Vassiliev knot invariants III. Forest algebra and weighted graphs

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The main tool used in the investigation of Vassiliev knot invariants is the Hopf algebra of chord diagrams [CDL1]. This algebra, simple as it seems at first sight, upon a closer examination proves to be a rather complicated object. It is sufficient to say that, up to now, the number of its primitive generators is known only in degrees no greater than 9.

A standard way to tackle a complex mathematical object O is through the study of its subobjects and its quotient objects. An ideal situation is when you can distinguish a simple subobject $S \subset O$ such that the corresponding quotient object O/S is also simple enough and so that the properties of the entire O are completely determined by the properties of S and O/S.

With the Hopf algebra of chord diagrams, we were not able to achieve this goal. The best we could do was to find a simple (but non-trivial) subalgebra $\mathcal{A} \subset \mathcal{M}$ and a simple quotient algebra $\mathcal{W}$ defined by an epimorphism $\mathcal{M} \to \mathcal{W}$. These two algebras, $\mathcal{A}$ and $\mathcal{W}$, are, however, not complementary to each other. Quite on the contrary, the composition map in the short *non-exact* sequence

$$0 \to \mathcal{A} \to \mathcal{M} \to \mathcal{W} \to 0$$

proves to be an isomorphism between $\mathcal{A}$ and $\mathcal{W}$.

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An explicit description of $\mathcal{W}$ shows that both algebras have precisely one primitive element in each dimension. This also means that they are both isomorphic to the Hopf algebra of Young diagrams that appears in the representation theory of symmetric groups (see [Gei], [Zel]). The adjoint inclusion $\mathcal{W}^* \to \mathcal{M}^*$ provides a family of easily calculable Vassiliev knot invariants, one primitive element in each dimension.

The paper consists of two sections. In section 1, we formulate the main results, and in section 2, we give their proofs.

Logically, the exposition proceeds as follows. We begin by introducing the *forest algebra* $\mathcal{A}$, which is, by definition, a subalgebra in $\mathcal{M}$ generated by all trees (see [CDL2]). We prove that $\mathcal{A}$ has not more than one primitive element in each dimension and give an explicit representation for a primitive element. Then, we give the construction of the Hopf algebra of weighted graphs and prove the corresponding structure theorem. Finally, we discuss the relation between the two algebras and prove that the primitive element of $\mathcal{A}$ constructed in section 1 differs from zero.

1 Main results

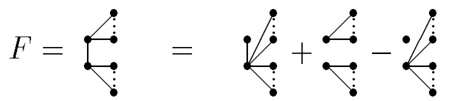
1.1 Forest algebra

Denote by $\mathcal{A}$ the subalgebra in the Hopf algebra $\mathcal{M}$ of chord diagrams generated by the classes (modulo only four-term relations) of all diagrams whose intersection graph is a forest.

Theorem 1. *The Hopf algebra $\mathcal{A}$ is isomorphic to the polynomial algebra $\mathbf{Z}[x_1, x_2, \ldots]$, where the grading of every x_n is n.*

The four-term relations induce the following relations in $\mathcal{A}$.

Proposition 1. *Let $F \in \mathcal{A}$ be an arbitrary forest. Consider three graphs F_0, F' and F_0' with the same set of vertices and with the set of edges modified with respect to the set of edges of F according to the choice of an edge e in F and one of its endpoints A. Namely, F_0 differs from F only in that the edge e is deleted, F' has all the edges incident with A in F shifted to the other endpoint of e, and F_0' is obtained from F' again by deleting e. Diagrammatically,*



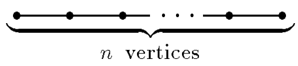
(1)

where the edge drawn vertically is e and its upper endpoint is A.

Proposition 1 follows from two statements proved in [CDL2]: Lemma 2 and Remark in section 3.6.

The two last terms in (1) are non-trivial products. Therefore, the two first terms are equivalent modulo the subspace spanned by the decomposable elements. It is easy to see that any two trees with n vertices are equivalent with respect to this equivalence relation, hence the dimension of n-th grade primitive space in $\mathcal{A}$ is at most 1. To finish the proof of the Theorem 1, it remains to specify a non-zero primitive element in each grading n.

Proposition 2. *Denote the tree*

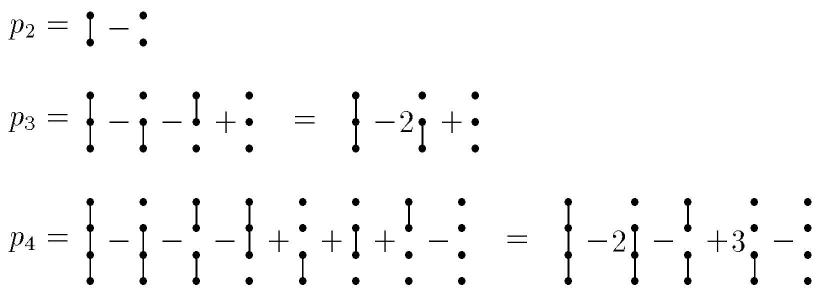


by a_n. Let J be a subset of edges of a_n, and $a_{n,J}$ be the forest obtained from a_n by deleting the edges from J. Then the element

$$p_n = \sum_J (-1)^{|J|} a_{n,J}$$

(sum over all subsets, including the empty one) is a primitive element in $\mathcal{A}_n$.

For example,



We postpone the proof of Proposition 2 until section 2.1. In section 3.3 we will show that $p_n \neq 0$ because the corresponding element in the algebra of weighted graphs $\mathcal{W} \cong \mathcal{A}$ is non-zero.

Corollary. *Either of the two series a_1, a_2, ..., and p_1, p_2, ... is a system of generators of the algebra $\mathcal{A}$. They are expressed through each other by the Newton-type formulas*

$$\begin{aligned} p_n &= a_n - \sum_{i+j=n} a_i a_j + \sum_{i+j+k=n} a_i a_j a_k - \dots \\ a_n &= p_n + \sum_{i+j=n} p_i p_j + \sum_{i+j+k=n} p_i p_j p_k + \dots \end{aligned}$$

Proof. The first formula is just another way to write the formula of Proposition 2. The second one easily follows by induction.

Remark. The main theorem of [CDL2] allows to give a description of the forest algebra $\mathcal{A}$ in terms of graphs. Its component $\mathcal{A}_n$ is generated by all forests with n vertices viewed modulo relations (1). The *multiplication* in $\mathcal{A}$ is the disjoint union of forests. The *comultiplication* in $\mathcal{A}$ consists in the following. Given a forest F, choose a subset J (possibly empty) of vertices of F. Let F_J be the subgraph of F containing the vertices from J and all edges of F that connect these vertices. Then

$$\delta(F) = \sum_J F_J \otimes F_{J'},$$

where J' is the complement of J in the set of all vertices of F. The *unity* in $\mathcal{A}$ is the empty graph. We will denote it by $\mathbf{1}$.

Identifying the primitive element $p_n \in \mathcal{A}_n$ with the Young diagram

$$\underbrace{\square\square\square\cdots\square\square\square}_{n\ \text{cells}}$$

we obtain an isomorphism of the forest algebra $\mathcal{A}$ with the Hopf algebra of Young diagrams (see [Gei], [Zel]).

1.2 Hopf algebra of weighted graphs

Assigning to every chord diagram its intersection graph, $D \mapsto \Gamma(D)$, we obtain a mapping Γ from the set of all chord diagrams to the set of graphs.

We have seen that in the module generated by all chord diagrams modulo the four term relations, there is a natural structure of a Hopf algebra, and this structure is important in the study of Vassiliev invariants.

It is thus a natural idea to define a structure of Hopf algebra in the module spanned by all graphs in such a way that the mapping Γ would become a homomorphism of bialgebras.

Now, there are obvious operations on graphs that mimic the multiplication and comultiplication of chord diagrams. The product of two graphs is just their disjoint union, while the coproduct of a graph G is obtained as the sum of all terms $G' \otimes G''$, where G' and G'' are two full subgraphs of G whose vertex sets constitute a splitting of the vertex set of G. This makes the free module generated by all graphs into a Hopf algebra $\mathcal{G}$.

Of course, the bare mapping Γ defines no algebra homomorphism because it is not compatible with the four term relations. To obtain a correctly defined homomorphism, we have to replace the target algebra $\mathcal{G}$ by a quotient $\mathcal{G}/S$, where S contains S_0, the ideal generated by the images of all four term relations. The problem is to find an ideal $S \supset S_0$ which allows a lucid description in terms of graphs and which is small enough so that the quotient $\mathcal{G}/S$ remains substantial.

One of the possible solutions is to consider the submodule in $\mathcal{G}$ generated by all *chromatic relations* (these are, up to a change of sign, relations satisfied by the chromatic polynomial, see below). It turns out, however, that in this case the remaining quotient is too small (as a Hopf algebra, it is generated by one primitive element). The reason of this phenomenon is that the chromatic relation is not compatible with the standard grading in the graph algebra (by the number of vertices). Quite remarkably, a close relation, considered in a larger algebra — the one freely generated by *weighted graphs* — becomes homogeneous, and the graded structure of the free algebra descends to the quotient modulo these relations.

In what follows, we call this quotient algebra *the algebra of weighted graphs* and denote it by $\mathcal{W}$. The algebra of weighted graphs is an example of a more ample quotient algebra of $\mathcal{M}$, having 1 primitive generator in every degree.

Another, and probably more important application of the algebra $\mathcal{W}$ is that it yields a series of pre-invariants for chord diagrams (see [CDL1]) through the notion of *weighted chromatic invariants*. This construction is an analog of the classical theory of Tutte invariants (see [T]) adjusted for graphs without multiple edges and loops, but having weighted vertices.

Example. Let D be a chord diagram. Consider the mapping $\chi : D \mapsto \chi(\Gamma(D))$, where $\Gamma(D)$ is the intersection graph of the chord diagram D, and $\chi(\Gamma(D)) \in \mathbf{Z}[t]$ its chromatic polynomial. *The mapping χ determines a pre-invariant $\chi : \mathcal{M} \to \mathbf{Z}[t]$ with values in the ring of one-variable polynomials.*

1.3 Weighted graphs and chromatic relations

Definition. A *weighted graph* is a graph G without loops and multiple edges given together with a mapping $w : V(G) \to \mathbf{N}$ called *weight*. The weight assigns a positive integer to each vertex of the graph. The *weight $w(G)$ of the graph G* is the sum of weights of all its vertices, $w(G) = \sum_{v \in V(G)} w(v)$.

Usual graphs without loops and multiple edges can be treated as weighted graphs with the weights of all vertices equal to 1.

We will use two natural operations on weighted graphs with a distinguished edge: *deletion* and *contraction*.

If e is an edge of the graph G, then the new graph G'_e is obtained from G by removing the edge e. The weights of the vertices do not change. This is what we call *deletion*.

The *contraction* G''_e of an edge e is defined as follows:

1) The edge e is contracted into a vertex v of the new graph G''_e;

2) If multiple edges arise they are replaced by unique edges;

3) The weight $w(v)$ of the vertex v is set to be equal to the sum of weights of the two ends of the edge e in G; weights of other vertices do not change.

Definiton. The *weighted chromatic relation* is the relation

$$G - G'_e - G''_e = 0,$$

where G is an arbitrary weighted graph and e its arbitrary edge.

We set $\mathcal{W}_n$ to be the $\mathbf{Z}$-module generated by all weighted graphs of weight n modulo all weighted chromatic relations. We set $\mathcal{W} = \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \ldots$.

Definition. The *Hopf algebra of weighted graphs* is the module $\mathcal{W}$ over $\mathbf{Z}$ with the following operations of multiplication and comultiplication:

1) multiplication

$$\mu : \mathcal{W}_n \otimes \mathcal{W}_m \to \mathcal{W}_{m+n}$$

comes from the disjoint union of graphs;

2) comultiplication

$$\delta : \mathcal{W}_n \to \mathcal{W}_0 \otimes \mathcal{W}_n \ \oplus \ \mathcal{W}_1 \otimes \mathcal{W}_{n-1} \ \oplus \ \dots \ \oplus \ \mathcal{W}_n \otimes \mathcal{W}_0$$

is defined on generators as follows. Consider a weighted graph G; let $J \subset V(G)$ be a subset in the vertices of G and $J' = V(G) \setminus J$ its complement. We set

$$\delta([G]) = [\sum_J G_J \otimes G_{J'}].$$

The sum is taken over all subsets J, and G_J is the full subgraph of G with vertices in J;

3) unity is represented by the empty graph;

4) counity is the coefficient of the empty graph in the canonical expansion of an element.

We omit a purely technical verification of the correctness of this definition.

Theorem 2. *The subgroup of primitive elements of $\mathcal{W}_n$ is isomorphic to $\mathbf{Z}$ and freely generated by the graph $\textcircled{n}$ with one vertex of weight n. The mapping*

$$C : \textcircled{n} \mapsto s_n$$

can be prolonged to the mapping from the set of all weighted graphs to the ring of polynomials in variables $s_1, s_2, \dots$. This prolongation descends to an isomorphism between the Hopf algebras $\mathcal{W}$ and $\mathbf{Z}[s_1, s_2, \dots]$, where the weight of every variable s_n is set to n.

Note. Theorem 2 gives a description of all weighted chromatic invariants, because the prolongation of C is just a universal weigted chromatical invariant. This means that an arbitrary weighted chromatic invariant can be obtained from C by substituting certain appropriate values for the variables s_k. For example, the conventional chromatic polynomial χ is obtained by the substitution $s_k = (-1)^{k-1}t$. Thus, the chromatic polynomial χ can be prolonged from the set of graphs without loops and multiple edges to a weigted chromatical polynomial on the set of all weighted graphs. Let $w\chi$ be the prolongation. Then

$$w\chi(G) = (-1)^{w(G)-|V(G)|}\chi(\bar{G}),$$

where $\bar{G}$ is obtained from G by forgetting the weights of the vertices. Note that not every classical Tutte polynomial [T] can be prolonged to a weigted chromatical invariant on the set of weighted graphs. For example, the Tutte dichromate has no such prolongation.

1.4 Constructing Vassiliev invariants from weighted chromatic invariants

Definition. A function f defined on the set of weighted graphs of weight n and satisfying the weighted chromatic relation

$$f(G) - f(G') - f(G'') = 0$$

will be called a *weighted chromatic invariant of order* n.

In other words, a weighted chromatic invariant of order n is just an element of $\mathcal{W}_n^*$.

Let D be a chord diagram. Its intersection graph $\Gamma(D)$ can be considered as a weighted graph with all weights set to 1. This yields a homomorphism of free $\mathbf{Z}$-modules.

Theorem 3. *The mapping* Γ *descends to a homomorphism of Hopf algebras* $\mathcal{M} \to \mathcal{W}$.

Corollary. *The dual mapping* $\Gamma^* : \mathcal{W}^* \to \mathcal{M}^*$ *determines a Hopf subalgebra of chromatic pre-invariants in the algebra of pre-invariants.*

1.5 Relation between forests and weighted graphs

Theorem 4. *The composition map*

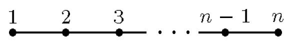
$$\mathcal{A} \to \mathcal{M} \to \mathcal{W},$$

where the first arrow is the natural inclusion and the second arrow is defined by intersection graphs, is an isomorphism of Hopf algebras. The image of the primitive element p_n is the graph n .

2 Proofs of statements

2.1 Proof of Proposition 2

Let us label the vertices of a_n consecutively:



This labelling induces the labelling of each summand $a_{n,J}$ of the formula

$$p_n = \sum_J (-1)^{|J|} a_{n,J}.$$

We have to prove that

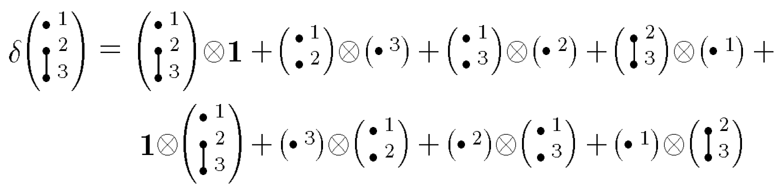
$$\delta(p_n) = p_n \otimes \mathbf{1} + \mathbf{1} \otimes p_n.$$

Note that

$$\delta(a_{n,J}) = \sum_{n'+n''=n} a_{n',J'} \otimes a_{n'',J''}.$$

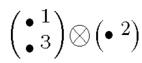
Moreover, each graph $a_{n',J'}$ and $a_{n'',J''}$ that appears in this formula has a natural labelling of its vertices induced by the labelling of $a_{n,J}$.

For example,



So, $\delta(p_n)$ is a sum of terms $a_{n',J'} \otimes a_{n'',J''}$ with coefficients ± 1, where the vertices of graphs $a_{n',J'}$ and $a_{n'',J''}$ are labelled by the indices $\{1, 2, \dots, n\}$.

Now fix a labelled monomial in the tensor product, for example



and let us look at all the summands $a_{n,J}$ which contribute a term $a_{n',J'} \otimes a_{n'',J''}$ equal to this labelled monomial.

A pair (v', v'') of vertices of the graphs $a_{n',J'}$ and $a_{n'',J''}$ will be called a *boundary* if v' and v'' are labelled by consecutive integers. Let B be the set of all boundary pairs. In our example $B = \{(1,2);(3,2)\}$. Let J''' be a subset of B (possibly empty). We will build the graph a from the graphs $a_{n',J'}$ and $a_{n'',J''}$ by adding certain edges that connect the vertices of the boundary pairs from J'''.

It is obvious that any graph $a_{n,J}$ such that the term $a_{n',J'} \otimes a_{n'',J''}$ appears in $\delta(a_{n,J})$ may be constructed as graph a for some J'''.

In our example, J''' is one of the four sets:

$$\emptyset; \qquad \{(1,2)\}; \qquad \{(3,2)\}; \qquad \{(1,2);(3,2)\}$$

So the term

$$\left(\begin{smallmatrix}\bullet\, 1\\ \bullet\, 3\end{smallmatrix}\right) \otimes \left(\bullet\, 2\right)$$

comes from the following graphs

Now suppose that the graph $a_{n,J}$ is constructed from the graphs $a_{n',J'}$ and $a_{n'',J''}$ with the help of some subset $J''' \subseteq B$. Then the sign of the graph $a_{n,J}$ in the formula

$$p_n = \sum_J (-1)^{|J|} a_{n,J}.$$

is equal to $(-1)^{1+|J'|+|J''|-|J'''|}$. So the term $a_{n',J'} \otimes a_{n'',J''}$ occurs in $\delta(p_n)$ with the coefficient

$$\sum_{J''' \subseteq B} (-1)^{1+|J'|+|J''|-|J'''|}$$

This coefficient is equal to zero due to the following fact: *The number of subsets of a given set B with an even number of elements is equal to the number of its subsets with an odd number of elements.*

This fact has the only exception, viz. when B is empty. This exception corresponds to the two terms $p_n \otimes \mathbf{1}$ and $\mathbf{1} \otimes p_n$ of $\delta(p_n)$ that do not cancel. This completes the proof of Proposition 2.

2.2 Proof of Theorem 2

First of all we give a more invariant definition for the mapping C extended to the set of all weighted graphs. Put

$$C(G) = \sum_{\gamma}(-1)^{b_1(\gamma)} \prod_{\gamma_i} s_{w(\gamma_i)},$$

where the sum is taken over all spanning subgraphs γ of the graph G and the product is taken over all connected components γ_i of γ. (A *spanning* subgraph is a subgraph whose set of vertices coincides with that of the whole graph. Thus, the spanning subgraphs of a graph G are in a one-to-one correspondence with the subsets in the set of edges of G).

Now we prolong C by linearity to the Z-module freely generated by all weighted graphs. We claim that this mapping descends to the quotient algebra $\mathcal{W}$ and defines an isomorphism between $\mathcal{W}$ and $\mathbf{Z}[s_1, s_2, \ldots]$ (also denoted by C in the sequel).

We have to check four things about the mapping C: 1) that it satisfies the weighted chromatic relation, 2) that it is multiplicative, 3) that it agrees with the grading in $\mathcal{W}$ and in the polynomial ring, where the degree of every x_k is k, and 4) that it is one-to-one.

1) Let G be an arbitrary weighted graph and e an arbitrary edge of G. We are going to prove that

$$C(G) = C(G'_e) + C(G''_e).$$

There exists a natural one-to-one correspondence between subgraphs of G'_e and those subgraphs of the graph G that do not contain the edge e. Thus we have

$$\sum_{\gamma'_e} t(\gamma'_e) = C(G'_e),$$

where the sum in the left hand side is taken over all spanning subgraphs γ'_e of the graph G that do not contain the edge e and $t(\gamma'_e)$ denotes the corresponding product.

We are going to prove that the other part of the sum for $C(G)$ is equal to $C(G''_e)$. Let γ'' be a spanning subgraph of the graph G''_e. Let b be an edge of the graph γ'' which is twice covered by the edges of graph G in the process of contraction. The pre-image of b under the contraction thus consists of two edges that form a triangle together with the edge e. A spanning subgraph γ of G which is contracted to γ'' and contains the edge e may either contain any of the two preimages of the edge b or both of them. So the spanning subgraph γ'' of G''_e corresponds to 3^k spanning subgraphs of G containing e, where k is the number of twice covered edges in γ''.

Mark an edge in each triangle in G that contains e. We have three possibilities for the preimage of each twice covered edge. Two of these possibilities correspond to one and the same first Betti number of the covering graph. The third one gives a graph whose first Betti number differs from this value by one. Thus, the products for two of the three possibilities mutually cancel, and we obtain a one-to-one correspondence between spanning subgraphs of G''_e and spanning subgraphs of G that contain e and a marked edge in each triangle containing e. So we have

$$\sum_{\gamma_e} t(\gamma_e) = C(G''_e),$$

where the sum is taken over all spanning subgraphs of G containing the edge e.

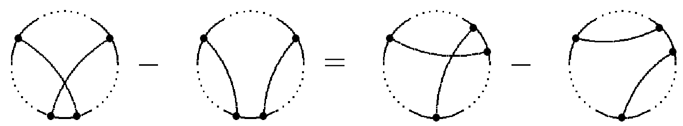
2) Multiplicativity of C follows from a straightforward direct calculation.

3) This is evident, if the grading of every s_k is set to k.

4) Since the polynomial algebra is free, the assignment $s_n \mapsto ⓝ$ defines a homomorphism $\mathbf{Z}[s_1, s_2, s_3, \ldots] \to \mathcal{W}$ which is obviously inverse to C.

2.3 Proof of Theorem 3

We have to prove that the mapping Γ satisfies the four-term relation. Write this relation in the form:



Denote the intersection graph of the first term in the left hand side by G_1. It contains two vertices A and B_1 corresponding to the two selected chords. These vertices are connected by an edge. The intersection graph of the second term is obtained from G_1 by deleting the edge AB_1. We denote this graph by G'_1.

In the algebra $\mathcal{W}$ the following relation holds:

$$G_1 - G_1' = G_1'',$$

where G_1'' is the result of contracting the edge AB_1 in G_1.

In the right hand side we similarly have:

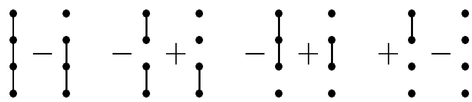
$$G_2 - G_2' = G_2'',$$

where G_2'' is obtained by contracting the edge AB_2 in G_2.

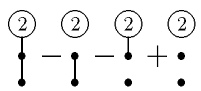
To accomplish the proof, it is sufficient to show that the two weighted graphs G_1'' and G_2'' coincide. In fact, there exists a natural one-to-one correspondence between the two sets of vertices $V(G_1'')$ and $V(G_2'')$. This correspondence preserves the weight function (the weight is equal to 1 for every vertex but one, for which it equals two). The edges of the two graphs are also in a one-to-one correspondence. Two chords different from A, B_1, B_2 intersect in G_1'' if and only if they intersect in G_2''. Suppose a chord C in G_1 intersects B_1 but not intersect B_2 in G_2; this means that C intersects A in both diagrams. Thus, the vertex C in both graphs G_1'' and G_2'' is connected to the new vertex by an edge, and we obtain a natural one-to-one correspondence between the edges of graphs G_1'' and G_2'' as well.

2.4 Proof of Theorem 4

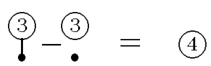
Split all graphs in the expansion of p_n in pairs in such a way that the two graphs in each pair differ only by one upper edge. For example, the pairing for p_4 is:

$p_4 =$ 

The difference of graphs in every pair is a graph with one vertex of weight 2. In our example:



We obtain a linear combination of graphs that looks like the corresponding expansion for p_{n-1} with the only difference that here the upper vertex is of weight 2. Repeating the process, we obtain the graph with one vertex of weight n. In our example



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