecture 8). If a tangle is closed into a circle, then the tangled 4 term relation becomes the usual 4 term relation for circular chord diagrams.

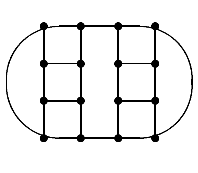
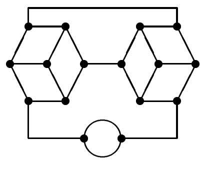
4. (Lecture 10). Prove that the mapping $\psi : \mathcal{B} \to S(\mathfrak{g})$ is well-defined, i.e. that the IHX relation follows from the Jacobi identity.

5. Prove the following identity in the space $\mathcal{F}$ of Feynman diagrams:

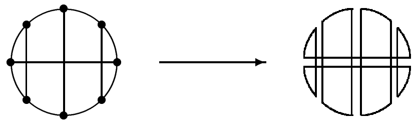
 $= (3/4)$  $- (1/12)$  $- (1/48)$  .

6. Prove the linear independence of the three elements in the right hand side of the last equality, using Lie algebra invariants.

7. Find explicitly a chain of IHX and AS relations that proves the following equality in the algebra Γ of 3-graphs:

 $=$ 

8. (*S. Lando*). Denote by $s(D)$ the number of connected components of the curve obtained by doubling all chords of a chord diagram D. Example:

for $D =$ 

we have $s(D) = 3$.

Let N be a formal variable. Prove that:

- $s(D) - 1$ is equal to the corank of the adjacency matrix of the intersection graph of D,
- $N^{s(D)-1}$ is a multiplicative weight system, i.e. defines an algebra homomorphism $\mathcal{A} \to \mathbb{Z}[N]$.
- Find the relation of this weight system with the Lie algebra weight system for the algebra $\mathfrak{gl}_N$.

9. (*S. Lando*). Let N be a formal variable. Prove that $N^{\operatorname{corank} A(G)}$ defines an algebra homomorphism $\mathcal{L} \to \mathbb{Z}[N]$, where $\mathcal{L}$ is the graph algebra of Lando and $A(G)$ stands for the adjacency matrix of the graph G.

10. (Lectures 2 and 13). Let A be the arrow diagram with two intersecting chords. Denote by $G(K)$ the Gauss diagram corresponding to any plane diagram of the knot K. Prove that $K \mapsto \langle A, G(K) \rangle$ is a Vassiliev invariant of order 2.

11. Let A be a graded commutative Hopf algebra and P its primitive subspace. Set $f(t) = \sum(\dim P_n)t^n$ and $g(t) = \sum(\dim A_n)t^n$. Express $g(t)$ through $f(t)$.

12. Let W be the algebra of weight systems, while W' be the subalgebra of strict weight systems (also satisfying 1-term relations). Find an explicit formula for the projection $\pi : W \to W'$ which splits the natural inclusion $\iota : W' \to W$, i.e. such that $\pi \circ \iota = \mathrm{id}$.

13. (Lecture 10). Prove that isomorphisms $\chi : \mathcal{B} \to \mathcal{A}$ (Bar-Natan) and PBW : $S(\mathfrak{g}) \to U(\mathfrak{g})$ (Poincaré-Birkhoff-Witt) carry the mapping $\psi_\mathfrak{g} : \mathcal{B} \to S(\mathfrak{g})$ into the mapping $\phi_\mathfrak{g} : \mathcal{A} \to U(\mathfrak{g})$.

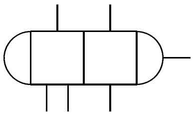
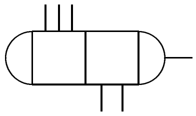
14. (Lecture 10). Prove the graphical rule to compute the invariant $\psi_\mathfrak{g}(b)$ for the Lie algebra $\mathfrak{g} = \mathfrak{so}_N$.

15. (Lecture 13). Call a Feynman diagram $f \in \mathcal{F}$ even (odd), if $\bar{f} = f$ (resp. $\bar{f} = -f$), where the bar denotes the change of orientation of the Wilson loop. Prove that under the isomorphism $\chi : \mathcal{B} \to \mathcal{F}$:

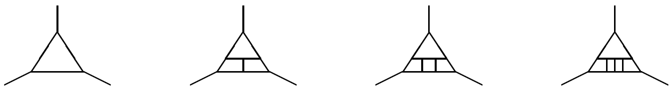
- the image of an even Feynman diagram in $\mathcal{B}$ is a linear combination of Chinese characters with an even number of legs,

- the image of an odd Feynman diagram in $\mathcal{B}$ is a linear combination of Chinese characters with an odd number of legs.

16. (*J. Kneissler*). Let $\mathcal{B}_n^u$ be the space of Chinese characters of degree n with u univalent vertices. Denote by $\omega_{i_1 i_2 \dots i_k}$ the element of $\mathcal{B}_{i_1+\dots+i_k+k-1, i_1+\dots+i_k}$ represented by a "caterpillar" diagram consisting of k body segments with $i_1, \dots,$ i_k "legs", respectively. Using AS and IHX relations, prove that $\omega_{i_1 i_2 \dots i_k}$ is well-defined, i.e. for inner segments it makes no difference on which side of the body they are drawn). For example,

$\omega_{0321} =$  $=$ 

17. Let t, x_3, x_4, x_5, ... be the following elements of the space $\mathcal{Y}^3$:



   - Prove that x_i's belong to Vogel's algebra Λ, i.e. that they are antisymmetric with respect to permutations of legs.
   - Prove that x_k with even k can be expressed through t, x_3, x_5, ...

18. Let $\mathcal{A}$ be the algebra of chord diagrams, while $\vec{\mathcal{A}}$ be the algebra of oriented (arrow) chord diagrams. Prove that the natural mapping $\mathcal{A} \to \vec{\mathcal{A}}$ is well-defined, i.e. that the 6T relation implies the 4T relation.

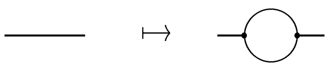
19. Let $\vec{\mathcal{A}}$ be the algebra of arrow diagrams, while $\vec{\mathcal{F}}$ be the algebra of acyclic arrow graphs (oriented Feynman diagrams). Prove that the natural mapping $\vec{\mathcal{A}} \to \vec{\mathcal{F}}$ is well-defined, i.e. that the $\vec{STU}$ relation implies the 6T relation.

Open problems

1. Is is true that any chord diagram of order 13 is equivalent to its mirror image modulo 4-term relations?

2. Is it true that a Chinese character with 3 univalent vertices is always equal to 0 as an element of the algebra $\mathcal{B}$?

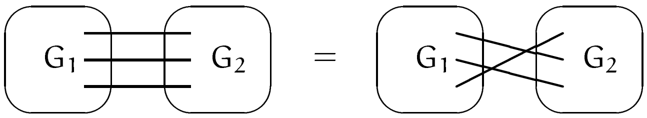
3. Is it true that the operator of multiplication by the "bubble"



(in the algebras $\mathcal{F}$, $\mathcal{B}$ and Γ) has no kernel?

4. Let Γ be the algebra of 3-graphs.

   - Is it true that Γ has no divisors of 0?
   - In particular, is it true that multiplication by Θ has no kernel?
   - Is it true that Γ is generated by plane graphs?
   - Find generators and relations of the algebra Γ.
   - Suppose that a graph $G \in \Gamma$ consists of two parts G_1 and G_2 connected by three edges. Is the following equality

     an identity in Γ?

5. Let $\mathcal{Y}^k$ be the space of 1- and 3-valent graphs with k numbered legs. Is it true that under any transposition of two legs every element of $\mathcal{Y}^k$ remains invariant (changes sign), if k is even (odd)? (The first nontrivial case is when $k = 3$).

6. Let Λ be Vogel's algebra, i.e. the subspace of $\mathcal{Y}^3$ consisting of all antisymmetric elements.

   - Is it true that $\Lambda = \mathcal{Y}^3$?
   - Is it true that Λ is generated by the elements t and x_k (see Exercise 17)?
   - Is it true that the operator of multiplication by t has no kernel?

7. Let $\mathcal{Y}^3$ be the space of 1- and 3-valent graphs with 3 numbered vertices. The group S_3 acts in $\mathcal{Y}^3$, splitting it into 3 subspaces:

   - symmetric, which is isomorphic to $\mathcal{B}^3$ (Chinese characters with 3 legs),
   - totally antisymmetric, which is Vogel's Λ by definition, and
   - some subspace Q, corresponding to the third irreducible representation of S_3.

   Question: is it true that $Q = 0$?

8. Define an analogue of the algebra of oriented Feynman diagrams $\vec{\mathcal{F}}$ generated by all graphs, not only acyclic. Denote it by $\vec{\mathcal{F}}_{\mathrm{all}}$. Is it true that the natural mapping $\vec{\mathcal{F}} \to \vec{\mathcal{F}}_{\mathrm{all}}$ is an isomorphism?

9. Construct the analogue of the algebra of Chinese characters $\mathcal{B}$ consisting of graphs with oriented edges.

10. Let $\lambda : \mathcal{A} \to \mathcal{L}$ be the natural homomorphism from the algebra of chord diagrams into the graph algebra of Lando.

- Find $\ker\lambda$ (unknown in degrees greater than 7).
- Find $\mathrm{im}\,\lambda$ (unknown in degrees greater than 7).
- Describe the primitive space $P(\mathcal{L})$.
- $\mathcal{L}$ is the analogue of the algebra of chord diagrams in the case of intersection graphs. Are there any counterparts of algebras $\mathcal{F}$ and $\mathcal{B}$?

11. (*J. Kneissler*) (See Exercise 16 for the definition of caterpillar diagrams). Is it true that any caterpillar diagram in the algebra $\mathcal{B}$ can be expressed through caterpillar diagrams with even indices $i_1, \ldots, i_k$? Is it true that the primitive space $\mathcal{P}(\mathcal{B})$ (i.e. the space spanned by connected Chinese characters) is generated by caterpillar diagrams?

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Index