The Kontsevich Integral

S. Chmutov[*], S. Duzhin[*†]

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Abstract

The paper contains a detailed exposition of the construction and properties of the Kontsevich integral invariant, crucial in the study of Vassiliev knot invariants.

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This expository paper grew out of several lectures given by the authors in different places (Moscow and Pereslavl-Zalessky, Russia; Pisa, Italy; Bath, UK; Aizu-Wakamatsu, Japan; Utrecht, Netherlands). Its purpose is to explain the famous contribution of Maxim Kontsevich to Vassiliev knot invariant theory. More specifically, our aim is to present a proof of Kontsevich's fundamental theorem, which states that for each weight system there exists a Vassiliev invariant whose symbol is exactly the given weight system, or, in other words, that the only relations in the graded algebra of Vassiliev knot invariants are the 1- and 4-term relations.

Recently, several papers dedicated to the Kontsevich integral have appeared: see [AF, BNS, Be, Hu] etc. We hope, however, that our text will be useful because we try to present a self-contained exposition clarifying some vague, but important points left in the shadow by previous authors.

In our approach, we mostly build on D. Bar-Natan's paper [BN1], the only essential difference being that in the proof of the main (Kontsevich's) theorem we follow the original paper [Ko1] using the plain Stokes theorem instead of the flatness of the Knizhnik–Zamolodchikov connection.

The paper consists of three sections. In Section 1 we give an introduction to Vassiliev invariant theory following [CD, CDL1]. Section 2 is about the Kontsevich integral: it contains Kontsevich's formula with comments, the proof of the invariance properties, the construction of the universal Vassiliev invariant via Kontsevich's integral and examples of computation. Finally, in Section 3 we discuss three operations on knots — mutation and changes of orientation of the ambient space and of the knot — in their relation to the Kontsevich integral; as an application, we give a counterexample to our intersection graph conjecture [CDL1] constructed by T. Q. T. Le using mutations.

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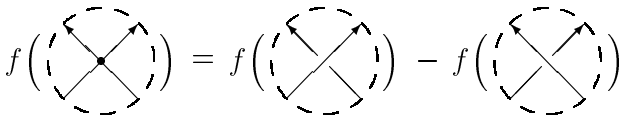
1 Introduction to Vassiliev's theory

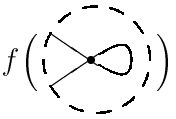
The ingenious idea of V. Vassiliev [V] was to prolong knot invariants to singular knots according to the rule (*Vassiliev's skein relation*)



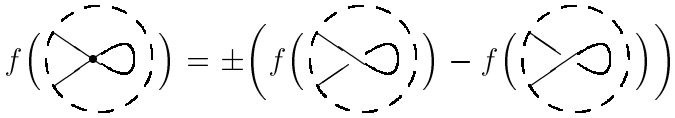
where f is a knot invariant initially defined only on nonsingular knots.

By a singular knot we understand a knot which may have a finite number of simple double points as singularities. We will mainly concern ourselves with $\mathbb{C}$-valued knot invariants, although many results carry over to any field of characteristic zero.

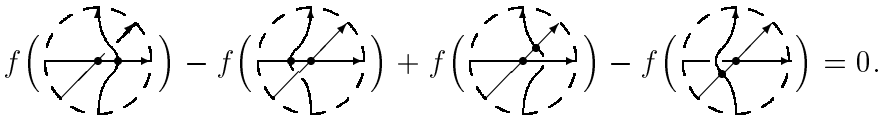
The prolongation of any knot invariant f satisfies two important sets of relations.

One-term relation for knots:  $= 0$.

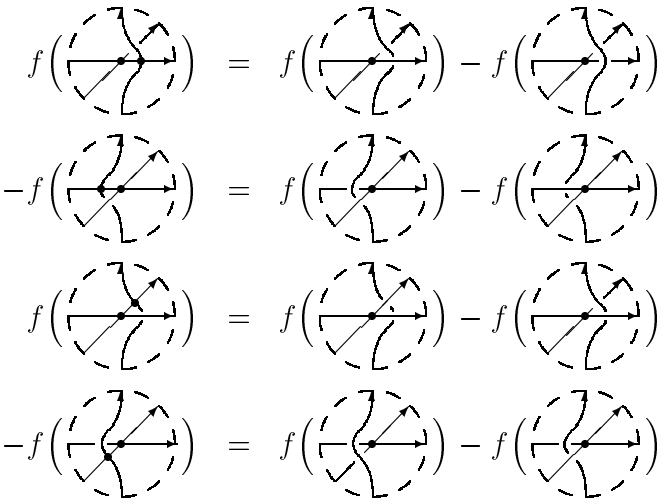
Indeed, two different resolutions of this double point yield one and the same knot:



Four-term relation for knots:



Indeed, resolving one of the two double points, we get:



In the right-hand parts of these equalities, every term appears twice with the opposite signs, so that everything cancels.

1.1 Vassiliev invariants

Definition. A *Vassiliev invariant* of order no greater than n is a knot invariant whose prolongation vanishes on any knot with more than n double points.

Remark 1. V. Vassiliev's original constructions were simplified and clarified by J. Birman and X.-S. Lin in [BL].

Remark 2. A theory of finite type invariants parallel to Vassiliev's theory was independently developed by M. Goussarov [G].

The set of all Vassiliev invariants of order no greater than n is a vector space. Denoting this space by $\mathcal{V}_n$, we obtain a filtration

$$\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \cdots \subseteq \mathcal{V}_n \subseteq \cdots \subset \mathcal{V}$$

in the space $\mathcal{V}$ of Vassiliev invariants of finite order. Using this filtration, we can study a more simple graded space

$$\operatorname{gr}\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1/\mathcal{V}_0 \oplus \mathcal{V}_2/\mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_n/\mathcal{V}_{n-1} \oplus \ldots$$

The structure of each component $\mathcal{V}_n/\mathcal{V}_{n-1}$ can be best understood in terms of *chord diagrams*.

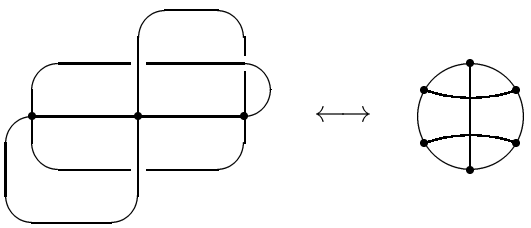
Definition. A *chord diagram* of order n is a circle with a distinguished set of n unordered pairs of points, regarded up to an orientation preserving diffeomorphism of the circle.

A chord diagram may be depicted as a circle with a set of n chords all of whose endpoints are distinct. The chords are drawn as curves whose actual shape is irrelevant; what matters is the way they bind their endpoints into pairs. A chord diagram of order n is thus the same thing as a cyclic word in n letters in which each letter appears twice. The use of diagrams is preferable because of their visual convenience.

Example. The total number of chord diagrams of order 1, 2 and 3 is 1, 2 and 5, respectively. We suggest that the reader who has never done it before should draw all of them now as an exercise, before proceeding with this text.

Definition. The *chord diagram of a singular knot* $S^1 \to \mathbb{R}^3$ is the oriented circle S^1 with the preimages of every double point connected by a chord.

Example. Here is a singular knot and its chord diagram:



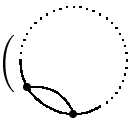
Let F_n be the linear space of all $\mathbb{C}$-valued functions on the set of chord diagrams of order n.

Definition. The *symbol* of a Vassiliev invariant $f \in \mathcal{V}_n$ is the function $[f] \in F_n$ defined by $[f](D) = f(K)$ where K is an arbitrary knot whose chord diagram is D.

The fact $f(K)$ depends only on the chord diagram of K can be checked as follows (see [CDL1] for details). If K_1 and K_2 are two knots with the same chord diagram, then they can be conventionally drawn in the plane in such a way that the pictures coincide everywhere except for a finite number of crossing points in which K_1 may have an undercrossing and K_2 an overcrossing or vice versa. But a flip between an undercrossing and an overcrossing does not change the value of $f(K)$ due to Vassiliev's skein relation.

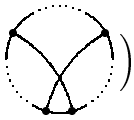
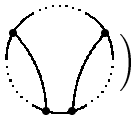
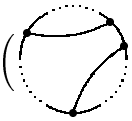
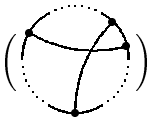
By definition, the kernel of the mapping $\mathcal{V}_n \to F_n : f \mapsto [f]$ coincides with the subspace $\mathcal{V}_{n-1} \subset \mathcal{V}_n$. We thus obtain an inclusion $\iota_n : \mathcal{V}_n/\mathcal{V}_{n-1} \to F_n$. What is the image of ι_n, or, in other words, what are the conditions that distinguish the symbols of Vassiliev invariants among all functions on the set of chord diagrams?

It turns out that any function that belongs to the image of ι_n satisfies the following two sets of relations:

One-term relation: $w\Big($  $\Big) = 0.$

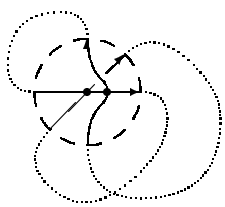
Here and below, dotted arcs suggest that there might be further chords attached to their points, while on the solid portions of the circle all the endpoints are explicitly shown.

Four-term relation:

$w\Big($  $\Big) - w\Big($  $\Big) + w\Big($  $\Big) - w\Big($  $\Big) \quad = \quad 0\ ,$

for an arbitrary fixed position of $(n-2)$ chords (which are not drawn here) and the two additional chords positioned as shown in the picture.

These two facts easily follow from the corresponding relations for knots. Indeed, the chord diagram of the one-term relation can be presented by a singular knot with a small curl like in the one-term relation for knots. (Remind that we have the freedom to choose the knot with the given chord diagram arbitrarily). The first diagram appearing in the four-term relation can be represented by the knot



which has a fragment shown above in the four-term relation for knots. The three remaining knots of that relation will produce the other diagrams in question.

1.2 Kontsevich's theorem

This is the main theorem of the article. It says that one- and four-term relations are the only relations for symbols of Vassiliev invariants.

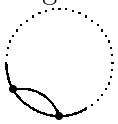
Definition. A *weight system* is a function on chord diagrams satisfying one- and four-term relations.

Let $W_n \subset F_n$ denote the space of all weight systems of order n. In the previous section we have seen that symbols of Vassiliev invariants are weight systems.

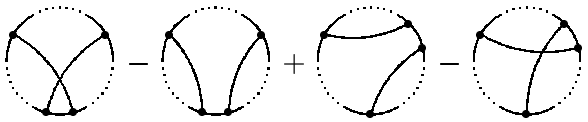
Kontsevich's theorem [Ko1]. *Each weight system is the symbol of an appropriate Vassiliev invariant.*

The theorem will be proved in Sec. 2.8. It gives a purely combinatorial description of the graded space $\operatorname{gr} \mathcal{V}$ as the space of weight systems.

For every n the space of weight systems W_n can be represented as the dual of the linear space $\mathcal{A}_n$ generated by chord diagrams of order n considered modulo

one-term relations:  $= \quad 0 \qquad$ and

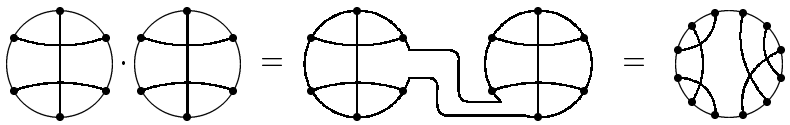
four-term relations:



The direct sum

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \dots$$

is in fact a graded algebra. The multiplication defined by the connected sum of chord diagrams:



is well-defined thanks to four-term relations. In fact $\mathcal{A}$ is even a Hopf algebra but we will not need the comultiplication here.

The proof of Kontsevich's theorem is based on the construction of the so called universal Vassiliev invariant. It is a knot invariant $I = \{I_n\}$ with values in the graded completion $\overline{\mathcal{A}}$ of the algebra $\mathcal{A}$ of chord diagrams with the following property: *for any weight system $w \in W_n$ the value of w on $I_n(K)$ is a Vassiliev invariant of K whose symbol is exactly w.* (By the *graded completion* we understand an extension of the graded algebra which, apart from the proper elements of the algebra, also contains series with an infinite number of nonzero terms.)

The main tool to construct the universal Vassiliev invariant is the famous Kontsevich integral described in the next section.

2 The Kontsevich integral

Represent the three-space $\mathbb{R}^3$ as a direct product of a complex line $\mathbb{C}$ with coordinate z and a real line $\mathbb{R}$ with coordinate t. Consider a knot K embedded in $\mathbb{R}^3 = \mathbb{C}_z \times \mathbb{R}_t$ in such a way that the coordinate t is a Morse function on K (all critical points are nondegenerate and all critical levels are different).

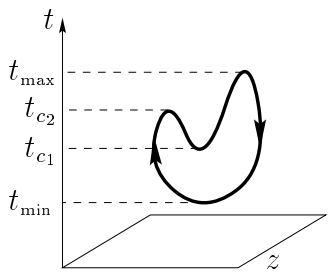
Definition. The *Kontsevich integral* of K is the following element of the graded completion $\overline{\mathcal{A}}$ of the algebra of chord diagrams:

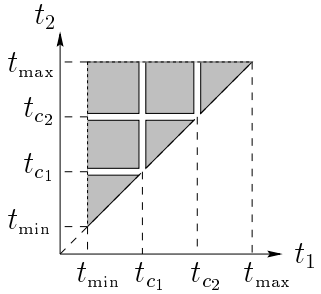
$$Z(K) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int\limits_{\substack{t_{\min}<t_1<\ldots<t_m<t_{\max} \\ t_j \text{ are noncritical}}} \sum_{P=\{(z_j,z'_j)\}} (-1)^{\downarrow} D_P \bigwedge_{j=1}^{m} \frac{dz_j - dz'_j}{z_j - z'_j} \ .$$

2.1 Who is who in the Kontsevich integral

The real numbers $t_{\min}$ and $t_{\max}$ are the minimum and the maximum of the function t on K.

The integration domain is the m-dimensional simplex $t_{\min} < t_1 < \ldots < t_m < t_{\max}$ divided by the critical values into a certain number of *connected components*. For example, for the following embedding of the unknot and $m = 2$ the corresponding integration domain has six connected components and looks like

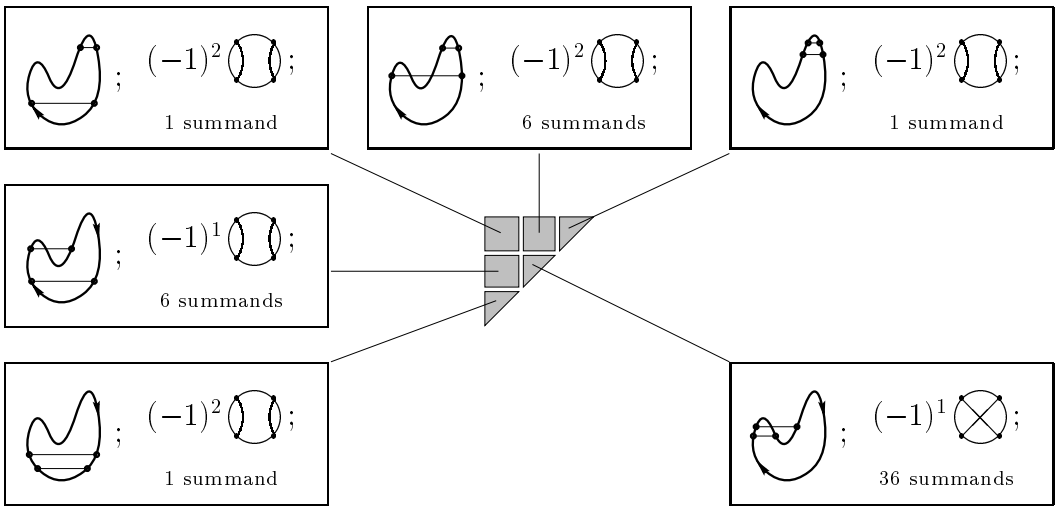




The number of summands in the integrand is constant in each connected component of the integration domain, but can be different for different components. In each plane $\{t = t_j\} \subset \mathbb{R}^3$ choose an unordered pair of distinct points (z_j, t_j) and (z'_j, t_j) on K, so that $z_j(t_j)$ and $z'_j(t_j)$ are continuous functions. We denote by $P = \{(z_j, z'_j)\}$ the set of such pairs for $j = 1, \ldots, m$. The integrand is the sum over all choices of P. In the example above for the component $\{t_{\min} < t_1 < t_{c_1},\ t_{c_2} < t_2 < t_{\max}\}$ we have only one possible pair of points on the levels $\{t = t_1\}$ and $\{t = t_2\}$. Therefore, the sum over P for this component consists of only one summand. Unlike this, in the component $\{t_{\min} < t_1 < t_{c_1},\ t_{c_1} < t_2 < t_{c_2}\}$ we still have only one possibility for the level $\{t = t_1\}$, but the plane $\{t = t_2\}$ intersects our knot K in four points. So we have $\binom{4}{2} = 6$ possible pairs (z_2, z'_2) and the total number of summands is six (see the picture below).

For a pairing P the symbol '$\downarrow$' denotes the number of points (z_j, t_j) or (z'_j, t_j) in P where the coordinate t decreases along the orientation of K.

Fix a pairing P. Consider our knot K as an oriented circle and connect the points (z_j, t_j) and (z'_j, t_j) by a chord. We obtain a chord diagram with m chords. The corresponding element of the algebra $\mathcal{A}$ is denoted by D_P. In the picture below, for each connected component in our example, we show one of the possible pairings, the corresponding chord diagram with the sign $(-1)^{\downarrow}$ and the number of summands of the integrand (some of which are equal to zero in $\mathcal{A}$ due to one-term relation; see the discussion in Sec. 2.6).



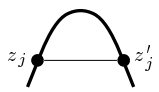
Over each connected component z_j and z'_j are smooth functions in t_j. By $\bigwedge_{j=1}^{m} \dfrac{dz_j - dz'_j}{z_j - z'_j}$ we mean the pullback of this form to the integration domain of variables $t_1, \ldots, t_m$. The integration domain is considered with the positive orientation of the space $\mathbb{R}^m$ defined by the natural order of the coordinates t_1, ..., t_m.

By convention, the term in the Kontsevich integral corresponding to $m = 0$ is the (only) chord diagram of order 0 with coefficient one. It represents the unit of the algebra $\mathcal{A}$.

2.2 Convergence of the integral

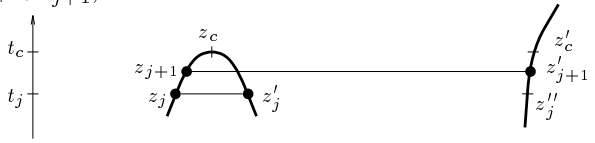
The integrand of the Kontsevich integral may have singularities near the boundaries of the connected components. This happens only in the following two cases.

1) An isolated chord (z_j, z'_j) tends to zero:



In this case the corresponding chord diagram D_P is equal to zero in $\mathcal{A}$ by the one-term relation.

2) A chord (z_j, z'_j) tends to zero near a critical point but is separated from that point by another, 'long' chord (z_{j+1}, z'_{j+1}):



In this case the integral for $(j+1)$-th chord can be estimated as follows:

$$\left| \int_{t_j}^{t_c} \frac{dz_{j+1} - dz'_{j+1}}{z_{j+1} - z'_{j+1}} \right| \leq C \left| \int_{t_j}^{t_c} d(z_{j+1} - z'_{j+1}) \right| = C \left| (z_c - z_j) - (z'_c - z''_j) \right| \leq C' |z_j - z'_j|$$

for some positive constants C and C'. So the integral for $(j+1)$-th chord is as small as $z_j - z'_j$ and this smallness compensates the smallness of the denominator corresponding to j-th chord.

This implies the convergence of the Kontsevich integral.

2.3 Invariance of the integral

Theorem. *The Kontsevich integral is invariant under the deformations in the class of knots all of whose critical points are nondegenerate .*

A deformation is a smooth family of diffeomorphisms of $\mathbb{R}^3$ which starts with the identity.

Any deformation that keeps a given Morse knot within the class of knots with nondegenerate critical points and hence preserves the number of critical points can be decomposed into *horizontal deformations* and *movements of critical points*.

A *horizontal deformation* is a deformation of $\mathbb{R}^3$ which preserves all horizontal planes $\{t = \text{const}\}$ and leaves all the critical points fixed. The invariance under horizontal deformations is the most essential point of the theory. We prove it in the next subsection.

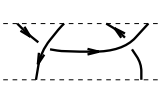
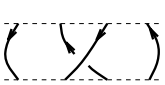
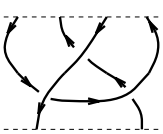
A *movement of the critical point* C is a deformation which is identical everywhere outside a small neighbourhood of C and does not introduce new critical points on the knot. In subsection 2.5 we consider invariance under the movements of critical points.

Unfortunately, the Kontsevich integral is not invariant under deformations that change the number of critical points. However, there is a formula that shows how the integral changes under such deformations, and this will lead to the construction of the universal Vassiliev invariant in subsection 2.7.

2.4 Invariance under horizontal deformations

To prove the horizontal invariance of the Kontsevich integral, we will use tangles and their properties.

Horizontal planes cut the knot into tangles. A *tangle* is a one-dimensional compact submanifold of the slice $[a, b] \times \mathbb{C} \subset \mathbb{R}^3$ whose boundary points belong to the boundary of the slice, considered up to an isotopy of the slice, identical on the boundary. We also assume that in the points where the tangle meets the boundary planes (the *top* and the *bottom*), it is not tangent to these planes. In the case when the top of one tangle T_1 coincides with the bottom of another tangle T_2, one can define the product $T_1 \cdot T_2$ by putting T_2 on top of T_1:

$T_1 =$  ; $T_2 =$  ; $T_1 \cdot T_2 =$  .

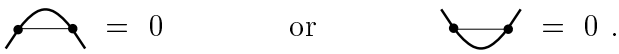
We will often enclose the tangle in a box by adding vertical sides or even in a ball depicted by a dashed disc as in one- and four-term relations for knots from the beginning of Sec. 1. This does not restrict generality, because tangles are compact.

Definition. A *tangled chord diagram* (*T-chord diagram*) is a tangle T supplied with a set of horizontal chords considered up to a diffeomorphism of the tangle that preserves the horizontal fibration. (The isotopy class of the embedding of the tangle into the slice of $\mathbb{R}^3$ is here irrelevant.)

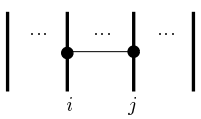
Multiplication of tangles induces a multiplication of tangled chord diagrams.

Definition. Let T be a tangle. The *space of T-chord diagrams* $\mathcal{A}_T$ is a vector space generated by all chord diagrams on T modulo the set of *tangled one- and four-term relations*.

Tangled one-term relation: A tangled chord diagram with an isolated chord is equal to zero in $\mathcal{A}_T$:



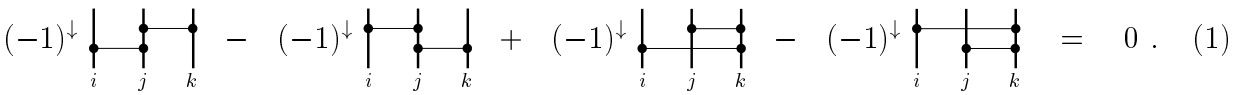
Tangled four-term relation: Denote by t_{ij} the following tangled chord diagram consisting of n parallel vertical strings and one chord between the i-th and j-th strings:

$t_{ij} = (-1)^{\downarrow}$ 

where $\downarrow$ stands for the number of endpoints of the chords in which the orientation is directed downwards. Then the tangled four-term relation reads:

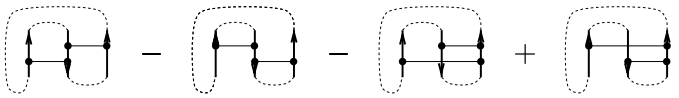
$$[t_{ij} + t_{ik}, t_{jk}] = 0$$

or, in pictures:

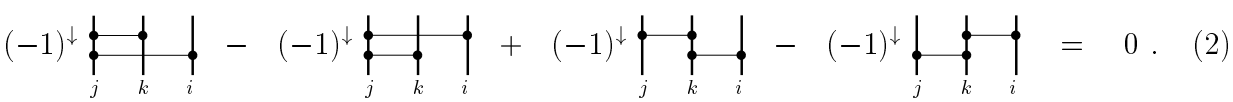


Exercise. If a tangle is closed into a circle in agreement with orientations of the strings, the tangled four-term relation carries over into the ordinary four-term relation of Sec. 1.2.

Here is an example:



Remark. Permuting the indices i, j, k we obtain another form of the tangled four-term relation:



If (top T_1) = (bottom T_2) then the multiplication of tangled chord diagrams descends to the multiplication of quotient spaces $\mathcal{A}_{T_1} \otimes \mathcal{A}_{T_2} \to \mathcal{A}_{T_1 \cdot T_2}$.

Now we can consider the Kontsevich integral for tangles. It is defined by the same formula only $t_{\min}$ must be the bottom and $t_{\max}$ the top of the tangle. The integral is multiplicative:

$$\boxed{Z(T_1) \cdot Z(T_2) = Z(T_1 \cdot T_2)}$$

whenever the product $T_1 \cdot T_2$ is defined. This important fact is a direct consequence of the Fubini theorem about multiple integrals. In particular it implies that the Kontsevich integral of the connected sum of knots is equal to the product (in sense of the algebra $\mathcal{A}$ of chord diagrams) of Kontsevich integrals of the summands.

Let us decompose the given knot into a product of tangles without critical points of the function t and (very thin) tangles containing the critical levels. Due to multiplicativity, it is enough to prove two facts:

1) the Kontsevich integral for a tangle without critical points is invariant under horizontal deformations that preserve the top and the bottom pointwise,

2) the Kontsevich integral for a tangle with a unique critical point is invariant under horizontal deformations that preserve the critical point.

These two facts are proved in the following two theorems.

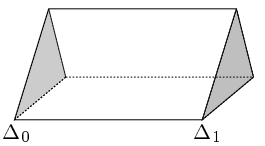
Theorem 1. Let T_0 be a tangle without critical points and T_λ, a horizontal deformation of T_0 to T_1 (preserving the top and the bottom of the tangle). Then $Z(T_0) = Z(T_1)$.

Proof. Denote by ω the integrand form in the m-th term of the Kontsevich integral:

$$\omega = \sum_{P=\{(z_j,z'_j)\}} (-1)^{\downarrow} D_P \quad \bigwedge_{j=1}^{m} \frac{dz_j - dz'_j}{z_j - z'_j} \ .$$

Our tangles do not contain the critical values of t; therefore the integration domain for $Z(T_\lambda)$ is the entire m-dimensional simplex $\Delta_\lambda = \{t_{\min} < t_1 < \ldots < t_m < t_{\max}\}$, the same for all values of λ. Consider the product of this simplex with the unit interval of λ:

$$\Delta = \Delta_0 \times I_\lambda = \quad .$$



By Stokes' theorem we have $\int\limits_{\partial\Delta} \omega \quad = \quad \int\limits_{\Delta} d\omega$.

The form ω is exact: $d\omega = 0$. The boundary of the integration domain is $\partial\Delta = \Delta_0 - \Delta_1 + \sum \{faces\}$. The theorem will follow from the fact that $\omega|_{\{face\}} = 0$. To show this consider two types of faces.

The first type corresponds to $t_j = t_{\min}$ or $t_j = t_{\max}$ for some j (of course, this j might be either 1 or m). In this situation $dz_j = dz'_j = 0$ because z_j and z'_j do not depend on λ.

The second type is that where we have $t_k = t_{k+1}$ for some k. In this case we have to choose the k-th and $(k+1)$-th chords on the same level $\{t = t_k\}$. In general, the endpoints of these chords may coincide and we don't get a chord diagram at all. Strictly speaking, we have to

define the prolongation of ω and D_P to such a face more carefully. It is natural to prolong D_P to this face as a locally constant function. This means that for the case in which some endpoints of k-th and $(k+1)$-th chords belong to the same string (and therefore coincide) we place k-th chord a little lower than $(k+1)$-th chord, so that its endpoint differs from the endpoint of $(k+1)$-th chord. This trick yields a well-defined prolongation of D_P and ω to the face, and we use it here.

All summands of ω are divided into three parts:

1) k-th and $(k+1)$-th chords connect the same two strings;

2) k-th and $(k+1)$-th chords are chosen in such a way that their endpoints belong to four different strings;

3) k-th and $(k+1)$-th chords are chosen in such a way that there exist exactly three different strings containing their endpoints.

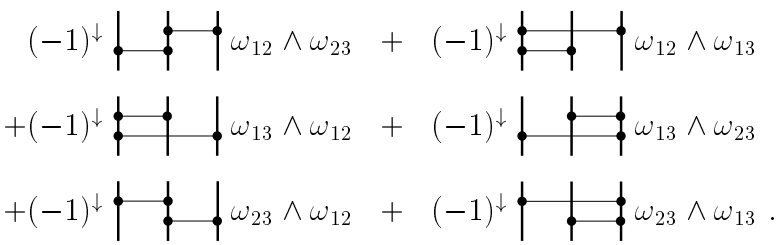
Consider all these cases one by one.

1) We have $z_k = z_{k+1}$ and $z'_k = z'_{k+1}$ or vice versa. So $d(z_k - z'_k) \wedge d(z_{k+1} - z'_{k+1}) = 0$ and therefore the restriction of ω to the face is zero.

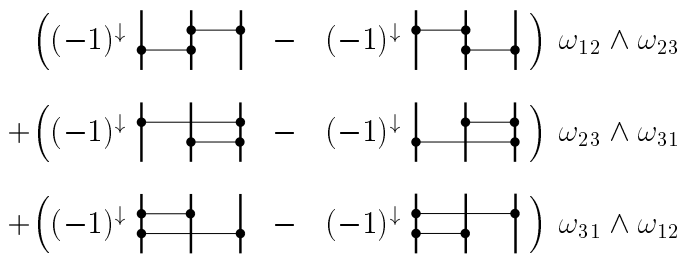
2) All choices of chords in this part of ω appear in mutually cancelling pairs. Fix four strings and number them by 1, 2, 3, 4. Suppose that for a certain choice of the pairing, the k-th chord connects the first two strings and $(k+1)$-th chord connects the last two strings. Then there exists another choice for which on the contrary the k-th chord connects the last two strings and $(k+1)$-th chord connects the first two strings. These two choices give two summands of ω differing by a sign:

$$\dots d(z_k - z'_k) \wedge d(z_{k+1} - z'_{k+1}) \dots + \dots d(z_{k+1} - z'_{k+1}) \wedge d(z_k - z'_k) \dots = 0.$$

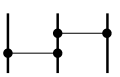
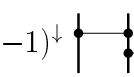
3) This is the most difficult case. There exist exactly one common string for the endpoints of k-th and $(k+1)$-th chords. Let the three strings be 1, 2, 3 and denote by ω_{ij} the form $\dfrac{dz_i - dz_j}{z_i - z_j}$. Then ω is the product of a certain $(m-2)$-form and the sum of the following six 2-forms:



Using the fact that $\omega_{ij} = \omega_{ji}$, we can rewrite this as follows:



The tangled four-term relations (1), (2) says that the expressions in parentheses are one and the same element of $\mathcal{A}_T$, hence the whole sum is equal to

$\Big((-1)^{\downarrow}$  $-\quad (-1)^{\downarrow}$  $\Big)\ (\omega_{12}\wedge\omega_{23}+\omega_{23}\wedge\omega_{31}+\omega_{31}\wedge\omega_{12}).$

The 2-form in parentheses is zero! This fact, known as *Arnold's identity*, can be verified directly:

$$\begin{aligned}
&\frac{dz_1-dz_2}{z_1-z_2}\wedge\frac{dz_2-dz_3}{z_2-z_3}\ +\ \frac{dz_2-dz_3}{z_2-z_3}\wedge\frac{dz_3-dz_1}{z_3-z_1}\ +\ \frac{dz_3-dz_1}{z_3-z_1}\wedge\frac{dz_1-dz_2}{z_1-z_2}\\
&\quad= \frac{dz_1\wedge dz_2}{(z_1-z_2)(z_2-z_3)}-\frac{dz_1\wedge dz_3}{(z_1-z_2)(z_2-z_3)}+\frac{dz_2\wedge dz_3}{(z_1-z_2)(z_2-z_3)}\\
&\qquad+\frac{dz_2\wedge dz_3}{(z_2-z_3)(z_3-z_1)}+\frac{dz_1\wedge dz_2}{(z_2-z_3)(z_3-z_1)}-\frac{dz_1\wedge dz_3}{(z_2-z_3)(z_3-z_1)}\\
&\qquad-\frac{dz_1\wedge dz_3}{(z_3-z_1)(z_1-z_2)}+\frac{dz_2\wedge dz_3}{(z_3-z_1)(z_1-z_2)}+\frac{dz_1\wedge dz_2}{(z_3-z_1)(z_1-z_2)}\\
&\quad= \left(\frac{1}{(z_1-z_2)(z_2-z_3)}+\frac{1}{(z_2-z_3)(z_3-z_1)}+\frac{1}{(z_3-z_1)(z_1-z_2)}\right)\\
&\qquad\qquad\times(dz_1\wedge dz_2-dz_1\wedge dz_3+dz_2\wedge dz_3)\quad=\quad 0\ .
\end{aligned}$$

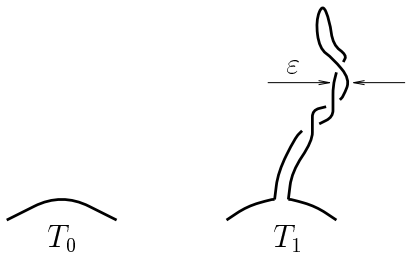
Theorem 1 is proved.

Theorem 2. Let T_0 and T_1 be two tangles, each with exactly one critical point, differing from each other by a horizontal deformation preserving the top and the bottom of the tangles and the critical point. Then $Z(T_0)=Z(T_1)$.

Proof. Assume for simplicity that the values of t at the bottom, the critical point and the top are -1, 0 and 1, respectively. Now consider a special horizontal deformation of T_0 into T_1 via a family of tangles T_λ, $0\le\lambda\le 1$, such that T_λ coincides with T_0 for $|t|<\lambda^2$ and coincides with T_0 for $|t|>\lambda$. Of course, we require that all T_λ be smooth tangles with only one critical point which does not depend on λ. By Theorem 1 we have that $Z(T_\lambda)=Z(T_0)$ for any λ between 0 and 1. But the limit of the tangle T_λ, as λ goes to 1, is T_1. This proves Theorem 2 and the horizontal invariance of the Kontsevich integral.

2.5 Moving the critical points

Let T_0 and T_1 be two tangles which are identical except a sharp tail of width ε, which may be twisted:



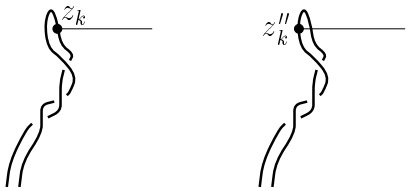
We assume that each level $\{t = \text{const}\}$ intersects the tail at most in two points and the distance between them is no greater than ε.

Theorem. $Z(T_0) = Z(T_1)$.

Proof. The difference between $Z(T_0)$ and $Z(T_1)$ can come only from terms with a chord ending on the tail.

If the highest of such chords connects the two sides of the tail, then the corresponding tangled chord diagram is zero by one-term relation.

So we can assume that the highest, k-th, chord is a 'long' chord, which means that it connects the tail with another part of T_1. Suppose the endpoint of the chord belonging to the tail is (z_k, t_k). Then there exists another choice for k-th chord which is almost the same but touches the tail at another point (z_k'', t_k) of the same horizontal level:



The corresponding two terms appear in $Z(T_1)$ with the opposite signs due to $(-1)^{\downarrow}$.

Let us estimate the difference of the integrals corresponding to such k-th chords:

$$\left| \int\limits_{t_{k-1}}^{t_c} d(\ln(z_k - z_k')) \; - \int\limits_{t_{k-1}}^{t_c} d(\ln(z_k'' - z_k')) \right| \;=\; \left| \ln\Big(\frac{z_{k-1}'' - z_{k-1}'}{z_{k-1} - z_{k-1}'}\Big) \right|$$

$$= \left| \ln\Big(1 + \frac{z_{k-1}'' - z_{k-1}}{z_{k-1} - z_{k-1}'}\Big) \right| \sim \left| z_{k-1}'' - z_{k-1} \right| \le \varepsilon$$

(here t_c is the value of t at the uppermost point of the tail).

Now, if the next $(k-1)$-th chord is also long, then, similarly, it can be paired with another long one so that they give a contribution to the integral proportional to $\left| z_{k-2}'' - z_{k-2} \right| \le \varepsilon$.

In the case the $(k-1)$-th chord is short (i.e. connects two points z_{k-1}'', z_{k-1} of the tail) we have the following estimate for the double integral corresponding to k-th and $(k-1)$-th

chords:

$$\left| \int\limits_{t_{k-2}}^{t_c} \left(\int\limits_{t_{k-1}}^{t_c} d(\ln(z_k - z'_k)) \; - \int\limits_{t_{k-1}}^{t_c} d(\ln(z''_k - z'_k)) \right) \frac{dz''_{k-1} - dz_{k-1}}{z''_{k-1} - z_{k-1}} \right|$$

$$\leq \; C \left| \int\limits_{t_{k-2}}^{t_c} \left| z''_{k-1} - z_{k-1} \right| \frac{dz''_{k-1} - dz_{k-1}}{\left| z''_{k-1} - z_{k-1} \right|} \right|$$

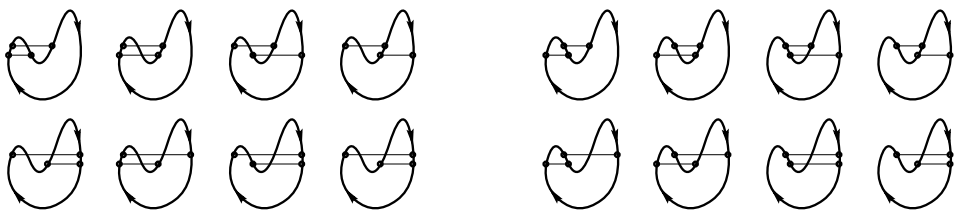
$$= \; C \left| \int\limits_{t_{k-2}}^{t_c} d(z''_{k-1} - z_{k-1}) \right| \sim \left| z''_{k-2} - z_{k-2} \right| \leq \varepsilon.$$

And again the integral is proportional to $\left| z''_{k-2} - z_{k-2} \right|$.

Continuing such considerations, we see that the difference between $Z(T_0)$ and $Z(T_1)$ is proportional to ε. Now by horizontal deformations we can make ε tend to zero. This proves the theorem and completes the proof of the Kontsevich integral's invariance in the class of Morse knots.

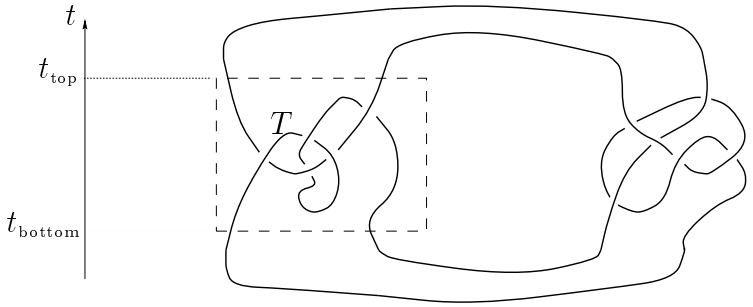
2.6 Reduction of the number of summands

We can use the one-term relation to reduce the number of summands in the Kontsevich integral. Indeed, in the definition of the integral we can consider only those pairings $P = \{(z_j, z'_j)\}$ for which the corresponding chord diagram D_P contains no isolated chords. In particular, for the example of Sec. 2.1, this leaves only one component $t_{c_1} < t_1 < t_2 < t_{c_2}$ of the integration domain and, in this component, leaves the following sixteen pairings out of the total number of thirty six:



The following useful observation belongs to the folklore; we learned it from T. Q. T. Le [Le]. It gives another way to reduce the number of summands and in fact is crucial for the construction of the universal Vassiliev invariant in the next subsection.

Suppose we have a Morse knot K with a distinguished tangle T.

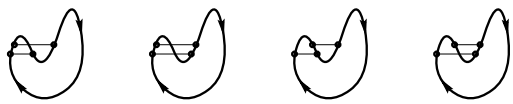


Let $t_{\rm top}$ and $t_{\rm bottom}$ be the maximal and minimal values of t on the tangle T. Consider the integral $Z_T(K)$ defined by the same formula as the Kontsevich integral $Z(K)$ but with a reduced number of summands in the integrand for those components of the integration domain whose projection on the coordinate axis t_j is entirely contained in the segment $[t_{\rm bottom}, t_{\rm top}]$. Namely, in $Z_T(K)$ we take into account only those pairings $P = \{(z_j, z'_j)\}$ on the level t_j ($t_{\rm bottom} < t_j < t_{\rm top}$) for which either both points (z_j, t_j) and (z'_j, t_j) belong to the tangle T or neither of them does. This means that we omit the summands of $Z(K)$ which contain a 'long' chord (connecting T with the remaining part of K). We call $Z_T(K)$ the *Kontsevich integral reduced with respect to the tangle* T.

Lemma [Le]. *Reduction with respect to any tangle does not change the value of the Kontsevich integral:* $Z_T(K) = Z(K)$.

Warning. The Lemma does not say anything about the integrals over a particular component of the integration domain. It only claims the coincidence of the total integrals.

Example. Consider the unknot from the example of Sec. 2.1 and the following tangle in it: . Then the lemma leaves only four summands for the component $t_{c_1} < t_1 < t_2 < t_{c_2}$ out of the initial (Sec. 2.1) 36 summands or out of 16 summands listed in the beginning of this subsection:



We shall explicitly compute the corresponding integral in Sec. 2.9.2, thus providing an example of how the Lemma can be used in practice.

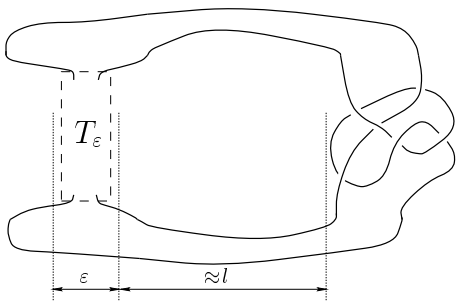
To prove the Lemma we need the following assertion which, as far as we know, is tacitly kept in mind in all texts about the Kontsevich integral but is not stated explicitly anywhere.

Sublemma. *Let (K_s, T_s) be a one-parameter deformation of the pair $(K = K_0, T = T_0)$ of a knot K with a distinguished tangle T in the class of Morse knots. Then the reduced integral is invariant:* $Z_{T_s}(K_s) = Z_T(K)$.

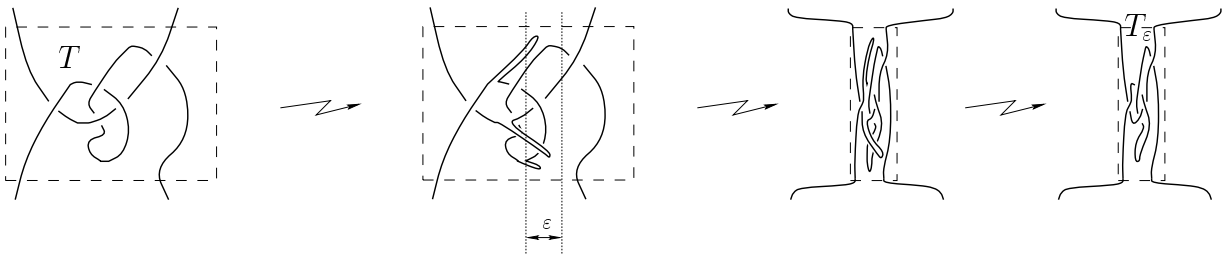
Proof of the sublemma. We can repeat step by step the proof of invariance of the ordinary Kontsevich integral given in Sec. 2.4 – 2.5. Invariance under the movement of a critical point proceeds in exactly the same way, word by word. In the proof of invariance under a horizontal deformation there is, however, a minor difference. It appears when we try to repeat the proof of the theorem from Sec. 2.4 in case (3) (see page 11). In this case we use the four-term relation, assuming that we have enough pairings: all of the six tangled chord diagrams displayed on page 11 have to be at our disposal. But, *a priori*, it is a question whether all of them participate in the reduced integral. An affirmative answer to this question follows from the next, rather obvious, fact. If one of the six tangled chord diagrams contains no 'long' chords and thus enters in the reduced integral, then all the remaining five diagrams have the same property, too. This proves the sublemma.

Proof of the lemma. We must prove that the difference $Z(K) - Z_T(K)$ is equal to zero.

Let us deform the knot K shrinking the tangle T sideways into a narrow rectangle box of width ε:



We will prove that the difference $Z(K_\varepsilon) - Z_{T_\varepsilon}(K_\varepsilon)$ tends to zero as $\varepsilon \to 0$. Since both integrals are invariant under the deformation, this immediately implies the lemma. Unfortunately this shrinking cannot be done by horizontal deformations only because we have to drag critical points into the narrow box. To perform the shrinking, we will first move all critical points of T into the box, using special movements of critical points. Then, by a horizontal deformation, we shrink the whole tangle to the box. After that return the critical points to their initial levels:



Applying a similar procedure to the portion of $K_\varepsilon \setminus T_\varepsilon$ that lies between the horizontal planes $t = t_{\text{bottom}}$ and $t = t_{\text{top}}$, we can squeeze it into a parallelepiped of width ε, too. (*Remark:* The process under study involves moving the critical values, hence it makes no sense to consider the integral over one particular component. This is why the lemma is about the integrals over the whole integration domain.)

Now the difference $Z(K_\varepsilon) - Z_{T_\varepsilon}(K_\varepsilon)$ consists of summands of $Z(K_\varepsilon)$ with 'long' chords. So it is enough to prove that the integral corresponding to one 'long' chord tends to zero. For

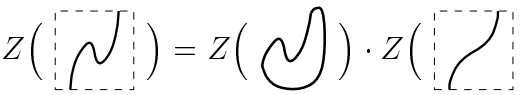
a long chord (z_j, z'_j) connecting T_ε with the remaining part of K_ε, the corresponding integral is equal to the logarithm of the quotient of the differences $z_j - z'_j$ at the upper and lower limits of integration, which obviously tends to zero as $\varepsilon \to 0$. The lemma is proved.

2.7 The universal Vassiliev invariant

We have mentioned before that the Kontsevich integral is not invariant under deformations of the knot that change the number of critical points of the function t. An elementary perestroika of this kind is the *insertion of a hump*. The next theorem shows how the Kontsevich integral changes under this operation.

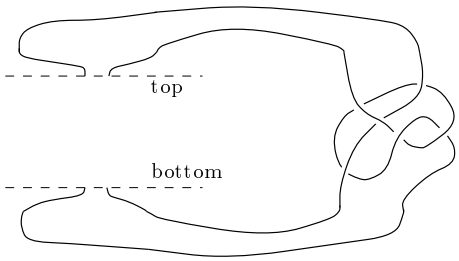
Theorem. *We have*

$$Z\Big(\quad\Big) = Z\Big(\quad\Big) \cdot Z\Big(\quad\Big),$$



where the first and the third pictures depict an arbitrary knot, differing only in the shown fragment, while the second picture represents an unknot embedded in $\mathbb{R}^3$ in the specified way and the product is the product in the completed algebra $\overline{\mathcal{A}}$ of chord diagrams.

To prove the theorem we will extend the construction of the Kontsevich integral for a tangle $Z(T) \in \overline{\mathcal{A}}_T$ to the complement of the tangle. For example, the complement $K \setminus T$ for the pair K, T from Sec. 2.6 looks like



We consider chord diagrams on $K \setminus T$ made up of horizontal chords modulo one- and four-term relations which look exactly like tangled ones. The Kontsevich integral $Z(K \setminus T) \in \overline{\mathcal{A}}_{K \setminus T}$ is defined in the same way as for the tangles.

If T' be a tangle whose top and bottom coincide with top and bottom of $K \setminus T$, then one can obviously define their product $T' \cdot (K \setminus T)$ and the product of chord diagrams on them (*concatenation product*):

$$\overline{\mathcal{A}}_{T'} \times \overline{\mathcal{A}}_{T \setminus K} \to \overline{\mathcal{A}}.$$

By definition, for any tangle T in the knot K

$$Z_T(K) = Z(T) \cdot Z(K \setminus T),$$

and the lemma of Sec. 2.6 implies

Lemma 1. (*Multiplication formula for the Konstevich integral*)

$$Z(K) = Z(T) \cdot Z(K \setminus T).$$

We will investigate a little further the properties of the Kontsevich integral for *one-string* tangles and their complements and then use them to prove the theorem.

A *one-string tangle* is a connected tangle that has only one string at the top and at the bottom. For such tangles, as well as for their complements, we have two naturally defined *closure* operations:

- For the tangle.
  1. Closure of the tangle $T \mapsto \overline{T}$. The tangle is converted into a knot by adding an arc with two nondegenerate critical points.
  2. Closure of chord diagrams: the mapping $\overline{\mathcal{A}}_T \to \overline{\mathcal{A}}$ defined by the identification of the two endpoints of T. Let $\overline{Z}(T) \in \overline{\mathcal{A}}$ stand for the Kontsevich integral of the tangle closed in this way.
- For the complement.
  1. Closure of the complement $K \setminus T \mapsto \overline{K \setminus T}$. The complement is converted into a knot by adding an arc without critical points.
  2. Closure of chord diagrams: the mapping $\overline{\mathcal{A}}_{K \setminus T} \to \overline{\mathcal{A}}$ defined by the identification of the two endpoints of $K \setminus T$. Let $\overline{Z}(K \setminus T) \in \overline{\mathcal{A}}$ stand for the Kontsevich integral of the complement closed in this way.

Here is a figure illustrating the closure operation for tangles:

As you can see, the closure is carried out by adding the arc .

Lemma 2. For a one-string tangle T, the following identities hold:

1. $\overline{Z}(T) = Z(\overline{T})$.
2. $\overline{Z}(K \setminus T) = Z(\overline{K \setminus T})$.

Proof.

1. Indeed, by Lemma 1, $Z(\overline{T}) = Z($ $) = Z(T) \cdot Z($ $)$. But each chord diagram on is equal to zero by one-term relation except for the diagram of order 0 (without any chords) which closes the chord diagrams on T.

2. The same argument holds for the complements, too, but it is even simpler, because in this case on the additional arc there is only one (empty) chord diagram.

Lemma 3. *For a one-string tangle T and its complement $K\setminus T$ the concatenation product $\overline{\mathcal{A}}_T\times\overline{\mathcal{A}}_{K\setminus T}\to\overline{\mathcal{A}}$ agrees with the product in the algebra $\overline{\mathcal{A}}$ in the sense that the following identity holds:*

$$Z(T)\cdot Z(K\setminus T)=\overline{Z}(T)\cdot\overline{Z}(K\setminus T).$$

Proof. This follows directly from the definitions.

Proof of the theorem. By Lemmas 2 and 3 we have:

$$Z\Big(\quad\Big) \;=\; \overline{Z}\Big(\quad\Big)\cdot\overline{Z}\Big(\quad\Big)$$

$$\;=\; Z\Big(\quad\Big)\cdot Z\Big(\quad\Big)\,.$$

The theorem is proved.

Definition. The theorem allows to define the universal Vassiliev invariant by the formula

$$\boxed{I(K)=Z(K)\Big/\Big(Z\Big(\quad\Big)\Big)^{c/2}},$$

where c denotes the number of critical points of K and quotient means division in the algebra $\overline{A}$: $(1+a)^{-1}=1-a+a^2-a^3+\ldots$. The invariance of $Z(K)$ in the class of Morse knots and the theorem above imply that $I(K)$ is invariant under an arbitrary deformation of K.

2.8 Proof of the Kontsevich Theorem

First of all we reformulate the Kontsevich theorem as follows.

Theorem. *Let w be a weight system of order n (i. e. vanishing on chord diagrams whose number of chords is different from n). Then there exists a Vassiliev invariant of order $\leq n$ whose symbol is w.*

Proof. The desired knot invariant is given by the formula

$$K\longmapsto w(I(K)).$$

Let D be a chord diagram of order n and K_D be a singular knot with chord diagram D. The theorem follows from the fact that $I(K_D)=D+(\text{terms of order}>n)$. Since the denominator of $I(K)$ always starts with the unit of the algebra $\mathcal{A}$, it is sufficient to prove that

$$\boxed{Z(K_D)=D+(\text{terms of order}>n)}$$

To see what happens at a single double point let us consider the difference of Z on two knots K_+ and K_- where an overcrossing in K_+ is changed to an undercrossing in K_-. These knots can be deformed as follows:

$Z\Big($ $\Big) \; - \; Z\Big($ $\Big) \qquad = \qquad Z\Big($ $\Big) \; - \; Z\Big($ $\Big) \; .$

The two knots differ only in that the first knot has one string that makes a small curl around another string. We assume that this curl is very thin and contained in a slice of thickness δ. Enclose the curl in a box of height δ and consider it as a tangle T_+. Let T_- be the similar tangle for K_- consisting of two parallel vertical strings:

$T_+ \; =$ $\qquad\qquad T_- \; =$.

By Lemma 1 of section 2.7 we have

$$Z(K_+) = Z(T_+) \cdot Z(K_+ \setminus T_+) \qquad\qquad Z(K_-) = Z(T_-) \cdot Z(K_- \setminus T_-) \; .$$

But $K_+ \setminus T_+$ coincides with $K_- \setminus T_-$, therefore,

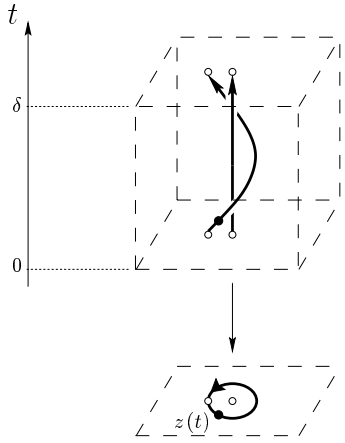
$$Z(K_+) - Z(K_-) = (Z(T_+) - Z(T_-)) \cdot Z(K_+ \setminus T_+) \; .$$

Now, for the tangle T_- both functions $z(t)$ and $z'(t)$ are constant, hence $Z(T_-)$ consists of only one summand: the trivial tangled chord diagram $D_-^{(0)}$ on T_- (here 'trivial' means with an empty set of chords) with coefficient one. The zero degree term $D_+^{(0)}$ of $Z(T_+)$ is also the trivial chord diagram on T_+ with coefficient one. But $D_-^{(0)} = D_+^{(0)}$ as chord diagrams. Thus the difference $Z(T_+) - Z(T_-)$, and therefore the difference $Z(K_+) - Z(K_-)$, starts with a first degree term, which comes from the chord connecting the two strings of T_+.

Now consider the n double points $p_1, \ldots, p_n$ of the knot K_D. Let $\varepsilon_1, \ldots, \varepsilon_n$ be a sequence of signs '+' or '−'. Denote by $K_{\varepsilon_1,\ldots,\varepsilon_n}$ the knot obtained from K_D by a positive (like in K_+ above) resolution of p_j if $\varepsilon_j = +$ or by a negative (like in K_- above) resolution of p_j if $\varepsilon_j = -$. Deforming all knots $K_{\varepsilon_1,\ldots,\varepsilon_n}$ near the points $p_1, \ldots, p_n$ in the same manner as above for K_+ and K_-, we get, for every $j = 1, \ldots, n$, two tangles $T_{j,+}$ and $T_{j,-}$ contained in a box of height δ near the point p_j. The tangle $T_{j,-}$ always consists of two vertical strings. Then we can write

$$\begin{aligned}
Z(K_D) &= \sum_{\varepsilon_1,\ldots,\varepsilon_n} \varepsilon_1 \cdot \ldots \cdot \varepsilon_n \cdot Z(K_{\varepsilon_1,\ldots,\varepsilon_n}) \\
&= \sum_{\varepsilon_1,\ldots,\varepsilon_{n-1}} \varepsilon_1 \cdot \ldots \cdot \varepsilon_{n-1} \cdot (Z(T_{n,+}) - Z(T_{n,-})) \cdot Z(K_{\varepsilon_1,\ldots,\varepsilon_{n-1},+} \setminus T_{n,+}) \\
&= \sum_{\varepsilon_1,\ldots,\varepsilon_{n-2}} \varepsilon_1 \cdot \ldots \cdot \varepsilon_{n-2} \cdot (Z(T_{n,+}) - Z(T_{n,-})) \cdot (Z(T_{n-1,+}) - Z(T_{n-1,-})) \\
&\qquad\qquad \cdot Z(K_{\varepsilon_1,\ldots,\varepsilon_{n-2},+,+} \setminus (T_{n,+} \cup T_{n-1,+})) \\
&\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&= (Z(T_{1,+}) - Z(T_{1,-})) \cdot \ldots \cdot (Z(T_{n,+}) - Z(T_{n,-})) \cdot Z(K_{+,\ldots,+} \setminus (\bigcup_j T_{j,+})) \; .
\end{aligned}$$

As above, each difference $Z(T_{j,+}) - Z(T_{j,-})$ starts with a term of degree 1. Hence the product $Z(K_D)$ starts from degree n. Moreover, the term of degree n is proportional to a chord diagram whose j-th chord connects the two strings of the tangle $T_{j,+}$. Since the two strings of $T_{j,+}$ correspond to the double point p_j in K_D, this chord diagram is precisely D. Now to calculate the order n part of $Z(K_D)$ we must compute the coefficient of D. It is equal to the product of coefficients of one chord terms in $Z(T_{j,+})$. We will show that all of them are equal to 1. Indeed, it is possible to choose the coordinates z and t is such a way that $z'(t) \equiv 0$ and when t varies from 0 to δ the point $z(t)$ makes one complete turn around zero:



So we have

$$\frac{1}{2\pi i} \int_0^\delta \frac{dz - dz'}{z - z'} = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} = 1$$

by the Cauchy theorem. The Kontsevich theorem is proved.

We will now state some properties of the universal Vassiliev invariant $I(K)$ which reveal the precise meaning of the word "universal". The first of these properties follows from the argument used in the proof of the Kontsevich's theorem above.

Proposition 1. *If $I_n(K)$ denotes the nth graded component of the series $I(K)$, then the function $K \mapsto I_n(K)$ is a Vassiliev invariant of order n.*

The converse assertion is contained in the following

Proposition 2. *Any Vassiliev invariant can be factored through I: for any $v \in \mathcal{V}$ there exists a function f on $\mathcal{A}$ such that $v = f \circ I$.*

Proof. Let $v \in \mathcal{V}_n$. By Kontsevich's theorem we know that there is a function f_0 such that v and $f_0 \circ I_n$ have the same symbol. Therefore, the difference $v - f_0 \circ I_n$ belongs to $\mathcal{V}_{n-1}$ and is thus representable as $f_1 \circ I_{n-1}$. Proceeding in this way, we will finally obtain:

$$v = \sum_{i=1}^{n} f_i \circ I_{n-i}.$$

Remark. The construction of the previous proof shows that the universal Vassiliev invariant induces a splitting of the filtred space $\mathcal{V}$ into a direct sum with summands isomorphic to the factors $\mathcal{V}_n/\mathcal{V}_{n-1}$.

The two last propositions, taken together, imply the following theorem.

Theorem. *The universal Vassiliev invariant I is exactly as strong as the set of all Vassiliev invariants: for any two knots K_1 and K_2 we have:*

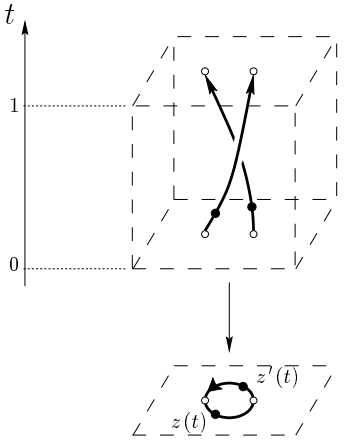
$$I(K_1) = I(K_2) \quad \Leftrightarrow \quad \forall v \in \mathcal{V} \quad v(K_1) = v(K_2).$$

2.9 Examples of calculation

Explicit computation of the Kontsevich integral for a given knot is a difficult problem. Let us note that, so far, no closed form expression for the entire Kontsevich integral is known even for the trefoil! In the examples below, we consider a simple tangle with two strings and an unknot with four critical levels.

2.9.1. Here we compute the Kontsevich integral of the tangle .

To start with introduce parametrizations $z(t)$ and $z'(t)$ of the two strings of the tangle according to the picture:



$$z(t) = e^{i(-\pi+\pi t)} \ ;$$

$$z'(t) = e^{i\pi t} \ ;$$

$$0 \le t \le 1 \ ;$$

$$z(t) - z'(t) = e^{i\pi t}(e^{-i\pi} - 1) \ ;$$

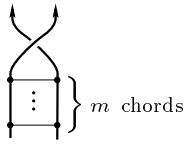
$$\ln(z(t) - z'(t)) = i\pi t + \ln(e^{-i\pi} - 1) \ ;$$

$$d(\ln(z(t) - z'(t))) = i\pi dt \ .$$

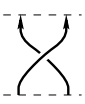
Now it is easy to compute the integral:

$$\begin{aligned} Z\left(\right) &= \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \cdot D_m \int\limits_{0 \le t_1 < \ldots < t_m \le 1} \bigwedge_{j=1}^{m} d(\ln(z(t_j) - z'(t_j))) \\ &= \sum_{m=0}^{\infty} \frac{1}{2^m} \cdot D_m \int\limits_{0 \le t_1 < \ldots < t_m \le 1} dt_1 \wedge \ldots \wedge dt_m \\ &= \sum_{m=0}^{\infty} \frac{D_m}{2^m \cdot m!} \ , \end{aligned}$$

where D_m is the following tangled chord diagram.

D_m = 

In terms of multiplication of tangled chord diagrams the answer can be written nicely as

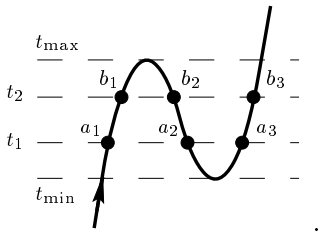
$Z\left(\right.$  $\left.\right) = \exp\left(\frac{1}{2}\cdot\right.$ $\left.\right)\cdot$,

where the last factor means a tangled chord diagram without chords.

2.9.2. We borrowed this example from [Ar]. It provides a good exercise in undergraduate calculus.

Let us calculate the coefficient of the chord diagram in $Z\Big($ $\Big)$.

Since $Z\Big($ $\Big) = \overline{Z\Big($ $\Big)}$, it is sufficient to calculate the integral corresponding to the choice of two levels t_1 and t_2 and pairs (z_1, z_1') and (z_2, z_2') such that the points of the first pair alternate with the points of the second one along the curve . Choose an orientation on the curve as shown in the picture. Denote by a_1, a_2, a_3 (resp. b_1, b_2, b_3) the three points of intersection of the curve with the level $\{t = t_1\}$ (resp. $\{t = t_2\}$):



.

The alternating condition leaves four possibilities for chosing (z_1, z_1') and (z_2, z_2'):

(z_2, z_2')	(b_1, b_3)	(b_2, b_3)	(b_1, b_3)	(b_2, b_3)
(z_1, z_1')	(a_1, a_2)	(a_1, a_2)	(a_1, a_3)	(a_1, a_3)

.

Put $a_{ij} = a_j - a_i$ and $b_{ij} = b_j - b_i$. Then our integral has the form

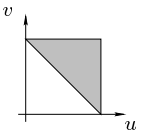
$$\frac{1}{(2\pi i)^2}\int\limits_{t_{\min}<t_1<t_2<t_{\max}} -d\ln a_{12}\wedge d\ln b_{13} + d\ln a_{12}\wedge d\ln b_{23} + d\ln a_{13}\wedge d\ln b_{13} - d\ln a_{13}\wedge d\ln b_{23}$$

$$= \frac{1}{-4\pi^2}\int\limits_{t_{\min}<t_1<t_2<t_{\max}} -d\ln\left(\frac{a_{12}}{a_{13}}\right)\wedge d\ln b_{13} + d\ln\left(\frac{a_{12}}{a_{13}}\right)\wedge d\ln b_{23}$$

$$= -\frac{1}{4\pi^2}\int\limits_{t_{\min}<t_1<t_2<t_{\max}} d\ln\left(\frac{a_{12}}{a_{13}}\right)\wedge d\ln\left(\frac{b_{23}}{b_{13}}\right) \quad .$$

The mapping $u(t_1,t_2)=\dfrac{a_{12}}{a_{13}},\ v(t_1,t_2)=\dfrac{b_{23}}{b_{13}}$ transforms the integration domain $t_{\min}<t_1<t_2<t_{\max}$ to the standard triangle:

$$u\le 1;\quad v\le 1;\quad u+v\ge 1$$



.

It is easy to see that the mapping $(t_1,t_2)\mapsto(u,v)$ is a diffeomorphism with a negative Jacobian (we assume that a_{12} is a decreasing function in t, while a_{23} is an increasing function). Therefore after the change of variables the integral becomes:

$$\frac{1}{4\pi^2}\int\limits_{\substack{u\le 1;\ v\le 1;\\ u+v\ge 1}} d\ln u\wedge d\ln v \;=\; \frac{1}{4\pi^2}\int\limits_0^1\left(\int\limits_{1-u}^1 d\ln v\right)\frac{du}{u} \;=\; -\frac{1}{4\pi^2}\int\limits_0^1\ln(1-u)\frac{du}{u}\ .$$

Taylorising the logarithm we get

$$-\frac{1}{4\pi^2}\sum_{k=1}^{\infty}\int\limits_0^1\frac{u^k}{k}\frac{du}{u} \;=\; -\frac{1}{4\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2} \;=\; -\frac{1}{4\pi^2}\zeta(2) \;=\; -\frac{1}{24}.$$

Calculating more complicated terms of the Kontsevich integral leads to some values of D. Zagier's multivariate generalization of ζ-function:

$$\zeta(a_1,\dots,a_n) \;=\; \sum_{0<k_1<k_2<\dots<k_n} k_1^{-a_1}\dots k_n^{-a_n}$$

(see [LM1]).

Remark. By definition, $Z\Big($ $\Big)$ is an element of $\overline{\mathcal{A}}$ inverse to the universal Vassiliev invariant of the unknot $I(O)$. A beautiful explicit formula for $I(O)$ was conjectured in [BNGRT] and recently proved in [BNLT] (see also [Ko2]).

3 Operations on knots and Kontsevich's integral

We saw in Sec. 2.4 that the Kontsevich integral is multiplicative with respect to the connected sum of knots. In this section we consider three other operations on knots and describe the effect of these operations on the Kontsevich integral. The operations in question are:

- R — mirror reflection (changing the orientation of the ambient space $\mathbb{R}^3$).
- S — changing the orientation of a knot.
- M — mutation of a knot with respect to a distinguished tangle T.

The effect of the operation R is closely related to the fact that the Kontsevich integral is real, proved in Sec. 3.1 below. Roughly speaking, this result means that odd order Vassiliev invariants detect the amphicheirality of knots, while even order invariants do not.

The operation S allows to reformulate the invertibility problem for knots in combinatorial terms, as a property of chord diagrams (Sec. 3.2).

Using the nice behaviour of the Kontsevich integral under mutations of knots, we describe (Sec. 3.3) a counterexample to the intersection graph conjecture from [CDL1] belonging to T. Q. T. Le [Le].

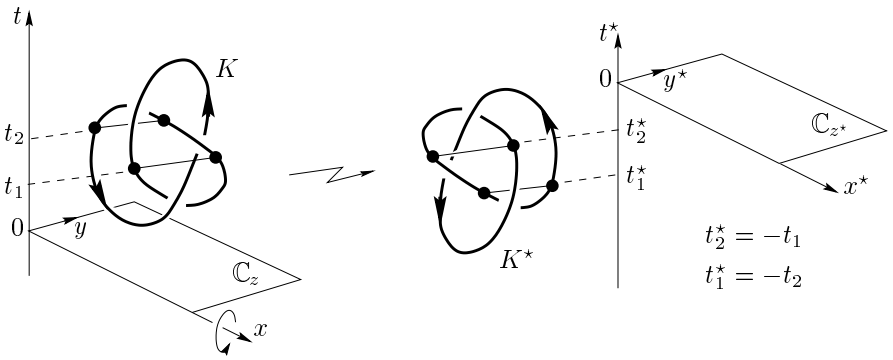
3.1 Mirror symmetry

3.1.1 Reality of the integral

Choose a countable family of chord diagrams $\{D_i\}$ that constitute a basis in the linear space $\mathcal{A}$. The Kontsevich integral of a knot K can be written as an infinite series $Z(K) = \sum c_i D_i$ whose coefficients c_i depend on the knot K. A priori these coefficients are complex numbers.

Theorem. *All coefficients c_i of the Kontsevich integral $Z(K) = \sum c_i D_i$ are real numbers.*

Proof. Rotate the knot K around the real axis x in $\mathbb{C}_z$ through 180° and denote the obtained knot by $K^\star$:



To distinguish the objects (space, coordinates etc.) related to the knot $K^\star$, we will tag the corresponding symbols with a star. Coordinates $t^\star$, $z^\star$ in the space that contains $K^\star$ are related to the coordinates t, z in the ambient space of K as follows: $t^\star = -t$, $z^\star = \overline{z}$.

The rotation can be realized by a smooth isotopy, hence the universal Vassiliev invariants of K and $K^\star$ coincide: $I(K) = I(K^\star)$. The two knots have the same number c of critical points of the height function. Therefore

$$Z(K) = I(K) \cdot Z(h)^{c/2} = I(K^\star) \cdot Z(h)^{c/2} = Z(K^\star),$$

where $h =$ is the plane unknot with one hump. Therefore, to prove the reality of $Z(K)$, it is enough to prove that each coefficient of $Z(K^\star)$ is the complex conjugate to the corresponding coefficient of $Z(K)$.

Fix the number of chords m. Then each pairing $P = \{(z_j, z'_j)\}$, $1 \le j \le m$ for the knot K corresponds to a pairing $P^\star = \{(z_j^\star, z_j^{\star\prime})\}$ for the knot $K^\star$, where $z_j^\star = \overline{z}_{m-j+1}$ and $z_j^{\star\prime} = \overline{z'}_{m-j+1}$. Note that the corresponding chord diagrams are equal: $D_P = D_{P^\star}$. Moreover, $\downarrow^\star = 2m - \downarrow$ and hence $(-1)^{\downarrow^\star} = (-1)^\downarrow$. The simplex $\Delta = t_{\min} < t_1 < \ldots < t_m < t_{\max}$ for the

variables t_i corresponds to the simplex $\Delta^\star = -t_{\max} < t^\star_m < \ldots < t^\star_1 < -t_{\min}$ for the variables $t^\star_i$. The coefficient of $D_{P^\star}$ in $Z(K^\star)$ is

$$c(D_{P^\star}) \;=\; \frac{(-1)^\downarrow}{(2\pi i)^m}\int \bigwedge_{j=1}^m d\ln(z^\star_j - z^{\star\prime}_j)\ ,$$

where $z^\star_j$ and $z^{\star\prime}_j$ are understood as functions in $t^\star_1$, ..., $t^\star_m$ and the integral is taken over a connected component in the simplex $\Delta^\star$. In the last integral we make the change of variables according to the formula $t^\star_j = -t_{m-j+1}$. The Jacobian of this transformation is equal to $(-1)^{m(m+1)/2}$. Therefore,

$$c(D_{P^\star}) \;=\; \frac{(-1)^\downarrow}{(2\pi i)^m}\int (-1)^{m(m+1)/2} \bigwedge_{j=1}^m d\ln(\overline{z}_{m-j+1} - \overline{z'}_{m-j+1})$$

(integral over the corresponding connected component in the simplex Δ). Now permute the differentials to arrange the subscripts in the increasing order. The sign of this permutation is $(-1)^{m(m-1)/2}$. Note that $(-1)^{m(m+1)/2}\cdot(-1)^{m(m-1)/2} = (-1)^m$. Hence,

$$c(D_{P^\star}) \;=\; \frac{(-1)^\downarrow}{(2\pi i)^m}(-1)^m\int \bigwedge_{j=1}^m d\ln(\overline{z}_j - \overline{z'}_j) \;=\; \frac{(-1)^\downarrow}{(-2\pi i)^m}\int \bigwedge_{j=1}^m \overline{d\ln(z_j - z'_j)} \;=\; \overline{c(D_P)}.$$

The theorem is proved.

Remark. M. Kontsevich [Ko1] claimed that the coefficients of $Z(K)$ are *rational*, i. e. belong to $\mathbb{Q}\subset\mathbb{R}$. He didn't give any proof but pointed to a rather difficult paper [Dr]. A proof can be found in [LM2] (see also [BN2]).

3.1.2 Mirror reflection

Let R be the operation sending a knot to its mirror image. Define the corresponding operation R on chord diagrams to be identical on the diagrams of even order (number of chords) and to be multiplication by (-1) on the diagrams of odd order.

Theorem. *The Kontsevich integral commutes with the operation R:*

$$Z(R(K)) = R(Z(K))\ ,$$

where by $R(Z(K))$ we mean simultaneous application of R to all the chord diagrams participating in $Z(K)$.

Proof. Let us realize the operation R on knots by the reflection of $\mathbb{R}^3 = \mathbb{R}_t\times\mathbb{C}_z$ coming from the complex conjugation in $\mathbb{C}_z$: $(t,z)\mapsto(t,\overline{z})$.

Consider the Kontsevich integral of K:

$$Z(K) = \sum_{m=0}^\infty \frac{1}{(2\pi i)^m}\int \sum_P (-1)^\downarrow D_P \bigwedge_{j=1}^m d\ln(z_j - z'_j)\ .$$

Then the Kontsevich integral for $R(K)$ will look as

$$\begin{aligned} Z(R(K)) &= \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int \sum_P (-1)^{\downarrow} D_P \bigwedge_{j=1}^{m} d\ln(\overline{z_j - z'_j}) \\ &= \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \overline{\int \sum_P (-1)^{\downarrow} D_P \bigwedge_{j=1}^{m} d\ln(z_j - z'_j)} \\ &= \overline{\sum_{m=0}^{\infty} \frac{(-1)^m}{(2\pi i)^m} \int \sum_P (-1)^{\downarrow} D_P \bigwedge_{j=1}^{m} d\ln(z_j - z'_j)} \ . \end{aligned}$$

We see that the terms of $Z(R(K))$ with an even number of chords coincide with those of $\overline{Z(K)}$ and terms of $Z(R(K))$ with an odd number of chords differ from the corresponding terms of $\overline{Z(K)}$ by a sign. Since $Z(K)$ is real, this implies the theorem.

A knot K is called *amphicheiral*, if it is equivalent to its mirror image: $K = R(K)$.

Corollary. The Kontsevich integral $Z(K)$ and the universal Vassiliev invariant $I(K)$ of an amphicheiral knot K consist only of even order terms.

Proof. Indeed, the theorem implies that $Z(K) = R(Z(K))$, hence all the odd order terms in the series $Z(K)$ vanish. The quotient of two even series in the graded completion $\overline{\mathcal{A}}$ is obviously even, therefore the same property holds for $I(K) = Z(K)/Z(h)^{c/2}$, too.

3.2 Orientations of knots

Let S be the operation on knots which inverts their orientation. The same letter will also denote the analogous operation on chord diagrams (inverting the orientation of the outer circle or, which is the same thing, axial symmetry of the diagram).

Theorem (cf. [BN1]). *The Kontsevich integral commutes with the operation S of inverting the orientation*:

$$Z(S(K)) = S(Z(K)).$$

Proof. The required identity follows directly from the definition of the Kontsevich integral in Sec. 2. Indeed, the number $\downarrow$ for the knot $S(K)$ with inverse orientation is equal to the number $\uparrow$ for the initial knot K. Here by $\uparrow$ we of course mean the number of points (z_j, t_j) or (z'_j, t_j) in a pairing P where the coordinate t increases along the orientation of K. Since the number of points in a pairing is always even, $(-1)^{\downarrow} = (-1)^{\uparrow}$. Hence the corresponding chord diagrams appear in $Z(S(K))$ and in $S(Z(K))$ with the same sign. The theorem is proved.

Corollary. *Vassiliev invariants do not distinguish the orientation of knots if and only if all chord diagrams are symmetric $D = S(D)$ modulo one and four-term relations.*

The calculations of [Knei] show that up to order 12 all chord diagrams are symmetric. For bigger orders the problem is still open.

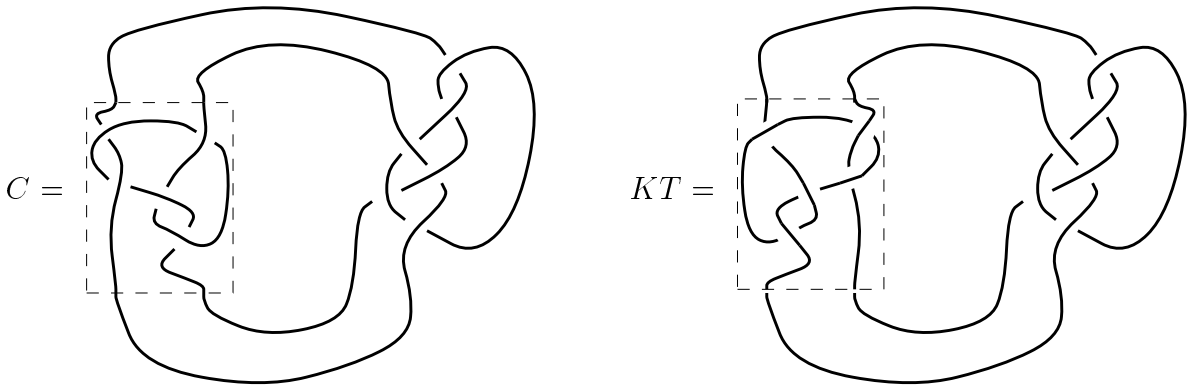
3.3 Mutation

The purpose of this subsection is to show that the Kontsevich integral commutes with the operation of mutation. We learned this fact from T. Q. T. Le [Le]. As an application, we will construct a counterexample to the intersection graph conjecture from [CDL1]. Finally we will formulate a conjecture about the combinatorial description of Vassiliev invariants that do not distinguish mutant knots.

3.3.1 Mutation of knots

Definition. Suppose we have a knot K with a distinguished tangle T which has two strings at the bottom and two strings at the top. Let us cut out the tangle, rotate it through 180° and insert it back. This operation M is called a *mutation* and the obtained knot $M(K)$ is called a *mutant* of K.

Here is the widely known example of a mutant pair of knots, the *Conway* and *Kinoshita–Teresaka* knots:



Theorem [MC]. *There exists a Vassiliev invariant v of order 11 such that $v(C) \neq v(KT)$.*

Morton and Cromwell produced the invariant v using the Lie algebra $\mathrm{gl}(N)$ with a non-standard representation (or, in other words, HOMFLY polynomial of certain cablings of the knots).

3.3.2 Mutation of the Kontsevich integral

Let K be a knot with a distinguished tangle T which has two strings at the bottom and two strings at the top. Let $M(K)$ be the mutant of K obtained by the rotation of T. We shall define the mutation $M(Z(K))$ of the Kontsevich integral as the simultaneous mutation of all chord diagram participating in $Z(K)$. Then the following theorem holds.

Theorem [Le]. *The Kontsevich integral commutes with the mutation operation M:*

$$Z(M(K)) = M(Z(K)).$$

By the lemma of Sec. 2.6 $Z(K)$ does not contain any chord diagram with a chord connecting T with the remaining part of K. This means that all chord diagrams which appear in $Z(K)$ are products (in the sense of Sec. 2.7) of tangled chord diagrams on T and chord diagrams on $K \setminus T$. In [CDL2], we called a tangled chord diagram on T, considered as a part of the whole chord diagram for K, a *share*.

Definition. A *share* is a part of the chord diagram consisting of two arcs of the outer circle and all chords whose endpoints belong to these arcs. Moreover, we require that the whole chord diagram contains no chords that connect these two arcs with the remaining part of the circle.

Examples. In the left picture below the collection of four chords inside the dotted oval is a share. But the collection of three fat chords is not a share because there exists a fourth chord that separates their ends. In the right picture there are two shares shown by dotted arcs.

The choice of a tangle T in a knot K distinguishes the T-shares (which correspond to the tangled chord diagrams of $Z(T)$) in all chord diagrams that appear in $Z(K)$.

Definition. The *mutation* $M(Z(K))$ of $Z(K)$ is the simultaneous rotation of the T-shares in all diagrams of $Z(K)$.

Mutants of the chord diagrams of last two examples look as follows

Proof of the theorem. Let T_r be a tangle obtained from T by rotation under the mutation M. Then $Z(K) = Z(T) \cdot Z(K \setminus T)$ and $Z(M(K)) = Z(T_r) \cdot Z(K \setminus T)$. So

$$M(Z(K)) = M(Z(T) \cdot Z(K \setminus T)) = M(Z(T)) \cdot Z(K \setminus T) = Z(T_r) \cdot Z(K \setminus T) = Z(M(K)) \ .$$

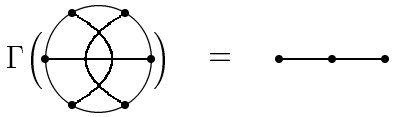
The theorem is proved.

3.3.3 Intersection graph conjecture

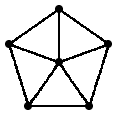
Here we introduce the notion of the intersection graph of a chord diagram and formulate the intersection graph conjecture [CDL2]. A variant of this notion was used in [BNG]. In the end we describe a counterexample to the intersection graph conjecture and formulate another conjecture instead.

Definition. *The intersection graph* $\Gamma(D)$ of a chord diagram D is a graph whose vertices correspond to the chords of D and two vertices are connected by an edge iff the corresponding chords intersect. (Two chords, a and b, are said to intersect if their endpoints a_1, a_2 and b_1, b_2 appear in the interchanging order a_1, b_1, a_2, b_2 along the circle.)

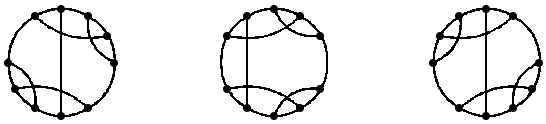
For example,



Note that not every graph can be obtained as the intersection graph of a chord diagram. The simplest example appears in degree 6 and is given by the graph



Different diagrams may have one and the same intersection graph. For example, there are three different diagrams



with the same intersection graph

Intersection graph conjecture. *If D_1 and D_2 are two chord diagrams whose intersection graphs are equal, $\Gamma(D_1) = \Gamma(D_2)$, then $D_1 = D_2$ as elements of $\mathcal{A}$ (i.e. modulo one- and four-term relations).*

This conjecture has been verified in the following situations:
1) for all diagrams D_1, D_2 with n chords up to $n = 8$ (we have checked this by computer);
2) for Vassiliev invariants v coming from the standard representations of Lie algebras $\mathrm{gl}(N)$ or $\mathrm{so}(N)$ according to Bar-Natan's construction [BN1] (this means that $\Gamma(D_1) = \Gamma(D_2)$ implies $v(D_1) = v(D_2)$);
3) when $\Gamma(D_1) = \Gamma(D_2)$ is a tree (see [CDL2]) or, more generally, D_1, D_2 belong to the forest subalgebra (see [CDL3]).

3.3.4 Counterexample to the Intersection Graph Conjecture

It is easy to see that the mutation of chord diagrams does not change the intersection graph. So, if the intersection graph conjecture were true, the Kontsevich integrals of mutant knots would coincide. Hence all Vassiliev invariants would take the same value on mutant knots. But this contradicts the result of Morton and Cromwell [MC] from Sec. 3.3.1. Therefore the intersection graph conjecture is false.

By the way, it follows from these arguments and our computer calculations that Vassiliev invariants up to order 8 do not distinguish mutants.

It might be interesting to find an explicit combinatorial description of a weight system that distinguishes chord diagrams with the same intersection graph (like the symbol of the Morton–Cromwell invariant v from Sec. 3.3.1).

Note that the previous argument proves the following proposition.

Proposition. *Let w be a weight system whose value on each chord diagram D depends only on the intersection graph $\Gamma(D)$. Then the Vassiliev invariant $w(I(K))$ does not distinguish mutants.*

So now we can refine the intersection graph conjecture as follows.

Conjecture. *If a Vassiliev invariant $f \in \mathcal{V}_n$ does not distinguish mutants, then its symbol $[f]$ depends only on the intersection graph.*

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Program Systems Institute,
Pereslavl-Zalessky, 152140, Russia
E-mail: `chmutov@botik.ru` `duzhin@botik.ru`